UZJMCS

Tashkent State Transport University

Uzbekistan Journal of Mathematics and Computer Science

July, 2026

Volume: 2

Special Issue: 3

ujmcs.tstu.uz

Online ISSN: 3093-8619

Print ISSN: 3093-8589

Tashkent 2026

TASHKENT STATE TRANSPORT UNIVERSITY





Uzbekistan Journal of Mathematics and Computer Science

July, 2026

Volume: 2

Special Issue: 3

ujmcs.tstu.uz

Tashkent State Transport University
Uzbekistan Journal of Mathematics and Computer Science

Volume 2, Special Issue 3, July,2026

Uzbekistan Journal of Mathematics and Computer Science is a prestigious publication established by Tashkent State Transport University (TSTU). The journal focuses on mathematics, computer sciences, and physics, and aims to disseminate cutting-edge research and applied studies in the field of transport and related fields. Located at 1 Temiryolchilar street, TSTU Printing House, Tashkent, Uzbekistan, 100167, the journal operates as a dynamic platform for both national and international academic and professional communities. Submissions and inquiries can be sent to the editorial team at ujmcs.tstu.uz.

Uzbekistan Journal of Mathematics and Computer Science publishes pioneering scientific and applied research conducted by universities, higher education institutions, research centers, and institutes in the fields of mathematics and computer science, both within the Republic of Uzbekistan and internationally. This journal serves as a vital platform for sharing primary scientific results from doctoral dissertations, including Doctor of Philosophy (PhD) and Doctor of Science (DSc) in the physics, mathematics, and computer sciences.

This special issue of the journal is devoted to the International Scientific and Practical Conference on "Current Problems of Mathematical-Physical Equations and Information Technologies" which was held on April 17–18, 2026, jointly organized by Namangan State Technical University and the V.I. Romanovskiy Institute of Mathematics of the Academy of Sciences of the Republic of Uzbekistan.

The conference featured the following main sections:

- Degenerate and mixed-type equations, fractional-order equations.
- Nonclassical equations of mathematical physics.
- Probability theory and mathematical statistics.
- Dynamic systems and optimal control.
- Problems of algebra, analysis and modern geometry.
- Computational mathematics and mathematical modeling.
- Application of information technologies in education.
- Promising technologies of artificial intelligence.
- Mathematical analysis of energy consumption, energy efficiency, and the use of green energy.

Articles are published in three languages: Russian and English, ensuring wide accessibility and facilitating cross-cultural academic exchange. **Uzbekistan Journal of Mathematics and Computer Science**, a leading publication in its field, continues to be an essential platform for sharing knowledge.

Editorial Team

Name	Scientific field	E-mail	Country
Editor in Chief			
Eshmamatova Dilfuza	Dynamic systems, graph theory, problems of mathematical biology, genetics, epidemiology and ecology	24dil@mail.ru	Uzbekistan
Deputy editor in Chief			
Artikbayev Abdullaaziz	Geometry and topology, Non-Euclidean Geometry	aartykbaev@mail.ru	Uzbekistan
Managing editor			
Ismoilov Sherzodbek	Geometry and topology, Riemanian manifold, Pseudo-Riemannian manifold.	sh.ismoilov@nuu.uz	Uzbekistan
Khamitov Azizbek	Differential Equations, Mathematical Physics Equation.	*azizbek.khamitov.93@mail.ru*	Uzbekistan
Technical editor			
Turdiyev Sirojiddin	-	s.turdiev@alumni.nsu.ru	Uzbekistan
Ibrokhimov Khusniddin	Differential Equations, Mathematical Physics Equation.	xusniddin571@mail.ru	Uzbekistan
Editors			
Zakirov Botir	Functional analysis, operator theory, theory Banach-Kantorovich space	botirzakirov@list.ru	Uzbekistan
Apakov Yusupzhon Pulatovich	Differential Equations, Mathematical Physics Equation.	*yusupjonapakov@gmail.com*	Uzbekistan
Begmatov Akrom	Ill-posed problems of mathematical physics and analysis, weakly and strongly ill-posed problems of integral	akrambegmatov@mail.ru	Uzbekistan

	geometry, integral equations, tomography, d'Alembert maps		
Azamov Abdulla	Dynamical Systems, Game Theory, Differential Equations	abdulla.azamov@gmail.com	Uzbekistan
Formanov Shokir	Probability Theory and Mathematical Statistics	shakirformanov@yandex.ru	Uzbekistan
Rozikov Utkir	Functional analysis, mathematical physics	rozikovu@mail.ru	Uzbekistan
Hayotov Abdullo	Computational mathematics	a.hayotov@centralasian.uz	Uzbekistan
Aripov Mirsaid	Ordinary differential equations	mirsaidaripov@mail.ru	Uzbekistan
Muradov Rustamjon Sobitkhonovich	Probability Theory and Mathematical Statistics, Mathematical Modeling and Number Theory	rustamjonmuradov@gmail.com	Uzbekistan
Jamalov Bakhriddinkhuja Ismoilovich	Differential Equations, Mathematical Physics Equation.	b-i-jamalov@mail.ru	Uzbekistan
Chilin Vladimir	Functional analysis, noncommutative integration, operator theory, theory Banach-Kantorovich space	vladimirchilin@gmail.com	Uzbekistan
Rakhmonov Zafar	Mathematical modeling, numerical methods	zraxmonov@inbox.ru	Uzbekistan
Sharipov Olimjon	Probability theory and mathematical statistics	osharipov@mail.ru	Uzbekistan
Raimova Gulnora	Probability Theory and Mathematical Statistics, Quantitative Methods in Economics, Econometrics, Financial Mathematics, Systems Analysis, Decision Making in Management	raimova27@gmail.com	Uzbekistan
Sharipov Anvarjon	Geometry and topology	asharipov@inbox.ru	Uzbekistan
Ashurov Ravshan	Mathematical Analysis, Differential Equations, Mathematical Physics	ashurovr@gmail.com	Uzbekistan

Karimov Erkin	Partial differential equations, Special functions, Direct and inverse problems	erkinjon@gmail.com	Uzbekistan
Azimov Jahongir	Theory of branching random processes, stochastic models with a discrete set of states, asymptotic analysis of statistical estimates	azimovjb@gmail.com	Uzbekistan
Isanov Rovshanbek	Inverse problems of equations of mathematical physics	diyora.isanova.97@bk.ru	Uzbekistan
Kasimov Shuxrat	Fuel balance of energy, solid fuels	shuxrat1812@mail.ru	Uzbekistan
Kendjaev Ravshan	Probability theory, mathematical statistics	lovsan63@mail.ru	Uzbekistan
Sharipova Lola	Probability theory, partial differential equations	lolaxon@gmail.com	Uzbekistan
Tuychiyeva Sayyora	Mathematical modeling of dynamic problems of porous media, Mathematical analysis	sayyora-tohirzoda@mail.ru	Uzbekistan
Irgashev Bahrom	Differential Equations, Mathematical Physics Equation.	bahromirgasev@gmail.com	Uzbekistan
Yusupov Farrux	Dynamic systems, graph theory, problems of mathematical biology, genetics, epidemiology and ecology	farrukhyusupovchambil@mail.ru	Uzbekistan
Turdiev Malikjon	Deformable solid mechanics	malikjon_ts@mail.ru	Uzbekistan
Tadjiyeva Moxbonu	Dynamic systems, graph theory, problems of mathematical biology, epidemiology and ecology	mohbonut@mail.ru	Uzbekistan
Mizomov Inomjon	Homological algebra, representation theory, algebraic geometry	mizomovinomjon@mail.ru	Uzbekistan
Davlatova Fatima	Computational mathematics	fotimadavlatova733@gmail.com	Uzbekistan
Mirsaburova Gulbahor	Differential Equations, Mathematical Physics Equation.	mirsaburovagulbahor26@gamil.com	Uzbekistan

Xudaykulov Shohruh	Differential Equations, Mathematical Physics Equation.	xudoykulov1194@gmail.com	Uzbekistan
Toshmatova Mohiniso	Mathematical Modeling; Differential Geometry; Mathematical Modeling of Railway Track Geometry.	toshmatova_mm@mail.ru	Uzbekistan
Kholmurodova Gulnoza	Geometry and topology	xolmurodovagulnoza3@gmail.com	Uzbekistan
Almuratov Shavkat	Mechanics of deformable solids	almuratovshavkat11@gmail.com	Uzbekistan
Hamroyeva Zilola	Mechanics of deformable solids	zilolaqaxramonova24@gmail.com	Uzbekistan
Abdusoatov Ulug'bek	Mathematics Education, Digital Educational Technologies	absoatov_ulugbek@tstu.uz	Uzbekistan
Shodiyev Oybek	Mathematics Education, Digital Educational Technologies	oybekshodiyev1993@gmail.com	Uzbekistan
International editors			
Aleksey Tujilin	Differential geometry, calculus of variations, graph theory and combinatorics	alexey.tuzhilin@yahoo.com	Russian Federation
Josef Mikeš	Riemannian geometry, geodesic mappings, holomorphically projective mappings, F-planar mappings	josef.mikes@upol.cz	Czech Republic
Umar Islambekov	Statistics, Machine Learning, Applied Topology, Data Science	yagol@physics.msu.ri	Russian Federation
Марат Гаязович Юмагулов	Differential equations, nonlinear oscillations, bifurcations	yum_mg@mail.ru	Russian Federation
Юрий Степанович Волков	Approximation theory, splines, numerical analysis	volkov@math.nsc.ru	Russian Federation
Fedor Sukochev	Theory noncommutative symmetric spaces, operator theory	f.sukochev@unsw.edu.au	Australia
Tursun Yuldashev	Differential equations, computational mathematics,	tursun.k.yuldashev@gmail.com	Kyrgyzstan

	approximation methods of optimal control		
Alexander Katz	Functional analysis, operator theory	katza@stjohns.edu	USA
Semyon Litvinov	Theory noncommutative symmetric spaces, operator theory	snl2@psu.edu	USA
Nurlan Temirbekov	Computational mathematics, mathematical modeling	temirbekov@rambler.ru	Kazakhstan
Farrukh Mukhamedov	Operator algebras, Functional analysis, Dynamical systems	farrukh.m@uaeu.ac.ae	UAE
Allaberen Ashyralyev	Delay differential equations, numerical functional analysis, stochastic differential equations	aallaberen@gmail.com	Turkey
Dilmurat Azimov	Trajectory optimization, guidance, navigation and control, analytical methods of integration	azimov@hawaii.edu	USA
Adakhimzhan Sopuevich Sopuev	Differential Equations, Mathematical Physics, Partial Differential Equations, Boundary Value Problems.	*sopuev@mail.ru*	Kyrgyzstan
Murat Ibraevich Ramazanov	Differential Equations, Mathematical Physics, Mathematical Analysis, Integral Equations.	*ramamur@mail.ru*	Kazakhstan

CONTENTS

Ю.П. Апаков. **Краткий очерк научной, педагогической и общественной деятельности профессора** 1

Ю.П. Апаков, З.Н. Нуритдинова. **Решение граничной задачи с несимметричными временными условиями для неоднородного уравнения четвертого порядка с постоянными коэффициентами методом функции Грина** 5

D.A. Tursunov, Z.M. Sulaimanov, B.A. Azimov. **Asymptotic Analysis of a Singularly Perturbed Boundary Value Problem with an Interior Turning Point** 15

A.A. Abdushukurov, R.S. Muradov. **Copula-Robust Cramér–Rao Lower Bound under Dependent Random Censoring** 24

Sh.T. Karimov, A.N. Boynazarov. **Properties of the Integer-Order Generalized Bessel Integral Operator** 32

M.M. Rahmatullaev, M.A. Rasulova. **Hidden Potts-Observed SOS Models on a Two-Layer Structure** 49

A.B. Babajonov. **Inverse Scattering Transform for the Fifth-Order Fractional KdV Equation with a Self-Consistent Source** 60

Sh.T. Karimov, R.A. Mamajonova. **The Cauchy problem for a nonhomogeneous time-fractional beam vibration equation involving the Gerasimov–Caputo operator** 71

I.U. Khaydarov, Sh.M. Khasanov. **Initial and Initial-Boundary Value Problems for Multi-Parameter Integral-Differential Equations** 87

Б.И. Жамалов, Д.М. Меликузиева. **Применение метода функции Грина к исследованию краевых задач в полубесконечных областях для уравнений четвертого порядка с кратными характеристиками** 102

B.Z. Tojiboev. **Periodic Gibbs measures for the soft-core Widom–Rowlinson model and their extremality** 119

А.А. Хамитов, Ж.С. Эрматов. **Построение решение краевой задачи для уравнения третьего порядка с кратными характеристиками в трехмерном пространстве** 136

B.A. Babajanov, A.K. Babadjanova, B. Fayzullaev. **The Inverse Spectral Problem Method for the Periodic Differential–Difference Sine–Gordon Equation** 145

Ю.П. Апаков, Х.К. Иброхимов. **Краевые задачи для вязко-трансзвукового уравнения в полуограниченных областях** 160

A.M. Tukhtabaev. **Uniqueness of 3-adic periodic Gibbs measures for the Ising model on a (2,3)-ary tree** 170

К.Т. Каримов, Д.С. Олимова. **Краевая задача с интегральным условием для уравнения смешанного типа с операторами Бесселя** 179



ЮСУПЖОН ПУЛАТОВИЧ АПАКОВ

доктор физико-математических наук, профессор

2 января 2026 года свой 70-летний юбилей отметил доктор физико-математических наук, профессор Юсупжон Пулатович Апаков.

Ю.П. Апаков - известный ученый и признанный специалист в области дифференциальных уравнений с частными производными. На протяжении многих лет он плодотворно работает в техническом вузе, где создал свою научную школу. Весомый вклад профессора в развитие отрасли снискали ему заслуженное уважение в научном сообществе.

Ю.П. Апаков родился 2 января 1956 года в семье тружеников села Хазратишох Чартакского (бывшего Янгикурганского) района Наманганской области. В 1972 году окончил среднюю школу №37 в селе Хазратишох и начал свою педагогическую деятельность, преподаванием истории и биологии в школе №44 того же села.

В 1974 году обучался на подготовительном курсе Андижанского государственного педагогического института. В 1975-1979 годах учился на математическом факультете Наманганского государственного педагогического института, который окончил с отличием. В 1979-1980 годах преподавал математику в средней школе №37 кишлака Хазратишох.

В 1980 году по приглашению своего наставника Юсуфжона Иргашева начал работать старшим лаборантом на кафедре высшей математики, недавно открытой в Наманганском филиале Ташкентского политехнического института. В 1981-1984 годах занимал должность заместителя директора филиала.

В 1984-1986 годах работал младшим научным сотрудником в Институте механики и сейсмостойкости сооружений АН РУз, а в 1986-1990 годах - научным сотрудником в Институте математики АН РУз. В этот период обучался в заочной аспирантуре под руководством академика Тухтамурада Джураевича Джураева.

В 1989 году на специализированном ученом совете Института математики защитил кандидатскую диссертацию на тему «Краевые задачи для уравнения смешанного параболо-гиперболического типа в трехмерных областях» и получил ученую степень кандидата физико-математических наук, а в 1993 году - ученое звание доцента.

За время своей деятельности работал: в 1990-1991 годах - старшим преподавателем кафедры Высшей математики Наманганского филиала Ташкентского политехнического института; в 1991-2003 годах – доцентом кафедры Высшей математики Наманганского индустриально-технологического (с 1989 года – инженерно-педагогического) института; в 2003-2005 годах - заведующим кафедрой Высшей математики; в 2005-2016 годах - доцентом кафедры Высшей математики Наманганского инженерно-строительного института.

В 2016 году под руководством академика Тухтамурада Джураевича Джураева успешно защитил докторскую диссертацию на тему «К теории уравнений третьего порядка с кратными характеристиками и некоторые задачи в трехмерном пространстве» на специализированном ученом совете Национального университета Узбекистана и получил ученую степень доктора физико-математических наук (DSc).

В 2017-2022 годах исполнял обязанности профессора кафедры Высшей математики Наманганского инженерно-строительного института. В 2022 году, после присвоения ученого звания профессора, в 2022-2025 годах работал профессором кафедры высшей математики Наманганского инженерно-строительного института. С 2025 года занимает должность профессора кафедры высшей математики Наманганского государственного технического университета.

По инициативе Ю.П. Апакова, 15 мая 2017 года, на кафедре Высшей математики Наманганского инженерно-строительного института был организован постоянно действующий научный семинар «Классические и неклассические задачи дифференциальных уравнений и их приложения». В нем участвуют докторанты, профессора-преподаватели и магистры высших учебных заведений Ферганской долины. Участники семинара обсуждают полученные новые результаты.

С 29 августа 2019 года решением Министерства высшего и среднего специального образования в Наманганском инженерно-строительном институте была открыта базовая докторантура (PhD) и докторантура (DSc) по специальности 01.01.02- Дифференциальные уравнения и математическая физика для подготовки научно-педагогических кадров.

Под руководством профессора Ю.П. Апакова защищены 4 диссертации на соискание степени доктора философии (PhD) по специальности 01.01.02 - Дифференциальные уравнения и математическая физика. В настоящее время еще 2 докторанта (PhD) и 1 докторант (DSc) завершают работу над диссертациями и готовятся к защите.

С 2020 года Ю.П. Апаков работает по совместительству ведущим научным сотрудником в Наманганском отделении Института математики АН РУз. Одновременно он читает лекционный курс «Современные методы математической физики» для магистров математического факультета Наманганского государственного университета.

Являясь членом Ученого совета PhD.03/2025.27.12.FM.10.04 по присуждению ученых степеней при Ферганском государственном университете, а также членом научного семинара при совете и комиссии по приему квалификационных экзаменов, он вносит достойный вклад в подготовку научно-педагогических кадров.

Ю.П. Апаков - автор 4 учебников, 1 учебного пособия и 2 монографий, в том числе: учебники «Высшая математика» (2022, 2026), том 1-2; двухтомный учебник «Примеры и задачи по высшей математике» (2022) в соавторстве с Б.И. Жамаловым и А.М. Тухтабаевым; учебное пособие «Примеры и задачи по обыкновенным дифференциальным уравнениям» (2009) в соавторстве с Н. Тургуновым и И. Гафаровым; монографии «К теории уравнений третьего порядка с кратными характеристиками» (2019) и «Краевые задачи для уравнений смешанного параболо-гиперболического типа в трехмерных областях» (2023).

За время своей деятельности Ю.П. Апаков отмечен следующими наградами: нагрудный знак II-степени «Ветеран труда» (2023); Почетная грамота Министра высшего и среднего специального образования (2022, 2025); Благодарственное письмо Республиканского центра духовности и просвещения (2022); неоднократно награждался благодарственными письмами хокима Наманганской области, хокима города Намангана и ректора института. На протяжении своей многолетней плодотворной научно-творческой деятельности Ю.П. Апаков проводит глубокие фундаментальные исследования в области дифференциальных уравнений, функционального анализа и математической физики, внося неоценимый вклад в развитие этих направлений. Научные исследования, выполненные в теории дифференциальных уравнений с частными производными, охватывают уравнения различных порядков - от модельных

уравнений второго порядка до сложных уравнений высокого порядка. В частности:

- задачи, поставленные в бесконечных цилиндрических областях трехмерного пространства для вырождающихся и смешанных параболо-гиперболических уравнений второго порядка, исследованы с помощью преобразования Фурье, строго обоснованы существование, единственность и другие свойства решений;

- построены новые фундаментальные решения для уравнений третьего порядка с кратными характеристиками (трансзвуковых уравнений), содержащих вторую производную по времени, с помощью вырождающихся гипергеометрических функций;

- проведен глубокий анализ аналитических свойств фундаментальных решений, установлены их асимптотические оценки при больших значениях аргумента;

- для таких уравнений исследованы корректно поставленные краевые задачи в прямоугольных, конечных и бесконечных геометрических областях, а также в трехмерном пространстве, построены функции Грина для краевых задач первого и второго рода, разработаны теоретические основы применения метода Фурье к уравнениям нечетного порядка;

- для уравнений четвертого порядка, содержащих вторую и третью производные по времени, также решены корректные краевые задачи в прямоугольной области на основе построения функции Грина, математически обоснованы существование и единственность решений;

- исследованы локальные и нелокальные краевые задачи для смешанных параболо-гиперболических уравнений третьего и четвертого порядков, доказана единственность решений рассмотренных задач.

Эти научные результаты опубликованы в высокорейтинговых научных журналах и получили высокое признание благодаря своей теоретической новизне, методологическому совершенству и практической значимости не только в нашей стране, но и в международном научном сообществе. На сегодняшний день опубликовано более 270 научных статей, из которых свыше 50 - в журналах, индексируемых в международных базах Scopus и Web of Science. h-index в этих базах составляет: Web of Science - 7, Scopus - 8, Google Scholar - 11.

Юсупжон Пулатович Апаков сегодня достойно встречает свой 70-летний юбилей. Желаем ему крепкого здоровья, семейного благополучия, долгих лет жизни в деле обучения и воспитания молодежи Нового Узбекистана, а также больших успехов в научно-творческой деятельности!

Ученики, коллеги и редакционная коллегия.

Uzbekistan Journal of Mathematics and Computer Science
Volume-2, Issue-3-special, Page 5–14 (2026)
DOI: https://doi.org/10.56143/ujmcs.v2i3s.1
Research article

Решение граничной задачи с несимметричными временными условиями для неоднородного уравнения четвертого порядка с постоянными коэффициентами методом функции Грина

Ю.П. Апаков * **З.Н. Нуритдинова**

Аннотация

В представленной статье исследуется корректность постановки краевой задачи в прямоугольной области для неоднородного уравнения четвертого порядка, содержащего младшие члены с постоянными коэффициентами. На основе метода энергетических неравенств (интегралов энергии) строго обоснована единственность решения рассматриваемой задачи. С использованием сконструированной функции Грина получено явное представление искомого решения. При исследовании вопросов равномерной сходимости рядов доказано, что возникающий в знаменателе «малый определитель» строго отличен от нуля, что гарантирует существование регулярного решения.

Ключевые слова: *Дифференциальное уравнение четвертого порядка; Функций Грина; Младших членов; Кратные характеристики; Единственность и существование решения; Постановка граничной задачи.*

Предметная классификацие AMS (2020): *Основная: 35C10; Дополнительная: 35J15.*

1. Введение

Значительная часть задач газовой динамики, теории упругости, а также механики пластин и оболочек ассоциируется с исследованием дифференциальных уравнений в частных производных высокого порядка. С позиции прикладного моделирования физических систем дифференциальные уравнения четвертого порядка также представляют серьезный научный интерес (см. [1-5]).

Классификации подобных уравнений и разработке методов решения некоторых краевых задач посвящена известная монография Т.Д. Джураева и А. Сопуева [6]. Данные исследования заложили основу для дальнейшего развития общей теории дифференциальных уравнений четвертого порядка.

В контексте этих изысканий, авторами Д. Амановым и М.Б. Мурзамбетовой в [7] было получено решение краевой задачи для неоднородного уравнения, содержащего один младший член. Вопросам начально-краевых задач, описывающих вынужденные колебания консольной балки, посвящена работа К.Б. Сабитова и О.В. Фадеевой [8]. В статье Б.Ю. Иргашева [9] с помощью построения функции Грина детально исследована краевая задача для уравнения высокого порядка с кратными характеристиками. Кроме того, задача для уравнения четвертого порядка с априори неизвестной правой частью успешно изучена в работе А.К. Уринова и М.С. Азизова [10].

Received : *28–april–2026,* ***Accepted :*** *31–july–2026*
*** Corresponding author***

2. Постановка задач

В области $D = \{(x, y) : \ 0 < x < p, \ 0 < y < q\}$ рассмотрим уравнение четвертого порядка вида

$$U_{xxxx} - U_{yy} + A_1 U_{xx} + A_3 U_y + A_2 U = F(x, y),$$

где $p, q, A_i \in R$, $i = \overline{1,3}$ va $F(x, y)$-заданная достаточно гладкая функция. Заменой

$$U(x, y) = u(x, y)\, e^{\left(\frac{A_3}{2} y\right)}$$

это уравнение можно привести к уравнению

$$L[u] = u_{xxxx} - u_{yy} + a_1 u_{xx} + a_2 u = f(x, y), \tag{2.1}$$

где

$$a_1 = A_1, \quad a_2 = A_2 + \frac{A_3^2}{4}, \quad f(x, y) = F(x, y) e^{\left(-\frac{A_3}{2} y\right)}, \quad f_{xyy}(x, y) \in C\left(\bar{D}\right).$$

Для уравнения (2.1) в области D изучим следующую задачу.

Задача A_1. Найти функцию $u(x, y)$ из классаа $C^{4,2}_{x,y}(D) \cap C^{3,1}_{x,y}\left(\bar{D}\right)$, удовлетворяющую уравнению (2.1) в области D и следующим краевым условиям:

$$u(x, 0) = 0, \quad \frac{\partial}{\partial y} u\,|_{y=q} = 0, \quad 0 \le x \le p, \tag{2.2}$$

$$\frac{\partial^m}{\partial x^m} u\Big|_{x=0} = \psi_m(y), \quad \frac{\partial^m}{\partial x^m} u\Big|_{x=p} = \psi_{m+1}(y), \quad m = 1, 3, \quad 0 \le y \le q, \tag{2.3}$$

здесь $\psi_i(y), i = \overline{1,4}$- заданные достаточно гладкие функции.

Следует указать, что в публикациях [11-12] спектральный подход был применен к изучению краевых задач для модельных уравнений четвертого порядка с кратными характеристиками. В свою очередь, в изысканиях [13-15] предмет исследования составили граничные задачи для уравнений четвертого порядка с постоянными коэффициентами при наличии симметричных условий относительно переменной y.

В статье [16] рассмотрена смешанная краевая задача для однородного уравнения третьего порядка с кратными характеристиками, в то время как в работе [17] исследована краевая задача для неоднородного уравнения третьего порядка с кратными характеристиками в трехмерном пространстве. В статье [18] изучена краевая задача в прямоугольной области для неоднородного уравнения четвертого порядка с кратными характеристиками, содержащего производную третьего порядка по времени.

3. Единственность решения

Теорема 3.1. *Если задача A_1 имеет решение, то при выполнении условий $a_1 \le 0$ и $a_2 \ge 0$, оно единственно.*

Доказательство. Теорему докажем методом интегралов энергии. Предположим обратное. Пусть задача A_1 имеет два решения $u_1(x, y)$ и $u_2(x, y)$.Тогда функция $u(x, y) = u_1(x, y) - u_2(x, y)$ удовлетворяет однородному уравнению (2.1) и однородным краевым условиям. Докажем, что $u(x, y) = 0$ в $\overline{D}$.

В области D справедливо тождество

$$uL[u] \equiv \frac{\partial}{\partial x}\left(uu_{xxx} - u_x u_{xx} + a_1 uu_x\right) - \frac{\partial}{\partial y}\left(uu_y\right) + u_{xx}^2 + u_y^2 - a_1 u_x^2 + a_2 u^2 = 0. \tag{3.1}$$

$$\iint\limits_D \left[\frac{\partial}{\partial x}\left(uu_{xxx} - u_x u_{xx} + a_1 uu_x\right) - \frac{\partial}{\partial y}\left(uu_y\right)\right] dxdy + \iint\limits_D \left(u_{xx}^2 + u_y^2 - a_1 u_x^2 + a_2 u^2\right) dxdy = 0,$$

$$\int_0^q [u(p,y)u_{xxx}(p,y) - u(0,y)u_{xxx}(0,y)]dy - \int_0^q [u_x(p,y)u_{xx}(p,y) - u_x(0,y)u_{xx}(0,y)]dy$$
$$+a_1 \int_0^q [u(p,y)u_x(p,y) - u(0,y)u_x(0,y)] - \int_0^p [u(x,q)u_y(x,q) - u(x,0)u_y(x,0)]dx$$
$$+ \iint_D [u_{xx}^2 + u_y^2 - a_1 u_x^2 + a_2 u^2]dxdy = 0.$$

Учитывая однородные краевые условия, получим

$$\int_0^p \int_0^q u_{xx}^2 dxdy + \int_0^p \int_0^q u_y^2 dxdy - a_1 \int_0^p \int_0^q u_x^2 dxdy + a_2 \int_0^p \int_0^q u^2 dxdy = 0. \tag{3.2}$$

В (3.2) если $a_1 < 0,\ \ a_2 > 0$, то из четвёртого слагаемого получим $u(x,y) \equiv 0$, $(x,y) \in \overline{D}$, а если $a_1 = a_2 = 0$, то из второго слагаемого имеем $u(x,y) = h(x)$. Здесь $h(x)$ является произвольной функцией, удовлетворяющей условиям задачи. Из условия $u(x,0) = 0$, получим $h(x) = 0$, тогда $u(x,y) \equiv 0$, $(x,y) \in \bar{D}$. В силу последнего получим $u_1(x,y) = u_2(x,y)$.

Теоема 3.1 доказана.

Замечание 3.1. Отметим, что при нарушении условий теоремы 3.1 однородная задача A_1 для однородного уравнения (2.1) может иметь нетривиальные решения. Например, когда $a_1 = \left(\frac{\pi k}{p}\right)^2, a_2 = -\left(\frac{\pi(2n-1)}{2q}\right)^2$ задача

$$\begin{cases} u_{xxxx}(x,y) + \left(\frac{\pi k}{p}\right)^2 u_{xx}(x,y) - \left(\frac{\pi(2n-1)}{2q}\right)^2 u(x,y) - u_{yy} = 0, \\ u(x,0) = u_y(x,q) = 0, \\ u_x(0,y) = u_x(p,y) = u_{xxx}(0,y) = u_{xxx}(p,y) = 0, \end{cases}$$

имеет нетривиальные решения вида

$$u_{k,n} = \left(C_1 + C_2 \cos\left(\frac{\pi k}{p}x\right)\right) \sin\left(\frac{\pi(2n-1)}{2q}y\right), \quad k,n \in N,\ C_1, C_2 \in R.$$

4. Существование решения

Теорема 4.1. *Если выполняются следующие условия:*

1) $E < \frac{\mu_1^3\left(1-e^{-2\mu_1 p}\right)^2}{p\left(1+2\mu_1^2\right)}$;

2) $\psi_i^{(j)}(0) = \psi_i^{(j)}(q) = 0, \psi_i(y) \in C^3[0,q], i = \overline{1,4}, j = 0,2$;

3) $f(x,0) = f(x,q) = 0, 0 \le x \le p$, *то решение задачи* A_1 *существует.*

Здесь,

$$E = \max\{|a_i|, i = 1,2\},\ \mu_1 = \sqrt{\frac{\pi}{2q}}.$$

Доказательство. Решение задачи A_1 ищем в виде :

$$u(x,y) = \sum_{n=1}^{\infty} X_n(x) \cdot Y_n(y). \tag{4.1}$$

Формально считая, что (4.1) удовлетворяет уравнению (2.1), и подставляя (4.1) в (2.1), учитивая условия (2.2), для $Y_n(y)$ приходим следующей задаче:

$$\begin{cases} Y''(y) + \lambda Y(y) = 0, \\ Y(0) = Y'(q) = 0. \end{cases} \tag{4.2}$$

Нетривиальное решение задачи (4.1) существует только при

$\lambda = \lambda_n = \left(\frac{\pi(2n-1)}{2q}\right)^2,\ n = 1, 2, 3, ...$ и называются собственными значениями задачи (4.2), а соответствующие им собственные функции имеют вид:

$$Y_n(y) = \sqrt{\frac{2}{q}} \sin\left(\frac{\pi(2n-1)}{2q} y\right),\ n = 1, 2, 3, ... \tag{4.3}$$

Отметим, что система собственных функций (4.3) задачи (4.2) является полной и ортонормированной в пространстве$L_2(0, q)$ [16]. Учитывая граничные условия (2.3), получим задачу для $X_n(x)$:

$$\begin{cases} X_n^{(4)}(x) + a_1 X''_n(x) + a_2 X_n(x) + \lambda_n X_n(x) = f_n(x), \\ X'_n(0) = \psi_{1n},\ \ X'_n(p) = \psi_{2n},\ \ X'''_n(0) = \psi_{3n},\ \ X'''_n(p) = \psi_{4n}, \end{cases} \tag{4.4}$$

где

$$\psi_{in} = \int_0^q \psi_{in}(\eta) Y_n(n)\, d\eta, \quad i = \overline{1,4}, \quad f_n(x) = \int_0^q f(x,\eta) Y_n(\eta)\, d\eta, \quad n \in N.$$

Решаем задачу (4.4) с помощью функции Грина. Для этого введем функцию

$$V_n(x) = X_n(x) - \rho_n(x) \tag{4.5}$$

здесь

$$\rho_n(x) = \frac{2px - x^2}{2p}\psi_{1n} + \frac{x^2}{2p}\psi_{2n} + \frac{4px^3 - x^4 - 4p^2x^2}{24p}\psi_{3n} + \frac{x^4 - 2p^2x^2}{24p}\psi_{4n}.$$

Подставляем (4.5) в (4.4), получим задачу

$$\begin{cases} V_n^{(4)} + \lambda_n V_n(x) = g_n(x) - a_1 V''_n(x) - a_2 V_n(x), \\ V'_n(0) = V'_n(p) = V'''_n(0) = V'''_n(p) = 0. \end{cases} \tag{4.6}$$

Решение задачи (4.6) будем искать в следующем виде:

$$V(x) = \int_0^p G_n(x,\xi)(g_n(\xi) - a_1 V_n''(\xi) - a_2 V_n(\xi))d\xi \tag{4.7}$$

где

$$g_n(x) = f_n(x) + \frac{\psi_{3n} - \psi_{4n}}{p} - a_1\left(\frac{\psi_{2n} - \psi_{1n}}{p} + \frac{6px - 3x^2 - 2p^2}{6p}\psi_{3n} + \frac{3x^2 - p^2}{6p}\psi_{4n}\right) -$$
$$- (a_2 + \lambda_n)\left(\frac{2px - x^2}{2p}\psi_{1n} + \frac{x^2}{2p}\psi_{2n} + \frac{4px^3 - x^4 - 4p^2x^2}{24p}\psi_{3n} + \frac{x^4 - 2p^2x^2}{24p}\psi_{4n}\right).$$

$G_n(x,\xi)$ функция Грина задачи

$$\begin{cases} V_n^{(4)} + \lambda_n V_n(x) = 0, \\ V'_n(0) = V'_n(p) = V'''_n(0) = V'''_n(p) = 0, \end{cases} \tag{4.8}$$

которая имеет вид [19],

$$G_n(x,\xi) = \begin{cases} G_n^1(x,\xi), & 0 \le x < \xi \le p, \\ G_n^2(x,\xi), & 0 \le \xi < x \le p, \end{cases}$$

и обладают следующими свойствами:

1. В каждой из интервалов $0 \le x \le \xi$ и $\xi \le x \le p$ функция $G_n(x,\xi)$ рассматриваемая как функция от переменной x, является решением уравнения

$$G_{nxxxx}(x,\xi) + \lambda_n G_n(x,\xi) = 0.$$

2. Функция $G_n(x,\xi)$, удовлетворяет краевые условия

$$G_{nx}^1(0,\xi) = G_{nxxx}^1(0,\xi) = G_{nx}^2(p,\xi) = G_{nxxx}^2(p,\xi) = 0.$$

3. В $0 \le x \le p$ функции $G_n(x,\xi)$, $G_{nx}(x,\xi)$, $G_{nxx}(x,\xi)$ непрерывные, т.е.

$$G_n^2(\xi,\xi) - G_n^1(\xi,\xi) = G_{nx}^2(\xi,\xi) - G_{nx}^1(\xi,\xi) = G_{nxx}^2(\xi,\xi) - G_{nxx}^1(\xi,\xi) = 0.$$

4. При $x = \xi$, функция $G_{nxxx}(x,\xi)$ имеет разрыв первого рода, т.е.

$$G_{nxxx}^2(\xi,\xi) - G_{nxxx}^1(\xi,\xi) = 1.$$

Здесь

$$\begin{aligned}
G_n^1(x,\xi) = \frac{1}{2\mu_n^3\Delta}\{&[\cos(\mu_n p)\,sh(\mu_n p)(\sin(\mu_n(\xi-p))\,sh(\mu_n(\xi-p)) - \\
&-\cos(\mu_n(\xi-p))\,ch(\mu_n(\xi-p))) - \\
&-\sin(\mu_n p)\,ch(\mu_n p)(\sin(\mu_n(\xi-p))\,sh(\mu_n(\xi-p)) + \\
&+\cos(\mu_n(\xi-p))\,ch(\mu_n(\xi-p)))]\cos(\mu_n x)\,ch(\mu_n x) + \\
&+[\sin(\mu_n p)\,ch(\mu_n p)(\sin(\mu_n(\xi-p))\,sh(\mu_n(\xi-p)) - \\
&-\cos(\mu_n(\xi-p))\,ch(\mu_n(\xi-p))) + \\
&+\cos(\mu_n p)\,sh(\mu_n p)(\sin(\mu_n(\xi-p))\,sh(\mu_n(\xi-p)) + \\
&+\cos(\mu_n(\xi-p))\,ch(\mu_n(\xi-p)))]\sin(\mu_n x)\,sh(\mu_n x)\},
\end{aligned}$$

$$\begin{aligned}
G_n^2(x,\xi) = \frac{1}{2\mu_n^3\Delta}\{&[\cos(\mu_n p)\,sh(\mu_n p)(\sin(\mu_n\xi)\,sh(\mu_n\xi) - \\
&-\cos(\mu_n\xi)\,ch(\mu_n\xi)) - \sin(\mu_n p)\,ch(\mu_n p)(\sin(\mu_n\xi)\,sh(\mu_n\xi) + \\
&+(\cos(\mu_n\xi)\,ch(\mu_n\xi))]\cos(\mu_n(x-p))\,ch(\mu_n(x-p)) + \\
&+[\sin(\mu_n p)\,ch(\mu_n p)(\sin(\mu_n\xi)\,sh(\mu_n\xi) - \\
&-\cos(\mu_n\xi)\,ch(\mu_n\xi)) + \cos(\mu_n p)\,sh(\mu_n p)(\sin(\mu_n\xi)\,sh(\mu_n\xi) + \\
&+\cos(\mu_n\xi)\,ch(\mu_n\xi))]\sin(\mu_n(x-p))\,sh(\mu_n(x-p))\},
\end{aligned}$$

$$\Delta = -e^{2\mu_n p}\bar{\Delta},\ \bar{\Delta} = \frac{1+e^{-4\mu_n p}}{2} - \cos(2\mu_n p)\,e^{-2\mu_n p},\ \mu_n = \sqrt{\frac{\pi n}{2q}}.$$

Для функция Грина имеет место следующие оценки:

$$|G_n(x,\xi)| \le \frac{1}{\mu_n^3(1-e^{-2\mu_1 p})^{-2}} \begin{cases} e^{-\mu_n(2p-x-\xi)}, & x,\xi \in (0,p), \\ e^{-2\mu_n p}, & x = \xi = 0, \\ 1, & x = \xi = p. \end{cases}$$

Лемма 4.1. *Основной определитель $\overline{\Delta}$ не равен нулю.*

Доказательство. Предположим, что $\overline{\Delta} = 0$, тогда получим

$$2\cos(2\mu_n p) = e^{2\mu_n p} + e^{-2\mu_n p}.$$

С другой стороны, система неравенств

$$\begin{cases} e^{2\mu_n p} + e^{-2\mu_n p} > 2, \\ 2\cos(2\mu_n p) \le 2, \end{cases}$$

не имеет решения, отсюда $\bar{\Delta} > 0$.

Чтобы оценить $\overline{\Delta}$ снизу будем считать, что $\cos(2\mu_n p) = 1$ тогда имеем,

$$\bar{\Delta} \geq \frac{1}{2}\left(1 - e^{-2\mu_n p}\right)^2.$$

Правая часть этого неравенства при натуральных значениях n возрастает, поэтому при $n = 1$ она принимает минимальное значение. Тогда получим:

$$\bar{\Delta} \geq \frac{1}{2}\left(1 - e^{-2\mu_1 p}\right)^2 > 0.$$

Это доказывает отсутствие «малого знаменателя». Итак, $\overline{\Delta} \neq 0$. Лемма 4.1 доказана.

Интегрируя (4.7) по частям и вводя следующие обозначения:

$$\alpha_n(x) = \int_0^p G_n(x,\xi)\, g_n(\xi) d\xi,$$

$$\tilde{G}_n(x,\xi) = -a_1 G_{n\xi\xi}(x,\xi) - a_2 G_n(x,\xi),$$

получим

$$V_n(x) = \alpha_n(x) + \int_0^p \tilde{G}_n(x,\xi)\, V_n(\xi)\, d\xi, \quad n \in N, \tag{4.9}$$

являющееся интегральным уравнением Фредгольма второго рода. Решаем (14) с помощью последовательных приближений:

$$V_n(x) = \alpha_n(x) + \int_0^p R_n(x,\xi)\, \alpha_n(\xi)\, d\xi, \tag{4.10}$$

здесь

$$R_n(x,\xi) = \tilde{G}_{0n}(x,\xi) + \sum_{m=1}^{\infty} \tilde{G}_{mn}(x,\xi),$$

$$\tilde{G}_{0n}(x,\xi) = \tilde{G}_n(x,\xi),$$

$$\tilde{G}_{mn}(x,\xi) = \int_0^P \tilde{G}_{1n}(x,s)\, \tilde{G}_{(m-1)n}(s,\xi)\, ds,\ m = 2,3...,\ n \in N.$$

В силу (4.3), (4.5) и (4.10) решение задачи A_1 имеет вид

$$u(x,y) = \sum_{n=1}^{\infty} \left(\alpha_n(x) + \int_0^p R_n(x,\xi)\, \alpha_n(\xi)\, d\xi + \rho_n(x) \right) Y_n(y). \tag{4.11}$$

Для оценки резольвенты составим мажорирующий ряд

$$\frac{1}{p} \sum_{m=0}^{\infty} (Jp)^{m+1}, \tag{4.12}$$

здесь

$$J = E\left(\frac{2}{\mu_1} + \frac{1}{\mu_1^3}\right)\left(1 - e^{-2\mu_1 p}\right)^{-2},$$

Из условия 1 теоремы 4.1 можно записать в виде

$$E\left(\frac{2}{\mu_1} + \frac{1}{\mu_1^3}\right)\left(1 - e^{-2\mu_1 p}\right)^{-2} p < 1,$$

отсюда

$$Jp < 1,$$

тогда мажорирующий ряд (4.12) представляет собой сумму членов бесконечно убывающей геометрической прогрессии. В этом случае резольвента равномерно сходится, и ее оценка имеет вид

$$|R_n(x,\xi)| \le \frac{J}{1-Jp} \le M. \tag{4.13}$$

Теперь оценим $\alpha_n(x)$. Сначала, учитывая условия 2 и 3 теоремы 4.1, интегрируем по частям ψ_{in} три раза, а $f_n(x)$ два раза, находим следующие оценки:

$$|\psi_{in}| \le \frac{M}{n^3}|\Psi_{in}|, \quad i=\overline{1,4}, \quad |f_n(x)| \le \frac{M}{n^2}|\Phi_n(x)|, \quad n \in N, \tag{4.14}$$

где

$$\Psi_{in} = \sqrt{\frac{2}{q}}\int_0^q \psi'''_i(\eta)\sin\left(\frac{\pi(2n-1)}{2q}\eta\right)d\eta, \quad i=\overline{1,4}, \quad \Phi_n(x) = \int_0^q f_{\eta\eta}(x,\eta)\,Y_n(\eta)\,d\eta.$$

Учитывая (4.14), и следующие оценки:

$$|g_n(x)| \le \frac{M}{n}\sum_{i=1}^4|\Psi_{in}| + \frac{M}{n^2}|\Phi_n(x)| + \frac{M}{n^3}\sum_{i=1}^4|\Psi_{in}|,$$

$$|g'_n(x)| \le \frac{M}{n}\sum_{i=1}^4|\Psi_{in}| + \frac{M}{n^2}|\Phi'_n(x)| + \frac{M}{n^3}\sum_{i=1}^4|\Psi_{in}|,$$

находим

$$|\alpha_n(x)| \le \frac{M}{n^3}\sum_{i=1}^4|\Psi_{in}| + \frac{M}{n^4}(1+|\Phi_n(x)|) + \frac{M}{n^5}\sum_{i=1}^4|\Psi_{in}|, \quad x\in[0,p]. \tag{4.15}$$

Согласно (4.13), (4.15) имеем

$$|V_n(x)| \le \frac{M}{n^3}\sum_{i=1}^4|\Psi_{in}| + \frac{M}{n^4}(1+|\Phi_n(x)|) + \frac{M}{n^5}\sum_{i=1}^4|\Psi_{in}|, \quad x\in[0,p]. \tag{4.16}$$

После некоторых вычислений для (4.16) находим оценку

$$|V_n(x)| \le \frac{M}{n^3}\sum_{i=1}^4|\Psi_{in}|, \quad x\in[0,p]. \tag{4.17}$$

Принимая во внимание (4.16) и (4.17), получим

$$|u(x,y)| \le M\sum_{n=1}^{\infty}\frac{1}{n^3}\sum_{i=1}^4|\Psi_{in}| + M\sum_{n=1}^{\infty}\frac{1}{n^4}(1+|\Phi_n(x)|) + \frac{M}{n^5}\sum_{i=1}^4|\Psi_{in}| < \infty, \quad (x,y)\in\overline{D},$$

Отсюда следует, что решение задачи A_1 сходится абсолютно и равномерно в $\overline{D}$. Теперь покажем равномерную сходимость $u_{xxxx}(x,y)$. Производная третьего порядка функции Грина имеет разрыв. Учитывая это, находим производную четвёртого порядка от $V_n(x)$:

$$V_n^{(4)}(x) = g_n(x) - a_1\left(\alpha''_n(x) + \int_0^p R_{nxx}(x,\xi)\,\alpha_n(\xi)\,d\xi\right) -$$
$$-a_2\left(\alpha_n(x) + \int_0^p R_n(x,\xi)\,\alpha_n(\xi)\,d\xi\right) - \lambda_n\left(\alpha_n(x) + \int_0^p R_n(x,\xi)\,\alpha_n(\xi)\,d\xi\right).$$

Используя следующие оценки:

$$|\alpha''_n(x)| \le \frac{M}{n^2}\sum_{i=1}^{4}|\Psi_{in}| + \frac{M}{n^3} + \frac{M}{n^4}\sum_{i=1}^{4}|\Psi_{in}|, \quad |R_{nxx}(x,\xi)| \le nM,$$

находим

$$\left|V_n^{(4)}(x)\right| \le \frac{M}{n}\sum_{i=1}^{4}|\Psi_{in}| + \varphi_{1n}(x),$$

где

$$\varphi_{1n}(x) = \tfrac{M}{n^2}\left(\sum_{i=1}^{4}|\Psi_{in}| + 1 + |\Phi_n(x)|\right) + \tfrac{M}{n^3}\left(\sum_{i=1}^{4}|\Psi_{in}| + 1\right) +$$
$$+\tfrac{M}{n^4}\left(\sum_{i=1}^{4}|\Psi_{in}| + 1 + |\Phi_n(x)|\right) + \tfrac{M}{n^{\frac{5}{2}}}\sum_{i=1}^{4}|\Psi_{in}| + \tfrac{M}{n^{\frac{7}{2}}} + \tfrac{M}{n^{\frac{9}{2}}}\sum_{i=1}^{4}|\Psi_{in}| = O\left(n^{-2}\right), \quad n\to\infty.$$

Тогда имеем

$$|u_{xxxx}(x,y)| \le M\sum_{n=1}^{\infty}\frac{1}{n}\left(|\Psi_{1n}| + |\Psi_{2n}| + |\Psi_{3n}| + |\Psi_{4n}|\right) + \sum_{n=1}^{\infty}\varphi_{1n}(x), \quad (x,y)\in\overline{D},$$

где второе слагаемое сходится абсолютно и равномерно в $\overline{D}$. Чтобы доказать абсолютную и равномерную сходимость первого слагаемого, мы используем неравенства Коши-Банковского и Бесселя

$$M\sum_{n=1}^{\infty}\frac{1}{n}\left(|\Psi_{1n}| + |\Psi_{2n}| + |\Psi_{3n}| + |\Psi_{4n}|\right) \le$$
$$\le M\left(\sqrt{\sum_{n=1}^{\infty}|\Psi_{1n}|^2} + \sqrt{\sum_{n=1}^{\infty}|\Psi_{2n}|^2} + \sqrt{\sum_{n=1}^{\infty}|\Psi_{3n}|^2} + \sqrt{\sum_{n=1}^{\infty}|\Psi_{4n}|^2}\right)\sqrt{\sum_{n=1}^{\infty}\frac{1}{n^2}} \le$$
$$\le M\left(\|\psi'''_1(y)\| + \|\psi'''_2(y)\| + \|\psi'''_3(y)\| + \|\psi'''_4(y)\|\right)\sqrt{\frac{\pi^2}{6}} < \infty,$$

здесь

$$\sum_{n=1}^{\infty}|\Psi_{in}|^2 \le \|\psi'''_i(y)\|^2_{L_2(0,q)}, \quad i=\overline{1,4}, \quad \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}.$$

Отсюда следует, что $u_{xxxx}(x,y)$ сходится абсолютно и равномерно в $\overline{D}$. Учитывая уравнение (2.1), заключаем, что и u_{yy} тоже сходится.

Теорема 4.1 доказана.

5. Заключение

В данной работе проведено комплексное исследование краевой задачи с несимметричными по времени условиями для одного класса неоднородных дифференциальных уравнений четвертого порядка с младшими членами. В ходе изысканий получены следующие основные результаты:

1. С помощью метода энергетических неравенств строго обоснована теорема единственности решения поставленной задачи.
2. Сконструирована соответствующая функция Грина, через которую выписано явное аналитическое представление искомого решения.
3. Доказана разрешимость задачи в терминах регулярных решений, при этом ключевым элементом обоснования равномерной сходимости рядов стало установление отличности от нуля «малого знаменателя».

Полученные результаты вносят вклад в общую теорию уравнений с кратными характеристиками и могут быть применены в задачах механики и математической физики.

Благодарность

Авторы выражают глубокую признательность редактору и анонимным рецензентам за внимательное рассмотрение рукописи, ценные замечания и конструктивные рекомендации, которые способствовали существенному улучшению качества статьи.

Финансирование

Данное исследование выполнено без привлечения внешних источников финансирования.

Список литературы

[1] Турбин М.В.: *Исследование начально-краевой задачи для модели движения жидкости Гершел-Балкли.* Вест. Воронеж. Гос. Ун-та. Сер. Физ. Мат., 2013, № 2, с.246-257.
[2] Уизем Дж.: *Линейние и нелинейные волны.* М.: Мир, 1977.
[3] Шабров С.А.: *Об оценках функций влияния одной математической модели четвертого порядка.* Вест. Воронеж. Гос. Ун-та. Сер. Физ. Мат., 2015, № 2, с.168-179.
[4] Benney D.J., Luke J.C.: *Interactions of permanent waves of finite amplitude*. J. Math. Phys., 1964, 43, pp.309-313.
[5] Tursunov D.A., Orozov M.O.: *Asymptotic solution of the dirichlet problem for a ring, when the corresponding unperturbed equation has a regular special circle.* (2019) Vestnik Tomskogo Gosudarstvennogo Universiteta, Matematika i Mekhanika, 2019 (63), pp. 37 - 46. DOI: 10.17223/19988621/63/4
[6] Джураев Т.Д., Сопуев А.: *К теории дифференциальных уравнений в частных производных четвертого порядка.* Ташкент: Фан, 2000, 144 с.
[7] Аманов Д., Мурзамбетова М. Б.: *Краевая задача для уравнения четвертого порядка с младшим членом.* Вестн. Удмуртск. ун-та. Матем. Мех. Компьют. науки, 2013, выпуск 1, с.3-10.
[8] Сабитов К. Б., Фадеева О. В.: *Начально-граничная задача для уравнения вынужденных колебаний консольной балки.* Вестн. Сам. гос. техн. ун-та. Сер. Физ.-мат. науки, 2021, 25, 1, с.51-66.
[9] Иргашев Б. Ю.: *Краевая задача для одного вырождающегося уравнения высокого порядка с младшими членами.* Бюллетень Института математики 2019, №6, с.23-30.
[10] Urinov A.K., Azizov M.S.: *Boundary Value Problems for a Fourth Order partial Equation with an Unknown Right-hand Part.* Lobachevskii Journal of Mathematics, 2021, Vol. 42, № 3, pp.632-640.
[11] Аманов Д., Бекиев А.Б., Отарова Ж.А.: *Об одной краевой задаче для уравнения четвертого порядка.* УзМЖ. 2015, № 4, с.11-18.х
[12] Сабитов К. Б.: *Обратные задачи для уравнения колебаний балки по определению правой части и начальных условий.* Дифференциальные уравнения, 2020, том 56, №6, с.761-774.
[13] Апаков Ю.П., Мамажонов С.М.: *Краевая задача для неоднородного уравнения четвёртого порядка с постоянными коэффициентами.* Челябинский физико-математический журнал. 2023, Т. 8, выпуск 2, с.157-172.
[14] Мамажонов С.М.: *Третья краевая задача для неоднородного уравнения четвертого порядка с постоянными коэффициентами.* Бюллетень Института математики. 2022, Vol. 5, № 6, с.100-109.
[15] Апаков Ю.П., Мамажонов С.М., Нуритдинова З. Н.: *О решение краевой задачи для неоднородного уравнение четвертого порядка с постоянными коэффициентами с помощью функции Грина.* Научный вестник Наманганского государственного университета. 2024, № 4, -с.72-77.
[16] Умаров Р.А.: *Краевые задачи для уравнения третьего порядка с кратными характеристиками с младшими членами.* Дисс. док. фил. по физ.-мат. наук. - Фергана: Ферганский государственный университет, 2025. 98 с.
[17] Apakov Yu. P., Ibrokhimov Kh. K.*Bitsadze-Samarskii Type Problem For The Viscous-Transonic Equation.* Journal of Mathematical Sciences. Published: 13 February 2026. https://doi.org/10.1007/s10958-026-08224-6.
[18] Apakov Yu.P., Hamitov A.A. *On Constructing a Solution to a Boundary Value Problem for a Third-Order Inhomogeneous Equation Using the Green's Function in Three-Dimensional Space.* Lobachevskii Journal of Mathematics, 2025, Vol. 46, No 11, pp. 5504-5513. DOI: 10.1134/S1995080225611786
[19] Апаков Ю.П., Меликузиева Д.М. *Решение краевой задачи для неоднородного уравнения четвертого порядка с кратными характеристиками построением функции Грина.* "ДАН РУз", 3-9 ст. №3, 2026 г
[20] Мамажонов С.М.: *Решение второй краевой задачи для неоднородного уравнения четвертого порядка с кратными характеристиками.* Научный вестник Наманганского государственного университета. 2020, № 11, -с.8-15.

Solution of a boundary value problem with time-asymmetric conditions for a nonhomogeneous fourth-order equation with constant coefficients

Apakov Yu.P.,* Nuritdinova Z.N

Abstract

This paper investigates the well-posedness of a boundary value problem in a rectangular domain for a fourth-order inhomogeneous equation with lower-order terms and constant coefficients. Based on the method of energy inequalities (energy integrals), the uniqueness of the solution to the problem under consideration is rigorously established. An explicit representation of the desired solution is derived by constructing the corresponding Green's function. Furthermore, in examining the uniform convergence of the series, it is proved that the "small denominator"arising in the denominator is strictly non-zero, thereby guaranteeing the existence of a regular solution.

Keywords

Fourth-order differential equation; Green's functions; Lower-order terms; Multiple characteristics; Uniqueness and existence of a solution; Formulation of a boundary value problem.

Affiliations

Apakov Yusupjon Pulatovich
Address: Namangan State Technical University, Dept. of Higher Mathematics, 160103, Namangan, Uzbekistan and Institute of Mathematics named after V.I.Romanovsky Academy of Sciences of the Republic of Uzbekistan.
e-mail: yusupjonapakov@gmail.com
ORCID ID: https://orcid.org/0000-0001-9611-1466

Nuritdinova Zuxraxon Nosirjon qizi
Address: Namangan State Technical University, Dept. of Higher Mathematics, 160103, Namangan, Uzbekistan.
e-mail: zuhraahmadzonova60@gmail.com

Uzbekistan Journal of Mathematics and Computer Science
Volume-2, Issue-3-special, Page 15–23 (2026)
DOI: https://doi.org/10.56143/ujmcs.v2i3s.2
Research article

Asymptotic Analysis of a Singularly Perturbed Boundary Value Problem with an Interior Turning Point

D.A. Tursunov * Z.M. Sulaimanov and B.A. Azimov

Abstract

This paper investigates a two-point boundary value problem for a singularly perturbed linear inhomogeneous second-order ordinary differential equation with turning point inside the interval of the form $\varepsilon y''(x) + (x - x_0)y'(x) - (x - x_0)y(x) = 0,\ 0 < x < 1,\ x_0 \in (0, 1)$, subject to the boundary conditions $y(0) = a,\ y(1) = b$. Problems of this type arise in various areas of science and engineering, including physics, biology, economics, and applied mathematics. The considered problem possesses two distinctive features: it belongs to the class of singularly perturbed differential equations, and the coefficient of the reduced equation vanishes at an interior point of the interval. The combination of singular perturbation and an interior turning point makes the problem bisingular and leads to a more complicated solution structure. Unlike classical singularly perturbed boundary value problems, boundary layers do not occur near the endpoints of the interval. Instead, a layer is formed only in a neighborhood of the interior singular point $x = x_0$. Consequently, the solution cannot be represented by a single asymptotic expression on the entire interval. To overcome this difficulty, separate asymptotic representations are constructed on the subintervals $[0, x_0]$ and $[x_0, 1]$, yielding a composite solution. Each local representation consists of the sum of a regular outer solution and an inner solution describing the layer behavior near the singular point. The main objective of the paper is to construct a uniformly valid asymptotic expansion of the solution on the interval ([0,1]). A comparison of the constructed asymptotic expansion with A.Nayfeh's result demonstrates their close agreement and confirms the validity of the proposed asymptotic construction.

Keywords: *singular perturbation, interior turning point, interior layer, matched asymptotic expansions, boundary function method, uniform asymptotic expansion.*

AMS Subject Classification (2020): *Primary: 34E05; Secondary: 34E15, 34E10.*

1. Introduction

The theory of singularly perturbed differential equations constitutes one of the most important branches of modern applied mathematics and mathematical physics. Differential equations containing a small parameter multiplying the highest-order derivative arise in the mathematical modeling of a wide variety of processes characterized by the coexistence of fast and slow dynamics. Typical examples include boundary-layer phenomena in fluid mechanics, contrast structures in chemical kinetics, transient processes in electrical circuits, and systems exhibiting localized internal transitions. In such problems, the solution may experience

Received : *28–april–2026,* ***Accepted :*** *31–july–2026*
**** Corresponding author***

rapid variations within a narrow region of the independent variable, giving rise to boundary layers near the endpoints of the domain or internal layers in the vicinity of interior points [1]-[6]. These localized structures significantly complicate both the qualitative analysis of solutions and the construction of accurate asymptotic approximations.

Of particular interest are singularly perturbed boundary value problems for second-order ordinary differential equations of the form

$$\varepsilon y'' = f(x, y, y', \varepsilon), \qquad x \in (a, b), \qquad \varepsilon \to 0,$$

subject to boundary conditions prescribed at the endpoints of the interval. A special class of such problems arises when the reduced equation possesses a turning point, that is, an interior point at which the coefficient of the reduced first-order equation vanishes. In the neighborhood of a turning point, the qualitative behavior of the solution may change significantly, and an interior layer can be formed. Within this narrow region, the solution exhibits rapid variation, while remaining slowly varying outside the layer [2, 3], [7]-[10].

The asymptotic behavior of solutions to singularly perturbed problems with turning points has been extensively investigated by A.B. Vasilieva, V.F. Butuzov, S.A. Lomov, and many other authors. The classical boundary function method provides a powerful tool for constructing uniformly valid asymptotic approximations by combining regular and rapidly varying components of the solution [1]–[6]. However, the presence of an interior layer generated by a turning point requires a separate asymptotic analysis in different subdomains and the subsequent matching of the corresponding expansions. This leads naturally to the use of asymptotic matching techniques and composite expansions.

In the present paper, we study a two-point boundary value problem for a singularly perturbed linear nonhomogeneous second-order ordinary differential equation whose reduced equation has a turning point located inside the interval under consideration. The interaction between the singular perturbation and the interior turning point gives rise to a nonstandard asymptotic structure and prevents the construction of a single asymptotic representation valid throughout the entire interval.

The main objective of this work is to construct a complete uniformly valid asymptotic expansion of the solution that accurately captures the interior layer generated by the turning point. To achieve this goal, we employ a combination of the generalized boundary function method and the method of matched asymptotic expansions. This approach yields asymptotic representations on both sides of the turning point and allows the construction of a composite approximation valid throughout the entire interval.

The principal results of the paper are as follows:

- conditions for the formation of an interior layer are established;
- the absence of boundary layers at the endpoints of the interval is demonstrated;
- the outer and inner components of the asymptotic expansion are constructed, including the interior-layer function in a neighborhood of the turning point $x = x_0$;
- the uniform validity of the resulting composite asymptotic expansion is rigorously justified.

The obtained results extend the asymptotic theory of singularly perturbed boundary value problems with interior turning points. They may be applied in the mathematical modeling of processes exhibiting sharp internal transitions and can serve as a basis for the development and validation of numerical methods for singularly perturbed differential equations.

2. Problem Statement

The objective is to construct a uniformly valid asymptotic expansion of the solution to problem (2.1) with arbitrary accuracy as $\varepsilon \to 0$:

$$\varepsilon y''(x) + (x - x_0)y'(x) - (x - x_0)y(x) = 0, \quad x \in (0,1), \quad y(0) = a, \quad y(1) = b, \tag{2.1}$$

where $0 < \varepsilon$ is a small parameter, $x_0 \in (0,1)$, a and b are given constant numbers.

Problem (2.1) in the particular case $x_0 = 1/2$ was discussed in Nayfeh's monograph [3]. A formal asymptotic approximation was obtained using a matching procedure; however, the construction was not accompanied by a rigorous justification. In the present work, we derive a complete asymptotic expansion and establish its uniform validity by means of the generalized boundary function method.

The main goal of this paper is to demonstrate the efficiency and quality of the proposed generalized boundary value function method.

The problem possesses two essential features. First, the small parameter multiplies the highest-order derivative, which makes the problem singularly perturbed in the sense of Tikhonov. Second, the coefficient of the reduced equation vanishes at the interior point $x = x_0$, creating an interior turning point. The interaction of these two mechanisms produces an interior layer and leads to a nonstandard asymptotic structure of the solution.

This discrepancy leads to the solution of the complete problem undergoing abrupt changes near the boundaries or within the domain, forming so-called boundary layers. Depending on the structure of the degenerate equation and the type of boundary conditions, such layers can arise either at the boundaries of the interval or at its interior points. In the latter case, we speak of an interior transition layer (or contrast structure).

To construct an asymptotic solution to singularly perturbed problems with boundary layers, boundary function methods (A.B. Vasil'eva, V.F. Butuzov) and the matching of asymptotic expansions (A.M. Ilyin, Van Dyke) are traditionally used. When the transition point is located within the domain, additional coordination of the expansions constructed to the left and right of it is required, which significantly complicates the problem.

When solving problem (2.1), we use methods for integrating ordinary differential equations with additional conditions, as well as asymptotic methods (the small parameter method, the generalized boundary value method, and the splicing method). The main idea, i.e., the research methodology, is to construct regular external solutions and splice them with internal solutions [7]-[10].

3. Problem solution

In equation (2.1), for $\varepsilon = 0$, we obtain a degenerate equation

$$(x - x_0)\tilde{y}'(x) - (x - x_0)\tilde{y}(x) = 0, \quad x \in (0,1), \tag{3.1}$$

For the resulting equation, the point $x = x_0$ is a singular point.

Therefore, the zeroth approximation of the external solution to problem (2.1) can be represented as [**?**]:

$$\tilde{y}(x) = \begin{cases} \tilde{y}_0^L(x), & x \in [0, x_0), \\ \tilde{y}_0^R(x), & x \in (x_0, 1], \end{cases}$$

where $\tilde{y}_0^L(x) = ae^x$, $\tilde{y}_0^R(x) = be^{x-1}$, *L-left*, *R-right*.

We begin by constructing the outer expansion. We will seek the outer solution in the form of the following series:

$$y^{\text{out}}(x) = \begin{cases} u^L(x), & 0 \le x < x_0, \\ u^R(x), & x_0 < x \le 1, \end{cases} \tag{3.2}$$

where $u^L(x) = \sum_{k=0}^{\infty} \varepsilon^k u_k^L(x)$, $u^R(x) = \sum_{k=0}^{\infty} \varepsilon^k u_k^R(x)$, $u_k^L(x)$, $u_k^R(x)$ are functions are still unknown.

Substituting the series (3.2) into problem (2.1) and applying the small parameter method, we have:

$$(x - x_0)u_0'(x) - (x - x_0)u_0(x) = 0, \quad u_0(0) = a, \quad u_0(1) = b, \tag{3.3}$$

$$(x - x_0)u_k'(x) - (x - x_0)u_k(x) = -u_{k-1}''(x), \quad u_k(0) = 0, \quad u_k(1) = 0, \quad k \in \mathbb{N}. \tag{3.4}$$

We write the solutions to problems (3.3) and (3.4) in the form:

$$u_0(x) = \begin{cases} u_0^L(x), & x \in [0, x_0), \\ u_0^R(x), & x \in (x_0, 1], \end{cases}$$

$$u_k(x) = \begin{cases} u_k^L(x), & x \in [0, x_0), \\ u_k^R(x), & x \in (x_0, 1], \end{cases}$$

where $u_0^L(x) = ae^x$, $u_0^R(x) = be^{x-1}$,

$$u_1^L(x) = -ae^x \int_0^x \frac{1}{\tau - x_0} d\tau = -ae^x \ln|x - x_0|, \quad u_1^R(x) = -be^{x-1} \ln|x - x_0|,$$

$$u_k^L(x) = -\int_0^x \frac{e^{x-\tau}}{\tau - x_0} \frac{d^2}{d\tau^2} u_{k-1}^L(\tau) d\tau, \quad u_k^R(x) = -\int_1^x \frac{e^{x-\tau}}{\tau - x_0} \frac{d^2}{d\tau^2} u_{k-1}^R(\tau) d\tau, \ k = 2, 3, \dots$$

In the vicinity of the singular point $x = x_0$ we will highlight the singularities of the series $u^L(x) = \sum_{k=0}^{\infty} \varepsilon^k u_k^L(x)$ and $u^R(x) = \sum_{k=0}^{\infty} \varepsilon^k u_k^R(x)$:

$$u^L(x) = ae^x - \varepsilon ae^x \ln|x - x_0| + \frac{\varepsilon^2}{(x - x_0)^2}(a_2(x) + \alpha_2(x - x_0)^2 \ln|x - x_0|) + \dots$$

$$+ \frac{\varepsilon^n}{(x - x_0)^{2(n-1)}}(a_n(x) + \alpha_n(x - x_0)^{2(n-1)} \ln|x - x_0|) + \dots,$$

$$u^R(x) = be^{x-1} - \varepsilon be^{x-1} \ln|x - x_0| + \frac{\varepsilon^2}{(x - x_0)^2}(b_2(x) + \beta_2(x - x_0)^2 \ln|x - x_0|) + \dots$$

$$+ \frac{\varepsilon^n}{(x - x_0)^{2(n-1)}}(b_n(x) + \beta_n(x - x_0)^{2(n-1)} \ln|x - x_0|) + \dots,$$

where $a_k,\ b_k \in C^\infty[0, 1]$, $b_k(1) = 0$, $a_k(0) = 0$, $k \in N_0 = \{0, 1, 2, \dots\}$, $\alpha_k, \beta_k = const$.

From the last constructed $u^R(x)$ and $u^L(x)$ series it follows that to construct an internal solution it is necessary to use the transformation $x - x_0 = \mu t$, where t is a new variable, $\mu = \sqrt{\varepsilon}$.

Thus, the external solution loses its asymptotic character at the singular point $x = x_0$ and in its vicinity, and cannot be a solution to the boundary value problem (2.1).

We will seek a solution to problem (2.1) including the singular point in the form:

$$y(x, \varepsilon) = v(x, \varepsilon) + w(t, \mu), \tag{3.5}$$

where $v(x, \varepsilon)$ is a regular exterior solution, has no singularities, and $w(t, \mu)$ is an interior solution in the neighborhood of the singular point $x = x_0$.

Substituting relation (3.5) into problem (2.1), we obtain:

$$\varepsilon v''(x) + (x - x_0)v'(x) - (x - x_0)v(x) = h(\varepsilon), \quad x \in (0, 1), \tag{3.6}$$

$$v(0) = a, \quad v(1) = b, \tag{3.7}$$

$$w''(t) + tw'(t) - t\mu w(t) = -h(\mu^2), \quad t \in (-\frac{x_0}{\mu}, \frac{1-x_0}{\mu}), \tag{3.8}$$

$$w\Big(-\frac{x_0}{\mu}\Big) = 0, \quad w\Big(\frac{1-x_0}{\mu}\Big) = 0. \tag{3.9}$$

Consider first problem (3.6)-(3.7). We will look for a solution to this problem in the form:

$$v(x) = \begin{cases} v^L(x), & 0 \le x \le x_0, \\ v^R(x), & x_0 \le x \le 1, \end{cases} \quad h(\varepsilon) = \begin{cases} h^L(\varepsilon), & 0 \le x \le x_0, \\ h^R(\varepsilon), & x_0 \le x \le 1, \end{cases} \tag{3.10}$$

where

$$v^L(x) = \sum_{k=0}^{\infty} \varepsilon^k v_k^L(x), \quad v^R(x) = \sum_{k=0}^{\infty} \varepsilon^k v_k^R(x), \quad h^L(\varepsilon) = \sum_{k=1}^{\infty} \varepsilon^k h_k^L, \quad h^R(\varepsilon) = \sum_{k=1}^{\infty} \varepsilon^k h_k^R.$$

Substituting relation (3.10) into problem (3.6)-(3.7) we have a simple system of recurrence relations:

$$(x-x_0)v_0'(x) - (x-x_0)v_0(x) = 0, \quad v_0(0) = a, \quad v_0(1) = b, \tag{3.11}$$

$$(x-x_0)v_k'(x) - (x-x_0)v_k(x) = h_k - v_{k-1}''(x), \quad v_k(0) = 0, \quad v_k(1) = 0, \quad k \in \mathbb{N}. \tag{3.12}$$

We obtain:

$$v_0(x) = \begin{cases} v_0^L(x), & x \in [0, x_0], \\ v_0^R(x), & x \in [x_0, 1], \end{cases}$$

$$v_k(x) = \begin{cases} v_k^L(x), & x \in [0, x_0], \\ v_k^R(x), & x \in [x_0, 1], \end{cases}$$

where

$$v_0^L(x) = ae^x, \quad v_0^R(x) = be^{x-1},$$

$$v_1^L(x) = \int_0^x \frac{h_1^L - ae^\tau}{\tau - x_0} e^{x-\tau} d\tau, \quad v_1^R(x) = \int_1^x \frac{h_1^R - be^{\tau-1}}{\tau - x_0} e^{x-\tau} d\tau,$$

$$v_k^L(x) = \int_0^x \frac{h_k^L - (v_{k-1}^L(\tau))''}{\tau - x_0} e^{x-\tau} d\tau, \quad v_k^R(x) = \int_1^x \frac{h_k^R - (v_{k-1}^R(\tau))''}{\tau - x_0} e^{x-\tau} d\tau, \ k = 2, 3, \ldots$$

Let

$$h_1^L = ae^{1/2}, \ h_k^L = \frac{d^2}{dx^2} v_{k-1}^L(x)\Big|_{x=x_0}, \quad h_1^R = be^{-1/2}, \ h_k^R = \frac{d^2}{dx^2} v_{k-1}^R(x)\Big|_{x=x_0}, \quad k \in N,$$

then

$$v_k^L \in C^\infty[0, x_0], \quad v_k^R \in C^\infty[x_0, 1].$$

Thus, all members of the unknown series are determined. $h^L(\varepsilon), h^R(\varepsilon)$:

$$h^L(\varepsilon) = \varepsilon a\sqrt{e} + \varepsilon^2 v_2^L + \cdots + \varepsilon^n v_n^L + \ldots, \quad v_k^L = \frac{d^2}{dx^2} v_k^L(x)\Big|_{x=x_0};$$

$$h^R(\varepsilon) = \varepsilon \frac{b}{\sqrt{e}} + \varepsilon^2 v_2^R + \cdots + \varepsilon^n v_n^R + \ldots, \quad v_k^R = \frac{d^2}{dx^2} v_k^R(x)\Big|_{x=x_0}.$$

We now turn to the construction of the inner expansion, i.e., to solving problem (3.8)-(3.9). We will seek a solution to this problem in the form:

$$w(t,\mu)=\begin{cases} w^L(t,\mu), & -x_0/\mu \le t \le 0,\\ w^R(t,\mu), & 0 \le t \le (1-x_0)/\mu, \end{cases} \tag{3.13}$$

where $w^L(t,\mu)=\sum\limits_{k=0}^{\infty}\mu^k w_k^L(t), \quad w^R(t,\mu)=\sum\limits_{k=0}^{\infty}\mu^k w_k^R(t).$

Substituting the series (3.13) into the relations (3.8)-(3.9) we have:

$$\begin{gathered} \tfrac{d^2}{dt^2}w_0^L(t)+t\tfrac{d}{dt}w_0^L(t)=0, \quad -x_0/\mu<t<0, \quad w_0^L(-x_0/\mu)=0;\\ \tfrac{d^2}{dt^2}w_0^R(t)+t\tfrac{d}{dt}w_0^R(t)=0, \quad 0<t<(1-x_0)/\mu, \quad w_0^R((1-x_0)/\mu)=0; \end{gathered} \tag{3.14}$$

$$\begin{gathered} \tfrac{d^2}{dt^2}w_{2k-1}^L(t)+t\tfrac{d}{dt}w_{2k-1}^L(t)=tw_{2k-2}^L(t), \quad -x_0/\mu<t<0, \quad w_{2k-1}^L(-x_0/\mu)=0;\\ \tfrac{d^2}{dt^2}w_{2k-1}^R(t)+t\tfrac{d}{dt}w_{2k-1}^R(t)=tw_{2k-2}^R(t), \quad 0<t<(1-x_0)/\mu, \quad w_{2k-1}^R((1-x_0)/\mu)=0; \end{gathered} \tag{3.15}$$

$$\begin{gathered} \tfrac{d^2}{dt^2}w_{2k}^L(t)+t\tfrac{d}{dt}w_{2k}^L(t)=tw_{2k-1}^L(t)-h_k^L, \quad -x_0/\mu<t<0, \quad w_{2k}^L(-x_0/\mu)=0;\\ \tfrac{d^2}{dt^2}w_{2k}^R(t)+t\tfrac{d}{dt}w_{2k}^R(t)=tw_{2k-1}^R(t)-h_k^R, \quad 0<t<(1-x_0)/\mu, \quad w_{2k}^R((1-x_0)/\mu)=0. \end{gathered} \tag{3.16}$$

For functions $w_k^L(t)$ and $w_k^R(t)$ we require additional conditions (merging conditions):

$$\begin{gathered} v_k^L(x_0)+w_{2k}^L(0)=v_k^R(x_0)+w_{2k}^R(0),\\ \frac{d}{dx}v_k^L(x)|_{x=x_0}+\frac{d}{\mu dt}w_{2k}^L(t)|_{t=0}=\frac{d}{dx}v_k^R(x)|_{x=x_0}+\frac{d}{\mu dt}w_{2k}^R(t)|_{t=0},\\ w_{2k+1}^L(0)=w_{2k+1}^R(0), \ \frac{d}{dt}w_{2k+1}^L(t)|_{t=0}=\frac{d}{dt}w_{2k+1}^R(t)|_{t=0}, \quad k=0,1,\dots \end{gathered} \tag{3.17}$$

Condition (3.17) ensures the continuity of the point $x=x_0$ and its neighborhood.

Let us prove the following auxiliary lemma.

Lemma 3.1. *There is a unique solution to problem:*

$$\begin{gathered} \frac{d^2}{dt^2}\xi^L(t)+t\frac{d}{dt}\xi^L(t)=G^L(t), \quad t\in(-x_0/\mu,0),\\ \frac{d^2}{dt^2}\xi^R(t)+t\frac{d}{dt}\xi^R(t)=G^R(t), \quad t\in(0,(1-x_0)/\mu),\\ \xi^L(0)+k_1\xi^R(0)=\alpha, \quad \frac{d}{dt}\xi^L(0)+k_2\frac{d}{dt}\xi^R(0)=\beta,\\ \xi^L(-x_0/\mu)=0, \quad \xi^R((1-x_0)/2\mu)=0, \end{gathered}$$

and represent it in the form:

$$\xi^L(t)=\frac{k_2(\alpha-\gamma B_1^{(-)}-k_1\gamma B_1^{(+)})+k_1\beta}{k_1+k_2}\int_{-x_0/\mu}^{t}e^{-\tau^2/2}d\tau+\int_{-x_0/\mu}^{t}e^{-\tau^2/2}\int_0^{\tau}e^{s^2/2}G^L(s)dsd\tau,$$

$$\xi^R(t)=\frac{k_1\beta-k_2(\alpha-\gamma B_1^{(-)}-k_1\gamma B_1^{(+)})}{k_1-k_2}\int_{(1-x_0)/\mu}^{t}e^{-\tau^2/2}d\tau+\int_{(1-x_0)/\mu}^{t}e^{-\tau^2/2}\int_0^{\tau}e^{s^2/2}G^R(s)dsd\tau,$$

where $k_1, k_2, (|k_1| \neq |k_2|)$, α, β *are given constant numbers,* $\gamma = \left(\int_{-x_0/\mu}^{0} e^{-\tau^2/2} d\tau\right)^{-1}$,

$$B_1^{(-)} = \int_{-x_0/\mu}^{0} e^{-\tau^2/2} \int_0^{\tau} e^{s^2/2} G^L(s) ds d\tau, \quad B_1^{(+)} = \int_{(1-x_0)/\mu}^{0} e^{-\tau^2/2} \int_0^{\tau} e^{s^2/2} G^R(s) ds d\tau.$$

Based on this lemma, we can assert that solutions to problems (3.14)-(3.17) exist and are unique.
In particular

$$w_0^L(t) = \frac{(b-ae)(\gamma+\mu)}{2\sqrt{e}} \int_{-x_0/\mu}^{t} e^{-s^2/2} ds, \quad w_0^R(t) = \frac{(b-ae)(\gamma-\mu)}{2\sqrt{e}} \int_{(1-x_0)/\mu}^{t} e^{-s^2/2} ds.$$

The asymptotic solution of the zeroth order of problem (2.1) will take the following form:

$$y_0^L(x) = ae^x + \frac{(b-ae)(\gamma+\mu)}{2\sqrt{e}} \int_{-x_0/\mu}^{t} e^{-s^2/2} ds,$$

$$y_0^R(x) = be^{x-1} + \frac{(b-ae)(\gamma-\mu)}{2\sqrt{e}} \int_{(1-x_0)/\mu}^{t} e^{-s^2/2} ds.$$

This result was presented in order to compare with the result of A.H. Naife [9, p.316]:

$$y_l^c = ae^x + (b-ae)\left(\frac{1}{2\sqrt{e}} + \frac{1}{\sqrt{2\pi e}} \int_0^{t} e^{-s^2/2} ds\right),$$

$$y_r^c = be^{x-1} + (ae-b)\left(\frac{1}{2\sqrt{e}} - \frac{1}{\sqrt{2\pi e}} \int_0^{t} e^{-s^2/2} ds\right).$$

If we take into account that $\int_{-x_0/\mu}^{t}(*) = \int_{-x_0/\mu}^{0}(*) + \int_0^t(*) = \gamma + \int_0^t(*)$ and $\mu \to 0$, then our result exactly coincides with A.H.Nayfeh's result.

Thus, we have proved the theorem.

Theorem 3.1. *For the solution to problem* (2.1) *the asymptotic expansion holds for* $\forall x \in [0,1]$*:*

$$y(x,\varepsilon) = \sum_{k=0}^{n} \varepsilon^k v_k(x) + \sum_{k=0}^{2n+1} \mu^k w_k(t) + O(\varepsilon^{n+1}), \ \varepsilon \to 0.$$

Here the estimate for the remainder term is obtained by applying the maximum principle [11].

4. Discussion.

The proposed approach, based on a combination of the generalized boundary function method and the method of matched asymptotic expansions, makes it possible not only to construct formal asymptotic representations but also to establish their uniform validity on the entire interval. A crucial step in the analysis is the introduction of the stretched variable

$$t = \frac{x - x_0}{\mu}, \qquad \mu^2 = \varepsilon,$$

which resolves the neighborhood of the interior turning point and enables the construction of the corresponding inner expansion. This transformation separates the slow and fast scales of the problem and provides an accurate description of the interior layer structure.

From a computational perspective, the derived asymptotic formulas may serve as effective approximations even for moderately small values of the perturbation parameter. Consequently, they can be employed

as a foundation for the development of efficient numerical algorithms and hybrid asymptotic–numerical methods. Numerical experiments performed in Maple demonstrate excellent agreement between the asymptotic approximations and the corresponding numerical solutions, thereby confirming the accuracy and practical applicability of the proposed approach.

An important feature of the method is that the structure of the interior layer emerges naturally from the asymptotic analysis and does not require prior assumptions regarding its location or form. This property makes the approach particularly attractive for symbolic implementation and automated asymptotic analysis of singularly perturbed problems.

The present study also raises several directions for future research. These include the extension of the proposed methodology to equations with more general variable coefficients, nonlinear singularly perturbed problems, systems of differential equations, and problems involving multiple interior turning points. Another important issue is the rigorous analysis of the remainder term and the derivation of sharp error estimates for the resulting composite asymptotic expansion.

Overall, the obtained results broaden the class of singularly perturbed problems that can be treated by asymptotic matching techniques and demonstrate that nontrivial interior layers may arise even in linear boundary value problems when the reduced equation possesses an interior turning point.

5. Conclusion

A singularly perturbed two-point boundary value problem with an interior turning point has been investigated. By combining the generalized boundary function method with the method of matched asymptotic expansions, a complete uniformly valid asymptotic expansion of the solution has been constructed. The analysis shows that, despite the absence of boundary layers at the endpoints of the interval, an interior layer is formed in a neighborhood of the turning point $x = x_0$. The solution therefore admits distinct outer expansions on either side of the turning point together with an inner expansion describing the rapid transition in the layer region.

Existence and uniqueness results for the auxiliary inner problem have been established, and the resulting composite asymptotic expansion has been shown to be uniformly valid on the entire interval. Numerical computations performed in Maple demonstrate excellent agreement between the asymptotic approximation and the exact solution.

The proposed approach can be extended to more general classes of singularly perturbed problems with interior turning points, including nonlinear equations, systems of differential equations, and problems containing multiple turning points.

Funding

This work is supported by grant of Osh State University

Availability of data and materials

Not applicable.

Competing interests

The authors declare that they have no competing interests.

Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

References

[1] Tikhonov, A. N.: *On the dependence of differential equation solutions on a small parameter.* Matematicheskii Sbornik **64** (22), 193–204 (1948).
[2] Il'in, A. M.: *Matching of Asymptotic Expansions of Solutions of Boundary Value Problems*. Nauka, Moscow (1989); AMS, Providence (1992).
[3] Nayfeh, A. H.: *Perturbation Methods*. Pure and Applied Mathematics. Wiley-Interscience, New York (1973).
[4] Apakov, Yu. P., Ibrokhimov, Kh. K.: *On solving the second boundary value problem for the Viscous Transonic Equation.* Bulletin of the Karaganda University. Mathematics Series **119** (3), 34–45 (2025). https://doi.org/10.31489/2025M3/34-45
[5] Apakov, Yu. P., Umarov, R. A.: *Solutions of a time-asymmetric boundary-value problem for a third-order equation with variable coefficients*. Ukrainian Mathematical Journal **78** (1-2), 77–78 (2026). https://doi.org/10.3842/umzh.v78i1-2.8975
[6] Arnold, V. I.: *Catastrophe Theory*. 3rd ed. Nauka, Moscow (1990).
[7] Alymkulov, K., Tursunov, D. A.: *A method for constructing asymptotic expansions of bisingularly perturbed problems*. Russian Mathematics **60** (12), 1–8 (2016). https://doi.org/10.3103/S1066369X1612001X
[8] Tursunov, D. A., Orozov, M. O.: *Asymptotics of the solution to the Roben problem for a ring with regularly singular boundary*. Lobachevskii Journal of Mathematics **41** (1), 89–95 (2020). https://doi.org/10.1134/S1995080220010126
[9] Tursunov, D. A., Kozhobekov, K.G., Shakirov, K. K.: *Asymptotics of the Solution of a Singularly Perturbed Problem with a Singular Line*. Lobachevskii Journal of MathematicsOpen source preview **46** (1), 535-–545 (2025). DOI: 10.1134/S1995080224608063
[10] Tursunov, D. A., Sulaimanov, Z. M., Khalmatov, A. A.: *Singularly Perturbed Ordinary Differential Equation with Turning Point and Interior Layer*. Lobachevskii Journal of MathematicsOpen source preview **42** (12), 3016-–3021 (2021). DOI: 10.1134/S1995080221120362
[11] Protter, M. H., Weinberger, H. F.: *Maximum Principles in Differential Equations*. Prentice-Hall, Englewood Cliffs, N.J. (1967).

Affiliations

D.A. Tursunov
Address: Osh State University, Osh-Kyrgyzstan.
e-mail: dtursunov@oshsu.kg
ORCID ID: https://orcid.org/0000-0002-6990-1742

Z.M. Sulaimanov
Address: Osh State University, Osh-Kyrgyzstan.
e-mail: zsulaimanov@oshsu.kg
ORCID ID: https://orcid.org/0009-0003-6485-1533

B.A. Azimov
Address: Osh State University, Osh-Kyrgyzstan.
e-mail: bazimovd@oshsu.kg
ORCID ID: https://orcid.org/0000-0001-5849-8583

Uzbekistan Journal of Mathematics and Computer Science
Volume-2, Issue-3-special, Page 24–31 (2026)
DOI: https://doi.org/10.56143/ujmcs.v2i3s.3
Research article

Copula-Robust Cramér–Rao Lower Bound under Dependent Random Censoring

A.A. Abdushukurov, R.S. Muradov*

Abstract

In survival analysis and reliability theory, the classical Cramér–Rao lower bound assumes independence between lifetime and censoring variables. This paper extends the bound to dependent censoring using a copula-based likelihood. We derive a robust Godambe information bound and prove a generalized Cramér–Rao inequality. The result reduces to the classical bound under independence and shows increased variance bounds with stronger dependence.

Keywords: *survival analysis; Cramér–Rao Lower Bound; Fisher infomation; copula function; robust Godambe information; Censoring.*

AMS Subject Classification (2020): *Primary: 62N01; Secondary: 62F10; 62H05; 62E20.*

1. Introduction.

Survival analysis and reliability theory are fundamental fields that study the time until an event of interest occurs — for example, patient death, machine failure, or loan default. In many practical studies, the event time (lifetime) is not fully observed for all subjects due to censoring some individuals leave the study before the event occurs, or the study ends before all events happen. The most common form is random right censoring, where we observe

$$Z = \min(\xi, \eta), \quad \delta = \mathbf{1}\{\xi \le \eta\},$$

where ξ is the lifetime variable of interest, η is the censoring time, Z is the observed time, δ indicates whether the event was observed (1) or censored (0).

1.1. Classical Cramér–Rao Lower Bound and Its Limitation

In classical mathematical statistics, the Cramér–Rao lower bound (CRLB) provides a fundamental lower bound on the variance of unbiased estimators. For a parametric model with density $f(x;\theta)$, under standard regularity conditions, for any unbiased estimator $\hat{\theta}$

$$\mathrm{Var}(\hat{\theta}) \ge I_F(\theta)^{-1}, \qquad I_F(\theta) = -E\left[\frac{\partial^2 \log f}{\partial \theta^2}\right],$$

where $I_F(\theta)$ is the Fisher information. In the context of random censoring, the classical CRLB is derived under the key assumption that ξ and η are independent [1]. This independence assumption simplifies the likelihood and allows the factorization of the joint distribution as

$$H(x, y; \theta) = F(x; \theta)G(y).$$

Received : *28–april–2026,* ***Accepted :*** *31–july–2026*
*** Corresponding author***

1.2. The Problem of Dependent Censoring

However, in many real-world applications, independence is unrealistic. Consider the following examples

- **Medical studies.** A patient's health deteriorates, making them more likely both to experience the event (e.g., death) and to withdraw from the study (censoring). Here, ξ and η are positively dependent.
- **Engineering reliability.** A machine's lifetime may be correlated with the censoring time caused by maintenance schedules — better machines may be censored earlier because they are still running at the study's end.
- **Economics.** Unemployment duration (lifetime) and survey dropout (censoring) may be correlated due to shared unobserved factors like motivation or health.

In such cases, ignoring dependence leads to biased estimation and incorrect inference. The classical Fisher information matrix becomes invalid because the likelihood function changes fundamentally.

1.3. Modeling Dependence with Copulas

To handle dependent censoring, we model the joint distribution of (ξ, η) using a copula [4]. A copula separates the marginal distributions from the dependence structure. Let $\xi \sim F(x;\theta)$, lifetime distribution (depends on parameter θ), $\eta \sim G(y)$, censoring distribution (possibly unknown, nuisance). By Sklar's theorem, there exists a copula C_a such that

$$H(x, y; \theta, a) = C_a(F(x;\theta), G(y)),$$

where $a > 0$ is a dependence parameter. In this paper, we focus on the Clayton copula

$$C_a(u, v) = (u^{-a} + v^{-a} - 1)^{-1/a}, \quad a > 0.$$

The Clayton copula captures lower tail dependence, meaning extreme small values of ξ and η tend to occur together — this is natural in survival analysis when early censoring is often associated with early event times.

1.4. Observed Data Likelihood Under Dependent Censoring

Under dependent censoring, the joint density of the observed pair (Z, δ) becomes

$$f_{Z,\delta}(z, 1) = f_\xi(z;\theta)\, \partial_2 C_a(F(z;\theta), G(z)),$$

$$f_{Z,\delta}(z, 0) = g_\eta(z)\, \partial_1 C_a(F(z;\theta), G(z)),$$

where $\partial_1 C_a = \frac{\partial C_a}{\partial u}$, $\partial_2 C_a = \frac{\partial C_a}{\partial v}$, f_ξ and g_η are densities of ξ and η, respectively. For n i.i.d. observations (Z_i, δ_i), the log-likelihood function is

$$\ell(\theta, a) = \sum_{i=1}^{n} \delta_i \left[\log f_\xi(Z_i;\theta) + \log \partial_2 C_a(u_i, v_i)\right] + (1 - \delta_i) \left[\log g_\eta(Z_i) + \log \partial_1 C_a(u_i, v_i)\right],$$

where $u_i = F(Z_i;\theta)$, $v_i = G(Z_i)$. This likelihood explicitly incorporates the dependence through the copula derivatives.

1.5. Misspecification and the Need for Godambe Information

If the assumed copula is correct, the Fisher information matrix derived from this likelihood gives the optimal asymptotic variance. However, in practice, the true dependence structure may deviate from the chosen copula (e.g., Clayton may not fit perfectly). Then, the Fisher information is no longer valid, and classical Cramér–Rao bound fails.

To address this, we turn to the Godambe information matrix [2], which remains valid under misspecification

$$H(\theta) = -E\left[\frac{\partial^2 \ell}{\partial \theta^2}\right], \quad J(\theta) = \mathrm{Var}\left(\frac{\partial \ell}{\partial \theta}\right), \quad I_G(\theta) = H(\theta)J(\theta)^{-1}H(\theta).$$

The robust Cramér–Rao bound then becomes

$$\mathrm{Var}(\hat{\theta}) \geq I_G(\theta)^{-1}.$$

2. Godambe Information Matrix Under Copula Misspecification

In parametric statistical inference, the Cramér–Rao lower bound is based on the Fisher information matrix, which relies on the assumption that the likelihood function is correctly specified. However, when the copula function is misspecified — meaning the true dependence structure between ξ and η differs from the assumed Clayton copula — the Fisher information matrix becomes invalid for obtaining valid variance bounds. This section introduces the Godambe information matrix, a robust alternative that remains valid under model misspecification.

2.1. Why Fisher Information Fails Under Misspecification

Consider a parametric model with log-likelihood function $\ell(\theta)$ based on n i.i.d. observations. The Fisher information matrix is defined as

$$I_F(\theta) = -E\left[\frac{\partial^2 \ell(\theta)}{\partial \theta^2}\right] = E\left[\left(\frac{\partial \ell(\theta)}{\partial \theta}\right)^2\right],$$

where the equality holds only if the model is correctly specified. Under correct specification, the score function has mean zero and its variance equals the negative expected Hessian. However, when the copula is misspecified, the likelihood function does not correspond to the true data-generating process. In this case, two important properties break down

1. The expected score is no longer zero: $E[\partial \ell / \partial \theta] \neq 0$.
2. The information equality fails: $-E[\partial^2 \ell / \partial \theta^2] \neq \mathrm{Var}(\partial \ell / \partial \theta)$.

Consequently, using the Fisher information for variance estimation leads to biased standard errors and incorrect confidence intervals. A more general framework is required.

2.2. Definition of the Godambe Information Matrix

The Godambe information matrix, also known as the sandwich information matrix or robust information matrix, provides a valid asymptotic variance for estimators derived from estimating equations, even when the model is misspecified [2]. For a parameter $\theta \in \mathbb{R}^d$, let

$$\psi(Z, \delta; \theta) = \frac{\partial \ell(\theta)}{\partial \theta}$$

be the score function. Define two key matrices

1. **Sensitivity matrix** (negative expected Hessian).

$$H(\theta) = -E\left[\frac{\partial^2 \ell(\theta)}{\partial \theta^2}\right] = -E\left[\frac{\partial \psi}{\partial \theta^T}\right].$$

This matrix measures the average curvature of the log-likelihood.

2. **Variability matrix** (variance of the score).

$$J(\theta) = \mathrm{Var}\left(\frac{\partial \ell(\theta)}{\partial \theta}\right) = E\left[\psi\psi^T\right] - E[\psi]E[\psi]^T.$$

Under misspecification, $E[\psi] \neq 0$, so centering is important.

The Godambe information matrix is defined as

$$I_G(\theta) = H(\theta)\, J(\theta)^{-1}\, H(\theta).$$

2.3. Intuition Behind the Godambe Information

The Godambe information generalizes the Fisher information in the following sense

- If the model is correctly specified, then $H(\theta) = J(\theta) = I_F(\theta)$, and thus $I_G(\theta) = I_F(\theta)$.
- If the model is misspecified, the sandwich form $HJ^{-1}H$ correctly adjusts the asymptotic variance by accounting for both the curvature (H) and the variability (J).

The name "sandwich" comes from the structure $HJ^{-1}H$, where J is "sandwiched" between two copies of H.

2.4. Asymptotic Distribution Under Misspecification

Let $\hat{\theta}_n$ be the maximum likelihood estimator (or more generally, an M-estimator) obtained by solving

$$\sum_{i=1}^{n} \psi(Z_i, \delta_i; \theta) = 0.$$

Under standard regularity conditions (differentiability, finite moments, and invertibility of H and J), the asymptotic distribution of $\hat{\theta}_n$ is

$$\sqrt{n}(\hat{\theta}_n - \theta^*) \xrightarrow{d} N(0, I_G(\theta^*)^{-1}),$$

where θ^* is the pseudo-true parameter value that minimizes the Kullback–Leibler divergence between the true and misspecified models. The asymptotic covariance matrix is $I_G(\theta^*)^{-1}$.

2.5. Copula-Robust Cramér–Rao Lower Bound

Theorem 2.1 (Copula-Robust CRLB)**.** *Assume the following regularity conditions hold*

1. *The parameter space Θ is compact and θ lies in its interior.*
2. *The log-likelihood $\ell(\theta)$ is twice continuously differentiable with respect to θ.*
3. *Differentiation under the integral sign is valid.*
4. *The matrices $H(\theta)$ and $J(\theta)$ are finite and positive definite for all $\theta \in \Theta$.*

Then, for any unbiased estimator $\hat{\theta}$ of θ,

$$\mathrm{Var}(\hat{\theta}) \geq I_G(\theta)^{-1}.$$

If the copula is correctly specified (i.e., the true dependence structure matches the assumed copula, which under independence reduces to $C(u, v) = uv$), then $H(\theta) = J(\theta) = I_F(\theta)$, and the bound reduces to the classical Cramér–Rao lower bound

$$\mathrm{Var}(\hat{\theta}) \geq I_F(\theta)^{-1}.$$

Furthermore, as the dependence parameter a in the Clayton copula increases (stronger dependence), the Godambe information decreases, implying a higher variance bound — an efficiency loss due to dependent censoring.

Proof. We prove the theorem in several steps, following the general theory of estimating functions [2].

Step 1. Unbiasedness and covariance identity. Let $\hat{\theta}$ be an unbiased estimator of θ, so that $E[\hat{\theta}] = \theta$. Consider the score function $\psi = \partial\ell/\partial\theta$. For the true data-generating distribution (even under misspecification), the following identity holds for any estimator with finite variance

$$\mathrm{Cov}(\hat{\theta}, \psi) = E\left[(\hat{\theta} - \theta)\psi^T\right] = I,$$

where I is the identity matrix. This follows from differentiating the unbiasedness condition

$$\frac{\partial}{\partial\theta} E[\hat{\theta}] = \int (\hat{\theta} - \theta)\frac{\partial f}{\partial\theta}\, d\mu = E[\hat{\theta}\psi^T] = I.$$

Step 2. Cauchy–Schwarz inequality. For any random vectors U and V, we have

$$\mathrm{Var}(U) \geq \mathrm{Cov}(U, V)\,\mathrm{Var}(V)^{-1}\,\mathrm{Cov}(V, U).$$

Set $U = \hat{\theta}$ and $V = \psi$. Then

$$\mathrm{Var}(\hat{\theta}) \geq \mathrm{Cov}(\hat{\theta}, \psi)\,\mathrm{Var}(\psi)^{-1}\,\mathrm{Cov}(\psi, \hat{\theta}) = I \cdot J(\theta)^{-1} \cdot I = J(\theta)^{-1}.$$

Thus we obtain

$$\mathrm{Var}(\hat{\theta}) \geq J(\theta)^{-1}.$$

Step 3. Adjusting for curvature. The above bound uses only the variability matrix $J(\theta)$. However, under misspecification, a sharper bound can be obtained by incorporating the sensitivity matrix $H(\theta)$. Consider a transformed estimating function

$$\psi^* = H(\theta)J(\theta)^{-1}\psi.$$

Then, applying the Cauchy–Schwarz inequality to ψ^* yields

$$\mathrm{Var}(\hat{\theta}) \geq \Big(H(\theta)J(\theta)^{-1}H(\theta)\Big)^{-1} = I_G(\theta)^{-1}.$$

Step 4. Verification of the special case. If the copula is correctly specified and the independence case holds ($C(u, v) = uv$), then

$$\partial_1 C = v, \quad \partial_2 C = u, \quad \partial_1\partial_2 C = 1.$$

It follows that the score function becomes

$$\psi = \frac{\partial\ell}{\partial\theta} = \sum_{i=1}^{n} \delta_i \frac{f'_\xi(Z_i;\theta)}{f_\xi(Z_i;\theta)} + \sum_{i=1}^{n}(1 - \delta_i)\frac{g'_\eta(Z_i)}{g_\eta(Z_i)} \cdot \frac{\partial F}{\partial\theta} \cdot (\cdots)$$

After simplification, the information equality holds $H(\theta) = J(\theta)$. Therefore, $I_G(\theta) = H(\theta)J(\theta)^{-1}H(\theta) = H(\theta) = I_F(\theta)$, and the bound reduces to the classical CRLB.

Step 5. Dependence effect. For the Clayton copula with $a > 0$, one can show that $H(\theta)$ decreases as a increases, while $J(\theta)$ increases. Consequently, $I_G(\theta) = HJ^{-1}H$ becomes smaller. Thus the lower bound $\mathrm{Var}(\hat{\theta}) \geq I_G(\theta)^{-1}$ becomes larger, indicating efficiency loss under stronger dependence. This completes the proof. □

2.6. Relationship to Estimating Equations

The Godambe information framework is particularly useful when the likelihood is misspecified. In such cases, the estimator $\hat{\theta}$ solves the estimating equation

$$\sum_{i=1}^{n} \psi(Z_i, \delta_i; \theta) = 0,$$

where ψ is any unbiased estimating function ($E[\psi] = 0$). The Godambe information for an estimating function is defined as

$$I_G(\theta) = \left(E\left[\frac{\partial \psi}{\partial \theta}\right]\right)^T \left(E[\psi\psi^T]\right)^{-1} \left(E\left[\frac{\partial \psi}{\partial \theta}\right]\right).$$

Our score-based definition is a special case where ψ is the score function.

2.7. Comparison: Fisher vs. Godambe Information

Property	**Fisher** $I_F(\theta)$	**Godambe** $I_G(\theta)$
Requires correct specification	Yes	No
Valid under misspecification	No	Yes
Based on expected Hessian	Yes	Yes (H)
Based on score variance	Yes	Yes (J)
Sandwich form	No	Yes ($HJ^{-1}H$)
Classical CRLB	Yes	Special case

Table 1. Comparison between Fisher and Godambe information matrices.

2.8. Implications for Dependent Censoring

Theorem 1 has important practical implications for survival analysis with dependent censoring

1. **Robust inference.** Even if the Clayton copula is not perfectly correct, the Godambe-based variance bound provides valid lower bounds.
2. **Efficiency loss quantification.** By comparing $I_G(\theta)^{-1}$ under different a values, one can quantify how much precision is lost due to dependence.
3. **Model selection.** The bound can guide the choice of copula — a copula that yields larger Godambe information (smaller variance bound) is more efficient.

2.9. Remark. When Does Censoring Increase Information?

As noted in [5], interestingly, censoring does not always reduce Fisher information. Under informative censoring — when the censoring time provides information about the lifetime — it is possible to construct models where information is preserved or even increased compared to the uncensored case. This phenomenon can be analyzed within the Godambe framework by comparing $I_G(\theta)$ under censored and uncensored designs.

3. Application to the Clayton Copula

For the Clayton copula

$$\partial_2 C_a(u,v) = u^{-(1+a)} \left(u^{-a} + v^{-a} - 1\right)^{-(1+1/a)}.$$

Substituting into the log-likelihood yields closed-form expressions for the score function and the Godambe information matrix.

Proposition 3.1. *If (ξ, η) are linked by a Clayton copula, then for any estimator $\hat{\theta}$*

$$\mathrm{Var}(\hat{\theta}) \geq I_G(\theta)^{-1}.$$

Proof. The proof follows directly from Theorem 1, since the Clayton copula satisfies the regularity conditions for the Godambe information bound. The specific form of $\partial_2 C_a$ ensures the score function is well-defined and the matrices $H(\theta)$ and $J(\theta)$ are finite and invertible for $a > 0$. □

4. Asymptotic Normality of Copula-Graphic Estimators

Abdushukurov and Muradov [6] showed that copula-graphic estimators remain asymptotically normal under dependent censoring

$$\sqrt{n}(\hat{F}_n(t) - F(t)) \xrightarrow{d} \mathcal{G}(t),$$

where $\mathcal{G}(t)$ is a Gaussian process with covariance structure depending on the Godambe information [6, 4].

5. Main Contributions

This work provides

1. A decomposition of Fisher information under dependent censoring using copula functions.
2. A new Cramér–Rao lower bound based on the Godambe information, robust to copula misspecification.
3. Quantification of efficiency loss due to dependence between lifetime and censoring variables.

Remark 5.1. Interestingly, censoring does not always reduce Fisher information [5]. In the case of informative censoring, it is possible to construct models where information is preserved or even increased.

Acknowledgements

The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

References

[1] Abdushukurov A. A.: *Extension of relative-risk power estimator under dependent random censored data.* CMFD **68** (1), 1-13, (2022).
[2] Godambe V. P.: Estimating Functions. Oxford University Press, (1991).
[3] Abdushukurov A. A., Muradov R. S.: *On estimation of conditional distribution function under dependent random right censored data*. Journal of Siberian Federal University. **7** (4), 409-416, (2014).
[4] Nelsen R. B.: An Introduction to Copulas.Springer. New York, (2006).

[5] Abdushukurov A. A.: *On Fisher-Bhattacharyya Information Function and Lower Bounds under Right Random Censoring*. Lobachevskii Journal of Mathematics. **45** (9), 1-10, (2024). DOI: 10.1134/S1995080224603011
[6] Abdushukurov A. A., Muradov R. S.: *On estimation of conditional distribution function under dependent random right censored data*. Journal of Siberian Federal University. **7** (4), 409-416, (2014).
[7] Abdushukurov A. A., Muradov R. S.: *Copula-Based Fisher Information and Cramér–Rao Lower Bound under Dependent Censoring*. Abstracts of International Conference on "Current Problems of Mathematical-Physical Equations and Information Technologies. **Part 1**, 298-300, (2026).

Affiliations

A.A. Abdushukurov
Address: Moscow State University Tashkent Branch, Tashkent, Uzbekistan.
e-mail: a_abdushukurov@rambler.ru
ORCID ID: https://orcid.org/0000-0002-0994-8127

R.S. Muradov
Address: Namangan State Technical University, Namangan, Uzbekistan.
e-mail: rustamjonmuradov@gmail.com
ORCID ID: https://orcid.org/0000-0001-9402-7957

Uzbekistan Journal of Mathematics and Computer Science
Volume-2, Issue-3-special, Page 32–48 (2026)
DOI: https://doi.org/10.56143/ujmcs.v2i3s.4
Research article

UZJMCS

Properties of the Integer-Order Generalized Bessel Integral Operator

Sh.T. Karimov* **A.N. Boynazarov**

Abstract

This article examines the properties of solutions obtained for the Cauchy problem associated with ordinary differential equations that contain a higher-order singular Bessel differential operator together with a lower-order term. As part of the research, an integral expression for the unique solution to the Cauchy problem is derived, one that incorporates both the Gauss hypergeometric function and the second-kind Humbert hypergeometric function. It is rigorously proved that the obtained solution completely satisfies the prescribed initial conditions and is unique. Furthermore, based on this integral representation, an integer-order generalized Bessel integral operator is introduced. Several of its fundamental properties, including the existence of an inverse operator, integration-by-parts formulas, and the semigroup property of the operator, are established through a series of lemmas.

Keywords: *higher-order ordinary differential equation; generalized Bessel differential operator; generalized Erdélyi–Kober operator; integer-order generalized Bessel integral operator.*

AMS Subject Classification (2020): *Primary:34A05 ; Secondary: 34A12, 34B05 .*

1. Introduction

Within differential equation theory, problems built around the Bessel operator hold considerable significance, since they arise naturally when modeling physical phenomena that possess radial symmetry, such as wave motion, heat transfer, and quantum-mechanical systems. Although classical Riemann–Liouville-type fractional integrals have been extensively studied for ordinary differential operators, the necessity of developing a special theory for higher-order Bessel operators of the form

$$B_\gamma = \frac{d^2}{dx^2} + \frac{2\gamma}{x}\frac{d}{dx}$$

is still relevant.

One of the first fundamental works in this direction is due to I. G. Sprinkhuizen-Kuyper, in which a class of integral operators related to negative powers of the operator D_ν and their connection with hypergeometric functions was studied [5]. I. A. Kipriyanov and his Voronezh school also investigated equations related to the Bessel operator extensively [3, 4, 7]. In particular, researchers from the Voronezh school—Elina Shishkina and Sergei M. Sitnik—studied the integer- and fractional-order Bessel operator and its integral operator in detail [7, 10, 8, 9, 15, 18].

Received : *28–april–2026,* ***Accepted :*** *31–july–2026*
*** Corresponding author***

However, modern problems of mathematical physics demand the investigation of singular differential equations that involve not only spectral parameters but also high-order singular Bessel differential operators with lower-order terms. Constructing solutions for such problems in the vicinity of the zero (singular) point poses significant complexities due to the absence of a universal method. Nevertheless, Sh. T. Karimov and his research group developed a method to construct solutions for these types of problems by employing a transmutation operator, which transforms the singular equations into non-singular ordinary differential equations [26, 25, 23, 24].

In this study, the solution to the considered initial value problem is also determined using this technique. This work sets out to accomplish two principal goals: establishing, with full mathematical rigour, that the constructed solution to the Cauchy problem both meets the prescribed initial conditions and is the only such solution, and carrying out a comprehensive study of the core properties exhibited by the generalized Bessel integral operator of integer order that is proposed here.

2. Supplementary Information: Hypergeometric Functions

In this section, we collect the formulas used in the main calculations.

The Gauss hypergeometric function is defined inside the disk $|z| < 1$ as the sum of the hypergeometric series [10, 11, 19]

$$ {}_2F_1(a,b;c;z) = F(a,b;c;z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}, \tag{2.1} $$

and for $z \in (-\infty, -1)$, it is defined by analytic continuation. In (2.1), the parameters a, b, c and the variable z may be complex, with $c \neq 0, -1, -2, \ldots$

The Pochhammer symbol is

$$ (a)_n = a(a+1)\cdots(a+n-1), \quad n = 1, 2, \ldots; \quad (a)_0 = 1, $$

and for $n = 2n$ the following doubling formula holds:

$$ (a)_{2n} = \left(\frac{a}{2}\right)_n \left(\frac{a+1}{2}\right)_n 2^{2n}. \tag{2.2} $$

The integral representation of ${}_2F_1$ is

$$ {}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\,\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\,dt. \tag{2.3} $$

Transformation formulas:

$$ {}_2F_1(a,b;c;z) = (1-z)^{c-a-b}\, {}_2F_1(c-a, c-b; c; z), \quad \mathrm{Re}(c-a-b) > 0, \tag{2.4} $$

$$ {}_2F_1(a,b;c;z) = (1-z)^{-a}\, {}_2F_1\left(a, c-b; c; \tfrac{z}{z-1}\right), \tag{2.5} $$

$$ {}_2F_1(a,b,a-b+1;z) = \frac{1}{(1-z)^a}\, {}_2F_1\left(\tfrac{a}{2}, \tfrac{a+1}{2} - b;\, a-b+1;\, -\tfrac{4z}{(1-z)^2}\right), \tag{2.6} $$

$$ {}_2F_1\left(a, a-b+\tfrac{1}{2}; b+\tfrac{1}{2}; \left(\tfrac{1-\sqrt{1-z}}{1+\sqrt{1-z}}\right)^2\right) = \left(\tfrac{2}{1+\sqrt{1-z}}\right)^{-2a} {}_2F_1(a,b;2b;z), \tag{2.7} $$

$$ {}_2F_1(a,b;b;z) = (1-z)^{-a}. \tag{2.8} $$

The integral representation of the Humbert function Ξ_2 is:

$$\Xi_2(a,b,c;z,t) =$$

$$= \frac{\Gamma(c)}{\Gamma(a)\,\Gamma(c-a)} \int_0^1 u^{a-1}(1-u)^{c-a-1}(1-zu)^{-b}\, \bar{I}_{c-a-1}\left[2\sqrt{t(1-u)}\right] du, \quad \mathrm{Re}(c) > \mathrm{Re}(a) > 0, \tag{2.9}$$

and its power-series representation is

$$\Xi_2(a,b,c;z,t) = \sum_{n=0}^{\infty} \frac{t^n}{(c)_n\, n!}\, {}_2F_1(a,b;c+n;z), \quad |z| < 1, \tag{2.10}$$

where $\Xi_2(a,b,c;x,y)$ is the Humbert second-kind hypergeometric function [17, 19].

The following differentiation formula holds for hypergeometric functions [5]:

$$\left(\frac{d^2}{dz^2} + \frac{2(a+b-c)}{z}\,\frac{d}{dz}\right) \frac{(1-z^2)^{c-1}}{2^c\,\Gamma(c)}\, {}_2F_1(a,b;c;1-z^2) = \frac{(1-z^2)^{c-3}}{2^{c-2}\,\Gamma(c-2)}\, {}_2F_1(a-1,b-1;c-2;1-z^2). \tag{2.11}$$

3. Initial-Value Problem

The results obtained in this section serve as the foundation for defining the integer-order generalized Bessel integral operator introduced in Section 4.

Lemma 3.1. *Let $\lambda \in \mathbb{R}$, $0 < \gamma < 1$, and $m \in \mathbb{N}$. Then the following identity holds:*

$$B_{\gamma,\lambda}\, K_m(x,s) = K_{m-1}(x,s),$$

where

$$B_{\gamma,\lambda} = B_\gamma + \lambda^2 = \frac{d^2}{dx^2} + \frac{2\gamma}{x}\,\frac{d}{dx} + \lambda^2$$

is the generalized Bessel differential operator, and

$$K_m(x,s) = \frac{1}{\Gamma(2m)}(x-s)^{2m-1}\left(\frac{s}{x}\right)^{\gamma} \Xi_2\left(\gamma, 1-\gamma;\, m+\tfrac{1}{2};\, -\frac{(x-s)^2}{4xs},\, -\frac{\lambda^2}{4}(x-s)^2\right), \tag{3.1}$$

where $\Xi_2\left(\gamma, 1-\gamma; m+\frac{1}{2}; -\frac{(x-s)^2}{4xs}, -\frac{\lambda^2}{4}(x-s)^2\right)$ is the Humbert second-kind hypergeometric function.

Proof. Using (2.10), we expand the Humbert function in (3.1):

$$\Xi_2\left(\gamma, 1-\gamma;\, m+\tfrac{1}{2};\, -\frac{(x-s)^2}{4xs},\, -\frac{\lambda^2}{4}(x-s)^2\right) = \sum_{p=0}^{\infty} \frac{\left(-\frac{\lambda^2}{4}(x-s)^2\right)^p}{\left(m+\frac{1}{2}\right)_p\, p!}\, {}_2F_1\left(\gamma, 1-\gamma;\, m+\tfrac{1}{2};\, -\frac{(x-s)^2}{4xs}\right).$$

For the Gauss hypergeometric function in the last series we apply first (2.5) and then (2.7), performing some calculations. Using the Pochhammer doubling formula (2.2):

$$\left(m+\tfrac{1}{2}\right)_p = \frac{(2m)_{2p}}{(m)_p\, 2^{2p}} = \frac{\Gamma(2m+2p)}{\Gamma(2m)(m)_p\, 2^{2p}},$$

we obtain the equivalent series representation $K_{m,p}(x,s)$ of $K_m(x,s)$ (the $p=0$ term being separated):

$$K_{m,p}(x,s) = 2(-s)^{2m-1}\frac{\left(1-\frac{x^2}{s^2}\right)^{2m-1}}{2^{2m}\,\Gamma(2m)}\,{}_2F_1\left(m+\gamma-\tfrac{1}{2},m;\,2m;\,1-\tfrac{x^2}{s^2}\right)+$$
$$+\,2(-s)^{2m-1}\sum_{p=1}^{\infty}\frac{(-\lambda^2s^2)^p\,(m)_p}{p!}\,\frac{\left(1-\frac{x^2}{s^2}\right)^{2m+2p-1}}{2^{2m+2p}\,\Gamma(2m+2p)}\,{}_2F_1\left(m+\gamma-\tfrac{1}{2}+p,m+p;\,2m+2p;\,1-\tfrac{x^2}{s^2}\right). \tag{3.2}$$

Applying the Bessel operator B_γ to the two parts and using (2.11), we eventually find

$$B_\gamma K_{m,p}(x,s) = B_\gamma K_{m-1,p}(x,s) - \lambda^2 K_{m,p}(x,s),$$

i.e.,

$$\left(B_\gamma+\lambda^2\right)K_{m,p}(x,s) = K_{m-1,p}(x,s). \tag{3.3}$$

Lemma 3.1 is proved.

Theorem 3.1 (see [1]). *Let $\Omega=\{x\in\mathbb{R}: x>0\}$, and let $f(x)\in C(0,b)$, $b>0$. Then the unique solution $y(x)\in C^{2m}(0,+\infty)\cap C[0,\infty)$ of the problem*

$$\left(\frac{d^2}{dx^2}+\frac{2\gamma}{x}\,\frac{d}{dx}+\lambda^2\right)^m y(x) = f(x),\quad x>0,\quad 0<\gamma<\tfrac{1}{2},\quad m\in\mathbb{N}, \tag{3.4}$$

$$y^{(j)}(0)=0,\quad j=0,1,\ldots,2m-1, \tag{3.5}$$

is given by

$$y(x) = \frac{1}{\Gamma(2m)}\int_0^x (x-s)^{2m-1}\left(\frac{s}{x}\right)^\gamma \Xi_2\left(\gamma,1-\gamma;\,m+\tfrac{1}{2};\,-\frac{(x-s)^2}{4xs},\,-\frac{\lambda^2}{4}(x-s)^2\right) f(s)\,ds. \tag{3.6}$$

Proof. We first establish the following relations for the integral kernel in (3.6). It is well known that $K_{m,p}(x,x)=0$, and we shall show that $K_{m,p}(x,0)=0$. For this purpose, utilizing relation (2.9) for the Humbert function, we perform the following calculations:

$$K_m(x,0) = \lim_{s\to 0}(x-s)^{2m-1}\left(\frac{s}{x}\right)^\gamma \Xi_2\left(\gamma,1-\gamma;m+\frac{1}{2};-\frac{(x-s)^2}{4xs},-\frac{\lambda^2}{4}(x-s)^2\right) =$$

$$= \lim_{s\to 0}(x-s)^{2m-1}\left(\frac{s}{x}\right)^\gamma \frac{\Gamma\left(m+\frac{1}{2}\right)}{\Gamma(\gamma)\,\Gamma\left(m+\frac{1}{2}-\gamma\right)}\times$$

$$\times\int_0^1 t^{\gamma-1}(1-t)^{\alpha-\frac{1}{2}-\gamma}\left(1+\frac{(x-s)^2}{4xs}t\right)^{\gamma-1} \bar{I}_{m-\frac{1}{2}-\gamma}\left[\lambda i\sqrt{(x-s)^2(1-t)}\right]dt =$$

$$= \left\langle \mathrm{Re}\left(m+\frac{1}{2}\right) > \mathrm{Re}\,(\gamma) > 0\right\rangle =$$

$$= \lim_{s\to 0} s x^{-\gamma}(x-s)^{2m-1}\frac{\Gamma\left(m+\frac{1}{2}\right)}{\Gamma(\gamma)\,\Gamma\left(m+\frac{1}{2}-\gamma\right)}\times$$

$$\times \int_0^1 t^{\gamma-1}(1-t)^{m-\frac{1}{2}-\gamma}\left(s+\frac{(x-s)^2}{4x}t\right)^{\gamma-1} \bar{I}_{m-\frac{1}{2}-\gamma}\left[\lambda i\sqrt{(x-s)^2(1-t)}\right]dt = 0,$$

where, according to formulas (2.9) and (2.10), the following equality holds when $s = 0$

$$x^{2m-1-\gamma}\frac{\Gamma\left(m+\frac{1}{2}\right)}{\Gamma(\gamma)\Gamma\left(m+\frac{1}{2}-\gamma\right)}\int_0^1 t^{\gamma-1}(1-t)^{m-\frac{1}{2}-\gamma}\left(\frac{x}{4}t\right)^{\gamma-1}\bar{I}_{m-\frac{1}{2}-\gamma}\left[\lambda i\sqrt{x^2(1-t)}\right]dt =$$

$$= x^{2m-2-\gamma}4^{1-\gamma}\frac{\Gamma\left(m+\frac{1}{2}\right)}{\Gamma(\gamma)\Gamma\left(m+\frac{1}{2}-\gamma\right)}\int_0^1 t^{2\gamma-2}(1-t)^{m-\frac{1}{2}-\gamma}(1-xt)^0\bar{I}_{m-\frac{1}{2}-\gamma}\left[\lambda i\sqrt{x^2(1-t)}\right]dt =$$

$$= x^{2m-2-\gamma}4^{1-\gamma}\frac{\Gamma\left(m+\frac{1}{2}\right)}{\Gamma(\gamma)\Gamma\left(m+\frac{1}{2}-\gamma\right)}\frac{\Gamma(2\gamma-1)\Gamma\left(m+\frac{1}{2}-\gamma\right)}{\Gamma\left(m-\frac{1}{2}+\gamma\right)}\Xi_2\left(0,2\gamma-1;m-\frac{1}{2}+\gamma;x,x^2\right) =$$

$$= x^{2m-2-\gamma}4^{1-\gamma}\frac{\Gamma\left(m+\frac{1}{2}\right)}{\Gamma(\gamma)}\frac{\Gamma(2\gamma-1)}{\Gamma\left(m-\frac{1}{2}+\gamma\right)}\bar{J}_{m-\frac{3}{2}+\gamma}(2x) < \infty, \quad m+\gamma > 1.$$

Applying the differential operator $B_{\gamma,\lambda}$ to both sides of (3.6) and using the Leibniz differentiation rule for integrals with variable limits, we have

$$B_{\gamma,\lambda}\, y(x) = \int_0^x B_{\gamma,\lambda}\, K_{m,p}(x,s)\, f(s)\, ds.$$

By (3.3) this equals

$$B_{\gamma,\lambda}\, y(x) = \int_0^x K_{m-1,p}(x,s)\, f(s)\, ds. \tag{3.7}$$

Applying $B_{\gamma,\lambda}$ to (3.7) successively $m-1$ more times, and we obtain:

$$B^m_{\gamma,\lambda} y(x) = B_\gamma \int_0^x K_{1,0}(x,s) f(s)\, ds + B_\gamma \int_0^x K_{1,p\geq 1}(x,s) f(s)\, ds + \lambda^2 \int_0^x K_{1,p}(x,s) f(s)\, ds =$$

$$= I_1(x,s) + I_2(x,s) + I_3(x,s).$$

Let us compute the first term:

$$I_1(x,s) = B_\gamma \int_0^x K_{1,0}(x,s) f(s)\, ds =$$

$$= \left(\frac{d^2}{dx^2} + \frac{2\gamma}{x}\frac{d}{dx}\right)\int_0^x \left(\frac{x^2-s^2}{2s}\right){}_2F_1\left(\gamma+\frac{1}{2},1;2;\frac{s^2-x^2}{s^2}\right) f(s)\, ds =$$

$$-\left(\frac{d}{dx} + \frac{2\gamma}{x}\right)\int_0^x \frac{d}{dx}\left(\frac{s^2-x^2}{s^2}\right){}_2F_1\left(\gamma+\frac{1}{2},1;2;\frac{s^2-x^2}{s^2}\right)\frac{s}{2} f(s)\, ds =$$

$$= \left(\frac{d}{dx} + \frac{2\gamma}{x}\right)\int_0^x \left(\frac{x}{s}\right){}_2F_1\left(\gamma+\frac{1}{2},1;1;\frac{s^2-x^2}{s^2}\right) f(s)\, ds.$$

According to formula (2.8)

$$I_1(x,s) = \left(\frac{d}{dx} + \frac{2\gamma}{x}\right)\int_0^x \left(\frac{s}{x}\right)^{2\gamma} f(s)\,ds = \left(\frac{s}{x}\right)^{2\gamma} f(s)\Big|_{s=x} - \int_0^x \left(\frac{d}{dx} + \frac{2\gamma}{x}\right)\left(\frac{s}{x}\right)^{2\gamma} f(s)\,ds =$$

$$= f(x) - \int_0^x \left(\frac{d}{dx}\left(\frac{s}{x}\right)^{2\gamma} + \frac{2\alpha}{x}\left(\frac{s}{x}\right)^{2\gamma}\right) f(s)\,ds =$$

$$= f(x) - \int_0^x \left(-\frac{2\gamma}{x}\left(\frac{s}{x}\right)^{2\gamma} + \frac{2\gamma}{x}\left(\frac{s}{x}\right)^{2\gamma}\right) f(s)\,ds = f(x).$$

Now, compute the second term:

$$I_2(x,s) = B_\gamma \int_0^x K_{1,p\geq 1}(x,s) f(s)\,ds =$$

$$= -B_\gamma \int_0^x 2\sum_{p=1}^{\infty} \frac{\left(-\frac{\lambda^2}{4}s^2\right)^p (1)_p}{p!} \frac{\left(\frac{s^2-x^2}{s^2}\right)^{2p+1}}{2^{2+2p}\Gamma(2+2p)} {}_2F_1\left(\gamma+\frac{1}{2}+p, 1+p; 2+2p; \frac{s^2-x^2}{s^2}\right) sf(s)\,ds.$$

According to formula (2.11)

$$I_2(x,s) = -\int_0^x 2\sum_{p=1}^{\infty} \frac{\left(-\frac{\lambda^2}{4}s^2\right)^p (1)_p}{p!} B_\gamma \frac{\left(\frac{s^2-x^2}{s^2}\right)^{2p+1}}{2^{2+2p}\Gamma(2+2p)} {}_2F_1\left(\gamma+\frac{1}{2}+p, 1+p; 2+2p; \frac{s^2-x^2}{s^2}\right) sf(s)\,ds =$$

$$= -\int_0^x 2\sum_{p=1}^{\infty} \frac{(-\lambda^2)^p \Gamma(p)}{(p-1)!\Gamma(1)} \frac{\left(\frac{s^2-x^2}{s^2}\right)^{2p-1}}{2^{2p}\Gamma(2p)} {}_2F_1\left(\gamma-\frac{1}{2}+p, p; 2p; \frac{s^2-x^2}{s^2}\right) s^{-1+2p} f(s)\,ds.$$

We replace the summation index p by $p+1$

$$I_2(x,s) = -\lambda^2 \int_0^x 2\sum_{p=0}^{\infty} \frac{(-\lambda^2)^p \Gamma(p+1)}{p!\Gamma(1)} \frac{\left(\frac{s^2-x^2}{s^2}\right)^{2p+1}}{2^{2p+2}\Gamma(2p+2)} \times$$

$$\times {}_2F_1\left(\gamma+\frac{1}{2}+p, p+1; 2p+2; \frac{s^2-x^2}{s^2}\right) s^{1+2p} f(s)\,ds = -\lambda^2 \int_0^x K_{1,p}(x,s) f(s)\,ds.$$

Now we compute the terms, and we get:

$$B^m_{\gamma,\lambda} y(x) = f(x),$$

where $I_3(x,s) = -I_2(x,s)$

Theorem 3.1 is proved.

3.1. Verification of Initial Conditions

We show that solution (3.6) satisfies initial conditions (3.5).

Condition $y(0) = 0$**.** Using representation (3.2), we rewrite solution (3.6) in the following form:

$$y(x) = \int_0^x \left(\frac{s}{x}\right)^{2\gamma} \left(\frac{x^2 - s^2}{2x}\right)^{2m-1} \sum_{p=0}^{\infty} \frac{\left(-\frac{\lambda^2}{4}\frac{(x^2-s^2)^2}{x^2}\right)^p (m)_p}{\Gamma(2m+2p)\, p!}$$

$$\times {}_2F_1\left(m + \gamma - \tfrac{1}{2} + p,\ m + p;\ 2m + 2p;\ \frac{x^2 - s^2}{x^2}\right) f(s)\, ds.$$

Applying the substitution $s = xt$ in the integral variable, we obtain

$$y(x) = 2^{1-2m} \sum_{p=0}^{\infty} \frac{\left(-\frac{\lambda^2}{4}\right)^p (m)_p}{\Gamma(2m+2p)\, p!}\, x^{2m+2p}$$

$$\times \int_0^1 t^{2\gamma}(1-t^2)^{2m+2p-1}\, {}_2F_1\left(m + \gamma - \tfrac{1}{2} + p,\ m + p;\ 2m + 2p;\ 1 - t^2\right) f(xt)\, dt.$$

We now estimate this integral. Taking absolute values and applying the triangle inequality, we get

$$|y(x)| leq 2^{1-2m} \sum_{p=0}^{\infty} \frac{\left|\left(-\frac{\lambda^2}{4}\right)^p (m)_p\right|}{\Gamma(2m+2p)\, p!}\, x^{2m+2p} \int_0^1 \left|t^{2\gamma}(1-t^2)^{2m+2p-1}\right|$$

$$\times \left|{}_2F_1\left(m + \gamma - \tfrac{1}{2} + p,\ m + p;\ 2m + 2p;\ 1 - t^2\right)\right| |f(xt)|\, dt.$$

The following convergence results for hypergeometric functions are known (see [28], pp. 24, 618):

$${}_2F_1\left(m + \gamma - \tfrac{1}{2} + p,\ m + p;\ 2m + 2p;\ 1 - t^2\right) = C_1 < \infty, \quad |1 - t^2| < 1, \quad 0 < t < 1,$$

$${}_1F_2\left(m,\ m + \gamma + \tfrac{1}{2};\ m + \gamma + 1;\ -\frac{\lambda^2 x^2}{16}\right) = C_2 < \infty, \quad \left|-\frac{\lambda^2 x^2}{16}\right| \in \mathbb{R}, \quad m \in \mathbb{N}, \quad 0 < \gamma < \tfrac{1}{2}.$$

Using these bounds and the continuity of $f(x) \in C[0, b]$, we carry out the following estimates for the last expression:

$$|y(x)| \le x^{2m} C_1 |f(xt)|\, \frac{2^{1-2m}}{\Gamma(2m)} \sum_{p=0}^{\infty} \frac{\left(-\frac{\lambda^2}{16}x^2\right)^p}{\left(m + \frac{1}{2}\right)_p\, p!}\, B(2\gamma + 1,\ 2m + 2p)$$

$$= x^{2m} C_1 |f(xt)|\, \frac{2^{1-2m}\,\Gamma(2\gamma+1)}{\Gamma(2\gamma+1+2m)} \sum_{p=0}^{\infty} \frac{(m)_p}{\left(\gamma + \frac{1}{2} + m\right)_p (m + \gamma + 1)_p\, p!} \left(-\frac{\lambda^2}{16}x^2\right)^p$$

$$= x^{2m} C_1 |f(xt)|\, \frac{2^{1-2m}\,\Gamma(2\gamma+1)}{\Gamma(2\gamma+1+2m)}\, {}_1F_2\left(m;\ m + \gamma + \tfrac{1}{2},\ m + \gamma + 1;\ -\frac{\lambda^2 x^2}{16}\right)$$

$$= x^{2m} C_1 C_2 |f(xt)|\, \frac{2^{1-2m}\,\Gamma(2\gamma+1)}{\Gamma(2\gamma+1+2m)} = x^{2m} C\, |f(xt)|,$$

where $C = C_1 C_2\, \frac{2^{1-2m}\,\Gamma(2\gamma+1)}{\Gamma(2\gamma+1+2m)}$.

Therefore, from the estimate above, it follows that $y(0) = 0$.

Condition $y'(0) = 0$**.** Differentiating (3.6) and writing it in terms of the integral representation (2.9), then substituting $s = xt$ and sending $x \to 0$, all terms vanish since the integrand contains a factor $x^{-2+2k+2n+2m}$ with $k + n + m > 0$, giving $y'(0) = 0$.

Condition $y''(0) = 0$**.** Differentiating equation (3.7) with respect to x and applying (3.3):

$$y''(x) + \frac{2\gamma}{x}\, y'(x) + \lambda^2 y(x) = \int_0^x K_{m-1,p}(x,s)\, f(s)\, ds = x^{2m-2} \int_0^1 K_{m-1,p}(1,t)\, f(xt)\, dt.$$

Applying L'Hôpital's rule:

$$\lim_{x\to 0}\left[(1+2\gamma)\, y''(x) + \lambda^2 y(x)\right] = 0 \;\Rightarrow\; y''(0) = 0.$$

Conditions $y^{(k)}(0) = 0$**,** $k < 2m-1$**.** We prove these by mathematical induction. Assume $y^{(k)}(0) = 0$ for all $k < 2m-1$. Then applying the operator $\dfrac{d^{k-1}}{dx^{k-1}}\left(y'' + \frac{2\gamma}{x}\, y'\right)$ to both sides, using L'Hôpital's rule, and noting the right-hand side tends to zero, gives $(C + 2\gamma)\, y^{(k+1)}(0) = 0$, hence $y^{(k+1)}(0) = 0$.

Thus, initial conditions (3.5) are completely satisfied by solution (3.6).

3.2. Uniqueness of Solution

Suppose $y_1(x)$ and $y_2(x)$ are two solutions of the problems (3.4)- (3.5). Then $u(x) = y_1(x) - y_2(x)$ satisfies the homogeneous equation:

$$(B_{\gamma,\lambda})^m\, u(x) = 0, \quad u^{(j)}(0) = 0, \quad j = 0, 1, 2, \ldots, 2m-1.$$

According to Theorem 1 in [1], for the problem stated above, the following substitution is valid:

$$\left(B_{\gamma,\lambda}\right)^m J_\lambda\left(-\tfrac{1}{2}, \gamma\right) v(x) = J_\lambda\left(-\tfrac{1}{2}, \gamma\right) \left(B_{0,0}\right)^m v(x),$$

where $B_{0,0} = \frac{d^2}{dx^2}$, and $u(x) = J_\lambda\left(-\frac{1}{2}, \gamma\right) v(x)$, with $J_\lambda\left(-\frac{1}{2}, \gamma\right)$ being the fractional-order generalized Erdélyi–Kober operator.

Consequently, we obtain the problem stated below.

$$\left(\frac{d}{dx}\right)^{2m} v(x) = 0, \quad v^{(i)}(0) = 0, \quad i = 0, 1, 2, \ldots, 2m-1.$$

By the Cauchy–Picard existence and uniqueness theorem for linear ODEs with constant coefficients (see [27, pp. 136–140]), the only solution to this homogeneous problem with zero initial conditions at the origin is the trivial solution $v(x) = 0$.

Hence $u(x) \equiv 0$, i.e., $y_1(x) = y_2(x)$, which proves the uniqueness of the solution.

4. Integer-Order Generalized Bessel Integral Operator

In this section we use the integral operator given by formula (3.6) in Theorem 3.1 to define the integer-order generalized Bessel integral operator.

Definition 4.1. Let $m \in \mathbb{N}$, $f(x) \in L_1(a,b)$, $a \geq 0$. The *right-sided integer-order generalized Bessel operator* $B^{-m}_{\gamma,\lambda,b-}$ and the *left-sided integer-order generalized Bessel operator* $B^{-m}_{\gamma,\lambda,a+}$ on the finite interval are defined,

respectively, by

$$B^{-m}_{\gamma,\lambda,b-}f(x) = \frac{1}{\Gamma(2m)}\int_x^b \left(\frac{s}{x}\right)^{\gamma}(s-x)^{2m-1}\Xi_2\left(\gamma, 1-\gamma;\, m+\tfrac{1}{2};\, 1-\frac{(x+s)^2}{4xs}, -\frac{\lambda^2}{4}(s-x)^2\right) f(s)\,ds,$$

$$B^{-m}_{\gamma,\lambda,a+}f(x) = \frac{1}{\Gamma(2m)}\int_a^x \left(\frac{s}{x}\right)^{\gamma}(x-s)^{2m-1}\Xi_2\left(\gamma, 1-\gamma;\, m+\tfrac{1}{2};\, 1-\frac{(x+s)^2}{4xs}, -\frac{\lambda^2}{4}(x-s)^2\right) f(s)\,ds,$$

or, equivalently, in series form:

$$B^{-m}_{\gamma,\lambda,b-}f(x) = \int_x^b \left(\frac{s^2-x^2}{2s}\right)^{2m-1}\sum_{p=0}^{\infty}\frac{\left(-\frac{\lambda^2}{4}\frac{\left(s^2-x^2\right)^2}{s^2}\right)^p (m)_p}{\Gamma(2m+2p)\,p!}\times$$

$$\times {}_2F_1\left(m+\gamma-\frac{1}{2}+p, m+p; 2m+2p; \frac{s^2-x^2}{s^2}\right) f(s)\,ds,$$

$$B^{-m}_{\gamma,\lambda,a+}f(x) = \int_a^x \left(\frac{s}{x}\right)^{2\gamma}\left(\frac{x^2-s^2}{2x}\right)^{2m-1}\sum_{p=0}^{\infty}\frac{\left(-\frac{\lambda^2}{4}\frac{\left(x^2-s^2\right)^2}{x^2}\right)^p (m)_p}{\Gamma(2m+2p)\,p!}\times$$

$$\times {}_2F_1\left(m+\gamma-\frac{1}{2}+p, m+p; 2m+2p; \frac{x^2-s^2}{x^2}\right) f(s)\,ds.$$

Definition 4.2. Let $m \in \mathbb{N}$, $f(x) \in L_1(0,\infty)$. The right-sided and left-sided integer-order generalized Bessel operators on $[0,\infty)$ are defined, respectively, by

$$B^{-m}_{\gamma,\lambda,-}f(x) = \frac{1}{\Gamma(2m)}\int_x^{\infty}\left(\frac{s}{x}\right)^{\gamma}(s-x)^{2m-1}\Xi_2\left(\gamma, 1-\gamma; m+\frac{1}{2}; -\frac{(s-x)^2}{4xs}, -\frac{\lambda^2}{4}(s-x)^2\right) f(s)\,ds \tag{4.1}$$

$$B^{-m}_{\gamma,\lambda,0+}f(x) = \frac{1}{\Gamma(2m)}\int_0^{x}\left(\frac{s}{x}\right)^{\gamma}(x-s)^{2m-1}\Xi_2\left(\gamma, 1-\gamma; m+\frac{1}{2}; -\frac{(x-s)^2}{4xs}, -\frac{\lambda^2}{4}(x-s)^2\right) f(s)\,ds \tag{4.2}$$

or equivalently, in series form (equations (4.1)–(4.2)):

$$B^{-m}_{\gamma,\lambda,-}f(x) = \int_x^{\infty}\left(\frac{s^2-x^2}{2s}\right)^{2m-1}\sum_{p=0}^{\infty}\frac{\left(-\frac{\lambda^2}{4}\frac{\left(s^2-x^2\right)^2}{s^2}\right)^p (m)_p}{\Gamma(2m+2p)\,p!}\times$$

$$\times {}_2F_1\left(m+\gamma-\frac{1}{2}+p, m+p; 2m+2p; \frac{s^2-x^2}{s^2}\right) f(s)\,ds, \tag{4.3}$$

$$B^{-m}_{\gamma,\lambda,0+}f(x) = \int_0^{x}\left(\frac{s}{x}\right)^{2\gamma}\left(\frac{x^2-s^2}{2x}\right)^{2m-1}\sum_{p=0}^{\infty}\frac{\left(-\frac{\lambda^2}{4}\frac{\left(x^2-s^2\right)^2}{x^2}\right)^p (m)_p}{\Gamma(2m+2p)\,p!}\times$$

$$\times {}_2F_1\left(m+\gamma-\frac{1}{2}+p, m+p; 2m+2p; \frac{x^2-s^2}{x^2}\right) f(s)\,ds \tag{4.4}$$

We may write (4.2) as

$$B^{-m}_{\gamma,\lambda,0+}f(x) = \int_0^x K_{\gamma,m}(x,s)\, f(s)\, ds, \tag{4.5}$$

where

$$K_{\gamma,m}(x,s) = \frac{1}{\Gamma(2m)}(x-s)^{2m-1}\left(\frac{s}{x}\right)^{\gamma} \Xi_2\left(\gamma, 1-\gamma;\, m+\tfrac{1}{2};\, -\frac{(x-s)^2}{4xs},\, -\frac{\lambda^2}{4}(x-s)^2\right), \quad 0<s<x<\infty.$$

Lemma 4.1. *If $f \in C([0,\infty])$, and $f(0) = 0$, then*

$$\lim_{m\to 0} B^{-m}_{\gamma,\lambda,0+}f(x) = f(x). \tag{4.6}$$

Proof. Employing the representation (4.4) for the generalized Bessel integral operator of integer order and using the convergence of the series, we interchange summation and integration, yielding

$$B^{-m}_{\gamma,\lambda,0+}y\,(x) = \sum_{p=0}^{\infty} \frac{\left(-\lambda^2\right)^p (m)_p 2^{1-2p-2m}}{\Gamma\,(2m+2p)\,p!} \times$$

$$\times \int_0^x \left(\frac{s^2-x^2}{s^2}\right)^{2m+2p-1} {}_2F_1\left(m+p+\frac{\gamma-1}{2}, m+p; 2m+2p; \frac{s^2-x^2}{s^2}\right)(-s)^{2m+2p-1} f\,(s)\, ds.$$

Applying (2.5) and integrating by parts, and sending $m \to 0$, only the $p = 0$ term survives (since $(0)_p = 0$ for $p \geq 1$), giving

$$\lim_{m\to 0} B^{-m}_{\gamma,\lambda,0+}f(x) = x^{-2}\int_0^x \frac{d}{ds}\left[s^2 f(s)\right] ds = x^{-2}\left[s^2 f(s)\right]_0^x = f(x).$$

Lemma 4.2. *Let $m \in \mathbb{N}$, $f \in C([0,\infty])$, be sufficiently smooth. Then $B^{m}_{\gamma,\lambda}$ is the inverse operator of $B^{-m}_{\gamma,\lambda,0+}$:*

$$B^{m}_{\gamma,\lambda}\, B^{-m}_{\gamma,\lambda,0+}f(x) = f(x), \quad \left(B^{m}_{\gamma,\lambda}\right)^{-1} = B^{-m}_{\gamma,\lambda,0+}. \tag{4.7}$$

Proof. The proof follows directly from Theorem 3.1 and Lemma 3.1.

Lemma 4.3. *Let $f \in C([0,\infty])$, and $m_1, m_2 \in \mathbb{N}$. Then the operator $B^{-m}_{\gamma,\lambda,0+}$ forms a semigroup with respect to m:*

$$B^{-m_1}_{\gamma,\lambda,0+}\, B^{-m_2}_{\gamma,\lambda,0+}f(x) = B^{-(m_1+m_2)}_{\gamma,\lambda,0+}f(x). \tag{4.8}$$

Proof. We use the series representation (4.4) and, since the integral and inner series are convergent, bring the series outside the integral:

$$B^{-m}_{\gamma,\lambda,0+}y(x) = 2^{1-2m}\sum_{p,l=0}^{\infty} A_{m,p,l}\, x^{1-2\gamma-2m-2p-2l}\int_0^x (x^2-s^2)^{2m+2p+l-1} s^{2\gamma} f(s)\, ds,$$

where

$$A_{m,p,l} = \frac{\left(-\frac{\lambda^2}{4}\right)^p (m)_p \left(m+p+\frac{2\gamma-1}{2}\right)_l (m+p)_l}{\Gamma(2m+2p)\, p!\, (2m+2p)_l\, l!}.$$

Computing $B^{-m_1}_{\gamma,\lambda,0+}B^{-m_2}_{\gamma,\lambda,0+}f(x)$

$$B^{-m_1}_{\gamma,\lambda,0+}B^{-m_1}_{\gamma,\lambda,0+}f\,(x) = 2^{2-2(m_1+m_2)}\sum_{p,l=0}^{\infty} A_{m_1,p,l}\sum_{k,n=0}^{\infty} A_{m_2,k,n} x^{1-2\gamma-2m_1-2p-2l} \times$$

$$\times \int_0^x t^{2\gamma} f(t)\, dt \int_t^x \left(x^2 - s^2\right)^{2m_1+2p+l-1} \left(s^2 - t^2\right)^{2m_2+2k+n-1} s^{1-2m_2-2k-2n} ds =$$

$$= 2^{2-2(m_1+m_2)} \sum_{p,l=0}^{\infty} A_{m_1,p,l} \sum_{k,n=0}^{\infty} A_{m_2,k,n} x^{1-2\gamma-2m_1-2p-2l} \int_0^x t^{2\gamma} f(t)\, g(x,t)\, dt,$$

where $g(x,t) = \int_t^x \left(x^2 - s^2\right)^{2m_1+2p+l-1} \left(s^2 - t^2\right)^{2m_2+2k+n-1} s^{1-2m_2-2k-2n} ds.$

First, we compute the integral $g(x,t)$ and use, get:

$$g(x,t) = \frac{1}{2}\left(x^2 - t^2\right)^{2m_2+2k+n-1+2m_1+2p+l} t^{-2m_2-2k-2n} \frac{\Gamma(2m_2+2k+n)\,\Gamma(2m_1+2p+l)}{\Gamma(2m_2+2m_1+2k+2p+n+l)} \times$$

$$\times {}_2F_1\left(m_2+k+n, 2m_2+2k+n; 2m_2+2m_1+2k+2p+n+l; \frac{t^2-x^2}{t^2}\right).$$

Substituting $g(x,t)$ into its place and performing the calculations, we obtain the following

$$B^{-m_1}_{\gamma,\lambda,0+} B^{-m_2}_{\gamma,\lambda,0+} f(x) =$$

$$= 2^{1-2(m_1+m_2)} \sum_{n=0}^{\infty} \frac{\left(-\lambda^2\left(\frac{x^2-t^2}{x}\right)^2\right)^n}{\Gamma(2m_2+2m_1+2n)\, n!} \sum_{p=0}^{n} \frac{(m_1)_p (m_2)_{n-p} n!}{p!\,(n-p)!} \times$$

$$\times \int_0^x \frac{t}{x} \left(\frac{x^2-t^2}{x}\right)^{2m_2+2m_1-1} {}_2F_1\left(m_1+m_2+n, m_1+m_2-\frac{2\gamma-1}{2}+n; 2m_2+2m_1+2n; \frac{x^2-t^2}{x^2}\right) f(t)\, dt$$

We reduce the multiple series in the last expression to a finite series. For this, we use the following formulas [19, 22, p. 44]

$$\sum_{p=0}^{n} \frac{(\beta_1)_p (\beta_2)_{n-p}\, n!}{p!(n-p)!} = \sum_{p=0}^{n} \binom{n}{p} (\beta_1)_p (\beta_2)_{n-p} = (\beta_1+\beta_2)_n$$

and we have

$$B^{-m_1}_{\gamma,\lambda,0+} B^{-m_2}_{\gamma,\lambda,0+} f(x) = \sum_{n=0}^{\infty} \frac{\left(-\lambda^2\left(\frac{x^2-t^2}{x}\right)^2\right)^n (m_1+m_2)_n}{\Gamma(2m_2+2m_1+2n)\, n!} \times$$

$$\times \int_0^x \left(\frac{t}{x}\right)^{2\gamma} \left(\frac{x^2-t^2}{2x}\right)^{2m_2+2m_1-1} {}_2F_1\left(m_1+m_2+\frac{2\gamma-1}{2}+n, m_1+m_2+n; 2m_2+2m_1+2n; \frac{x^2-t^2}{x^2}\right) f(t)\, dt.$$

Lemma 4.3 is proved.

Lemma 4.4. *Let $m \in \mathbb{N}$, $f \in C^{2m}([0,\infty))$, and $f^{(i)}(0) = 0$ for $i = 0, 1, \ldots, 2m-1$. Then the following equality hold:*

$$B^{-m}_{\gamma,\lambda,0+} B^{m}_{\gamma,\lambda} f(x) = f(x). \tag{4.9}$$

Proof. We prove this lemma using the generalized Bessel integral operator (4.2) and its equivalent series representation (4.4).

$$B^{-m}_{\gamma,\lambda,0+}B^{m}_{\gamma,\lambda}f\,(x)=\int\limits_0^x K_m\,(x,s)\left(B^{m}_{\gamma,\lambda}f\right)(s)\,ds, \tag{4.10}$$

where $K_m\,(x,s)=\frac{1}{\Gamma(2m)}(x-s)^{2m-1}G\,(x,s)$, $G\,(x,s)=\left(\frac{s}{x}\right)^{\gamma}\Xi_2\left(\gamma,1-\gamma;\frac{3}{2};-\frac{(x-s)^2}{4xs},-\frac{\lambda^2}{4}(x-s)^2\right)$.

Using integration by parts on the right-hand side of (4.10), together with the equality proved within Theorem 3.1, we get:

$$K_{m,1}(x,x)=K_{m,1}(x,0)=0,$$

and we rewrite right site of (4.10) for the case $m=1$:

$$\begin{aligned}\int\limits_0^x K_1\,(x,s)\left(B_{\gamma,\lambda}f\right)(s)\,ds&=\int\limits_0^x (x-s)\,G\,(x,s)\left(\frac{d^2}{ds^2}+\frac{2\gamma}{s}\frac{d}{ds}+\lambda^2\right)f\,(s)\,ds\\&=\int\limits_0^x (x-s)\,G\,(x,s)\frac{d^2}{ds^2}f\,(s)\,ds+2\gamma\int\limits_0^x (x-s)\,s^{-1}G\,(x,s)\frac{d}{ds}f\,(s)\,ds\\&\quad+\lambda^2\int\limits_0^x (x-s)\,G\,(x,s)f\,(s)\,ds=I_1+I_2+I_3.\end{aligned}$$

We integrate the first term by parts:

$$\begin{aligned}I_1&=\int\limits_0^x (x-s)\,G\,(x,s)\frac{d^2}{ds^2}f\,(s)\,ds\\&=(x-s)\,G\,(x,s)\,\frac{d}{ds}f\,(s)\Big|_0^x-\int\limits_0^x\frac{d}{ds}\,(x-s)\,G\,(x,s)\frac{d}{ds}f\,(s)\,ds\\&\quad-\int\limits_0^x\frac{d}{ds}\,(x-s)\,G\,(x,s)\frac{d}{ds}f\,(s)\,ds\\&=-\frac{d}{ds}\,(x-s)\,G\,(x,s)\;f\,(s)|_0^x+\int\limits_0^x\frac{d^2}{ds^2}\,(x-s)\,G\,(x,s)f\,(s)\,ds\\&=\int\limits_0^x\frac{d^2}{ds^2}\,(x-s)\,G\,(x,s)f\,(s)\,ds.\end{aligned}$$

Now integrate the second term by parts:

$$I_2 = 2\gamma \int_0^x (x-s)\, s^{-1} G(x,s) \frac{d}{ds} f(s)\, ds$$

$$= 2\gamma (x-s)\, s^{-1}\, G(x,s)\, f(s)|_0^x - 2\gamma \int_0^x \frac{d}{ds} (x-s)\, s^{-1} G(x,s) f(s)\, ds$$

$$= -\int_0^x \frac{d}{ds} \frac{2\gamma}{s} (x-s)\, G(x,s) f(s)\, ds.$$

Summing the contributions we obtain

$$\int_0^x (x-s)\, G(x,s) \left(\frac{d^2}{ds^2} + \frac{2\gamma}{s} \frac{d}{ds} + \lambda^2 \right) f(s)\, ds =$$

$$= \int_0^x \left(\frac{d^2}{ds^2} - \frac{d}{ds} \frac{2\gamma}{s} + \lambda^2 \right) (x-s)\, G(x,s) f(s)\, ds =$$

$$= \int_0^x \left(\frac{d^2}{ds^2} - \frac{d}{ds} \frac{2\gamma}{s} + \lambda^2 \right) K_1(x,s) f(s)\, ds. \tag{4.11}$$

Denote $B^*_{\gamma,\lambda} = \left(\frac{d^2}{ds^2} - \frac{d}{ds} \frac{2\gamma}{s} + \lambda^2 \right)$ and we calculate the following equation:

$$B^{-m}_{\gamma,\lambda,0+} B^*_{\gamma,\lambda} f(x) = \int_0^x B^*_{\gamma,\lambda} K_{m,p}(x,s)\, f(s)\, ds.$$

Separate the term with $p = 0$:

$$B^{-m}_{\gamma,\lambda,0+} B^*_{\gamma,\lambda} f(x) = \int_0^x B^*_{\gamma} K_{m,0}(x,s) f(s)\, ds + \int_0^x B^*_{\gamma} K_{m,p\geq 1}(x,s) f(s)\, ds + \lambda^2 \int_0^x K_{m,p}(x,s)\, f(s)\, ds =$$

$$= L_1 + L_2 + L_3. \tag{4.12}$$

First compute L_1:

$$L_1 = \int_0^x \left(\frac{d^2}{ds^2} - \frac{d}{ds}\frac{2\gamma}{s}\right)\left(\frac{s}{x}\right)^{2\gamma}\left(\frac{x^2-s^2}{2x}\right)^{2m-1} {}_2F_1\left(m+\gamma-\frac{1}{2}, m; 2m; \frac{x^2-s^2}{x^2}\right) f(s)\, ds$$

$$= 2\int_0^x x^{2m-1}\left(\frac{s}{x}\right)^{2\gamma}\left(\frac{d^2}{ds^2} + \frac{2\gamma}{s}\frac{d}{ds}\right)\frac{\left(\frac{x^2-s^2}{x^2}\right)^{2m-1}}{2^{2m}\Gamma(2m)} {}_2F_1\left(m+\gamma-\frac{1}{2}, m; 2m; \frac{x^2-s^2}{x^2}\right) f(s)\, ds$$

$$= 2\int_0^x x^{2m-3}\left(\frac{s}{x}\right)^{2\gamma}\frac{\left(\frac{x^2-s^2}{x^2}\right)^{2m-3}}{2^{2m-2}\Gamma(2m-2)} {}_2F_1\left(m+\gamma-\frac{3}{2}, m-1; 2m-2; \frac{x^2-s^2}{x^2}\right) f(s)\, ds$$

$$= \frac{1}{\Gamma(2(m-1))}\int_0^x \left(\frac{s}{x}\right)^{2\gamma}\left(\frac{x^2-s^2}{2x}\right)^{2(m-1)-1} {}_2F_1\left(m-1+\gamma-\frac{1}{2}, m-1; 2(m-1); \frac{x^2-s^2}{x^2}\right) f(s)\, ds$$

$$= \int_0^x K_{m-1,0}(x,s) f(s)\, ds.$$

Now compute L_2:

$$L_2 = 2\sum_{p=1}^{\infty}\frac{\left(-\lambda^2x^2\right)^p (m)_p}{p!}\int_0^x x^{2m-1}\left(\frac{s}{x}\right)^{2\gamma}\left(\frac{d^2}{ds^2}+\frac{2\gamma}{s}\frac{d}{ds}\right)\frac{\left(\frac{x^2-s^2}{x^2}\right)^{2m+2p-1}}{2^{2m+2p}\Gamma(2m+2p)}$$

$$\times {}_2F_1\left(m+\gamma-\frac{1}{2}+p,\ m+p;\ 2m+2p;\ \frac{x^2-s^2}{x^2}\right) f(s)\, ds$$

$$= 2\sum_{p=1}^{\infty}\frac{\left(-\lambda^2x^2\right)^p (m)_p}{p!}\int_0^x x^{2m-3}\left(\frac{s}{x}\right)^{2\gamma}\frac{\left(\frac{x^2-s^2}{x^2}\right)^{2m+2p-3}}{2^{2m+2p-2}\Gamma(2m+2p-2)}$$

$$\times {}_2F_1\left(m+\gamma-\frac{3}{2}+p,\ m+p-1;\ 2m+2p-2;\ \frac{x^2-s^2}{x^2}\right) f(s)\, ds$$

$$= \int_0^x \left(\frac{s}{x}\right)^{2\gamma}\left(\frac{x^2-s^2}{2x}\right)^{2m-3}\sum_{p=1}^{\infty}\frac{\left(-\frac{\lambda^2}{4}x^2\left(\frac{x^2-s^2}{x^2}\right)^2\right)^p (m)_p}{\Gamma(2m+2p-2)\, p!}$$

$$\times {}_2F_1\left(m+\gamma-\frac{3}{2}+p,\ m+p-1;\ 2m+2p-2;\ \frac{x^2-s^2}{x^2}\right) f(s)\, ds.$$

Using the identity

$$(m)_p = (m-1)_p + p\,(m)_{p-1},$$

we split L_2 into two parts:

$$L_2 = \int_0^x \left(\frac{s}{x}\right)^{2\gamma} \left(\frac{x^2-s^2}{2x}\right)^{2m-3} \sum_{p=1}^{\infty} \frac{\left(-\frac{\lambda^2}{4}x^2\left(\frac{x^2-s^2}{x^2}\right)^2\right)^p (m-1)_p}{\Gamma(2m+2p-2)\,p!}$$
$$\times {}_2F_1\left(m+\gamma-\frac{3}{2}+p, m+p-1; 2m+2p-2; \frac{x^2-s^2}{x^2}\right) f(s)\,ds$$
$$+ \int_0^x \left(\frac{s}{x}\right)^{2\gamma} \left(\frac{x^2-s^2}{2x}\right)^{2m-3} \sum_{p=1}^{\infty} \frac{\left(-\frac{\lambda^2}{4}x^2\left(\frac{x^2-s^2}{x^2}\right)^2\right)^p (m)_{p-1}}{\Gamma(2m+2p-2)\,(p-1)!}$$
$$\times {}_2F_1\left(m+\gamma-\frac{3}{2}+p, m+p-1; 2m+2p-2; \frac{x^2-s^2}{x^2}\right) f(s)\,ds.$$

The first part corresponds to replacing m by $m-1$; in the second part we shift the summation index $p \to p+1$. This yields

$$L_2 = \int_0^x K_{m-1,p\geq 1}(x,s)\, f(s)\,ds - \lambda^2 \int_0^x K_{m-1,p}(x,s)\, f(s)\,ds.$$

Substituting the obtained expressions into (4.12) we finally get

$$B^{-m}_{\gamma,\lambda,0+} B^{*}_{\gamma,\lambda} f(x) = \int_0^x K_{m-1,p}(x,s)\, f(s)\,ds.$$

The last formula, together with Lemmas 4.1 and 4.3, proves equality (4.9)
This completes the proof.

5. Conclusion

In this study, a unified analytical approach for higher-order ordinary differential equations involving the generalized Bessel differential operator with a spectral parameter has been developed. By employing the transmutation operator method together with the analytical properties of the Gauss and Humbert hypergeometric functions, an explicit integral representation of the solution to the corresponding Cauchy problem has been derived. The obtained integral representation has been rigorously verified to satisfy both the differential equation and the prescribed initial conditions, and the uniqueness of the solution has been established. These results provide a solid mathematical foundation for the analysis of singular differential equations associated with generalized Bessel operators. Based on the derived solution, a new integer-order generalized Bessel integral operator has been introduced. In contrast to the classical Erdélyi–Kober and Bessel-type integral operators, the proposed operator combines a higher-order generalized Bessel differential structure with a spectral parameter through a kernel represented by the Humbert hypergeometric function. Furthermore, its fundamental properties, including the inverse operator, left- and right-inverse identities, semigroup property, limiting behavior, and integration-by-parts formulas, have been rigorously established. The obtained results extend the existing theory of Bessel and Erdélyi–Kober integral operators and provide a theoretical basis for further investigations of generalized transmutation operators and fractional generalized Bessel integral operators. Future work will focus on the construction of fractional-order generalized Bessel

integral operators, the investigation of their mapping properties in functional spaces, the derivation of composition and factorization formulas, and their applications to boundary value problems, inverse problems, and singular differential equations arising in mathematical physics.

Acknowledgements

The authors have no acknowledgements to declare.

Funding

The authors received no financial support for this research.

Availability of data and materials

No datasets were generated or analyzed during the current study.

Competing interests

The authors declare that they have no competing interests.

Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

References

[1] Sitnik, S., Karimov, Sh., & Boynazarov, A. (2025). Solving the Cauchy problem for an ordinary differential equation with an integer power of the Bessel operator using transmutation operators. *Journal of Mathematical Analysis and Applications*. Received: 30 July 2025; Accepted: 7 October 2025. https://doi.org/10.1007/s10958-025-xxxx-x

[2] Samko, S. G., Kilbas, A. A., & Marichev, O. I. (1987). *Fractional integrals and derivatives: Theory and applications*. Minsk: Nauka i Tekhnika. 688 p.

[3] Urinov, A. K., Sitnik, S. M., Shishkina, E. L., & Karimov, Sh. T. (2022). *Fractional integrals and derivatives (Generalizations and applications)*. [In Russian]. Fergana: Publishing House "Farg'ona". 192 p.

[4] Urinov, A. K., & Karimov, Sh. T. (2021). *Erdélyi–Kober operators and applications to partial differential equations*. Fergana: Publishing House "Farg'ona". 202 p.

[5] Sprinkhuizen-Kuyper, I. G. (1979). A fractional integral operator corresponding to negative powers of a certain second-order differential operator. *Journal of Mathematical Analysis and Applications*, 72(2), 674–702. https://doi.org/10.1016/0022-247X(79)90278-9

[6] Lowndes, J. S. (1970). A generalization of the Erdélyi–Kober operators. *Proceedings of the Edinburgh Mathematical Society*, 17(2), 139–148. https://doi.org/10.1017/S001309150000856X

[7] Kiryakova, V. (1986). Applications of the generalized Poisson transformation for solving hyper-Bessel differential equations. *Godishnik VUZ. Applied Mathematics*, 22(4), 129–140.

[8] Kiryakova, V. (1994). *Generalized fractional calculus and applications*. Harlow: Longman Scientific & Technical. 360 p.

[9] Kiryakova, V. (2008). Transmutation method for solving hyper-Bessel differential equations based on the Poisson–Dimovski transformation. *Fractional Calculus and Applied Analysis*, 11(3), 299–316.

[10] Kiryakova, V. S. (1994). *Generalized fractional calculus and applications*. New York: John Wiley & Sons. 360 p.

[11] Urinov, A. K. (2012). *Special functions and special operators*. [In Uzbek]. Fergana: "Farg'ona" nashriyoti. 112 p.

[12] Lebedev, N. N. (1972). *Special functions and their applications*. New York: Dover Publications. 358 p.

[13] Sitnik, S. M., & Skoromnik, O. V. (2020). Transmutation operators that generalize the Sonine and Poisson transforms. *Journal of Mathematical Sciences*, 249(4), 543–558. https://doi.org/10.1007/s10958-020-04987-2

[14] Luchko, Y., & Gorenflo, R. (1998). Fractional derivatives of Erdélyi–Kober type and some of their applications. *Fractional Calculus and Applied Analysis*, 1(1), 63–78.

[15] Kiryakova, V. (2021). The interplay of generalized fractional calculus and integral transforms. *Mathematics*, 9(15), 1789. https://doi.org/10.3390/math9151789

[16] Karimov, Sh. T. (2024). On a method for solving the Cauchy problem for a high-order equation with a Bessel operator. *Izvestiya Vysshikh Uchebnykh Zavedeniy. Matematika*.

[17] Hasanov, A. S., & Ergashev, T. G. (2023). New properties of the Humbert functions and their applications. *Integral Transforms and Special Functions*, 34(5), 412–428. https://doi.org/10.1080/10652469.2022.2156789
[18] Shishkina, E., & Sitnik, S. M. (2017). On fractional powers of the Bessel operator on a semiaxis. *arXiv preprint* arXiv:1706.01928. https://doi.org/10.48550/arXiv.1706.01928
[19] Sneddon, I. N. (1972). *The use of integral transforms*. New York: McGraw-Hill. 540 p.
[20] Bateman, H., & Erdélyi, A. (1954). *Tables of integral transforms* (Vol. 1). New York: McGraw-Hill. 391 p.
[21] Erdélyi, A., Magnus, W., Oberhettinger, F., & Tricomi, F. G. (1953). *Higher transcendental functions* (Vol. I). New York: McGraw-Hill. 302 p.
[22] Comtet, L. (1974). *Advanced Combinatorics: The Art of Finite and Infinite Expansions*. D. Reidel Publishing Company, Dordrecht.
[23] Sh. T. Karimov, "The Cauchy Problem for the Iterated Klein–Gordon Equation with the Bessel Operator," *Lobachevskii Journal of Mathematics*, vol. 41, no. 5, pp. 772–784, 2020.
[24] Sh. T. Karimov and Sh. A. Oripov, "On a Method for Constructing the Riemann Function for Partial Differential Equations with a Singular Bessel Operator," *Lobachevskii Journal of Mathematics*, vol. 41, no. 6, pp. 1087–1093, 2020.
[25] Sh. T. Karimov and E. L. Shishkina, "Some methods of solution to the Cauchy problem for a inhomogeneous equation of hyperbolic type with a Bessel operator," *Journal of Physics: Conference Series*, vol. 1203, no. 1, art. no. 012096, 2019.
[26] Sh. T. Karimov, "On One Method for the Solution of an Analog of the Cauchy Problem for a Polycaloric Equation with Singular Bessel Operator," *Ukrainian Mathematical Journal*, vol. 69, no. 10, pp. 1593–1606, 2018.
[27] M. S. Salokhiddinov and G. N. Nasriddinov, *Oddiy differensial tenglamalar* [Ordinary Differential Equations], Tashkent: Ozbekiston, 1994 (in Uzbek).
[28] Dunaev A.S., Shlychkov V.I. Gipergeometricheskie Funktsii (Hypergeometric Functions).*Educational Electronic Text Resource.* Yekaterinburg, 2017.

Affiliations

Karimov Shakhobiddin Tuychiboyevich
Address: Fergana State University. Fergana, Uzbekistan
e-mail: shaxkarimov@gmail.com
ORCID ID: https://orcid.org/0000-0001-8325-998X

Boynazarov Akhrorjon Numonjonovich
Address: Fergana State University. Fergana, Uzbekistan
e-mail: ahror010185@gmail.com
ORCID ID: https://orcid.org/0009-0002-7422-4368

Uzbekistan Journal of Mathematics and Computer Science
Volume-2, Issue-3-special, Page 49–59 (2026)
DOI: https://doi.org/10.56143/ujmcs.v2i3s.5
Research article

Hidden Potts-Observed SOS Models on a Two-Layer Structure

M.M. Rahmatullaev *M.A. Rasulova

Abstract

In this paper, we introduce a two-layer hidden Potts-observed SOS model on Cayley trees of arbitrary order. The model is constructed through a hidden Markov structure in which the observable SOS configuration is generated from an underlying latent Potts field by means of a local stochastic transformation. For the proposed model, we derive the compatibility conditions for Gibbs measures on Cayley trees of arbitrary order and investigate translation-invariant Gibbs states. In particular, explicit forms of translation-invariant Gibbs measures are obtained for Cayley trees of orders one and two, and conditions for their existence are established. Moreover, for these Gibbs measures, the hidden states are reconstructed and predicted from the observable layer. The obtained results provide a rigorous probabilistic framework for multi-state hidden spin systems and extend hidden Markov methods to Potts-SOS interactions on hierarchical graphs. The proposed approach may be useful in the study of spatial stochastic systems, graphical models, and related inference problems involving latent structures.

Keywords: *Cayley tree; Potts model; SOS model; hidden Markov structure; Gibbs measure; probabilistic inference.*

AMS Subject Classification (2020): *Primary: 82B20; Secondary: 60K35, 82B26, 60J05, 05C05, 68T05.*

1. Introduction

Models on hierarchical graphs have attracted considerable attention in statistical mechanics due to their ability to describe complex interaction mechanisms while remaining mathematically tractable. Among such graphs, the Cayley tree occupies a special position since many questions concerning Gibbs measures, phase transitions, extremality, and ground states can be studied rigorously. Classical foundations of the theory of Gibbs measures and phase transitions were established in [13, 12, 23, 9], while structural properties of Cayley trees and related graph-theoretical aspects were discussed in [11]. Comprehensive accounts of Gibbs measures on Cayley trees and their applications can be found in the monographs [20, 21, 1].

Among multi-state lattice systems, the Potts model and its various generalizations have been extensively investigated. Exact methods and thermodynamic properties of such models were developed in the classical work [3]. Subsequently, Potts models on Cayley trees were studied from different viewpoints, including phase transitions, competing interactions, periodic Gibbs measures, and ground-state structures [2, 5, 19, 4, 22]. Another important direction concerns Potts-SOS models, which combine discrete Potts interactions with solid-on-solid (SOS) type interactions. Gibbs measures, extremality properties, limiting Gibbs measures, and

Received : *28–april–2026,* ***Accepted :*** *31–july–2026*
*** Corresponding author***

ground states for Potts-SOS models on Cayley trees were investigated in [14, 15, 17, 18], revealing a rich variety of equilibrium phenomena.

In parallel with these developments, hidden-variable models have become an important tool in machine learning, probabilistic inference, and graphical modeling. Hidden Markov structures provide a natural framework for describing situations in which the observed data arise from an underlying latent process. Recently, Gibbsian approaches to hidden Markov models have been proposed, establishing connections between statistical mechanics and modern inference methods [7, 1]. However, most existing studies have focused on binary-state systems, while multi-state hidden models remain much less understood. Recently, a hidden Markov framework for multi-state Potts systems was proposed in [16]. That work established the Gibbsian formalism for hidden Potts models on Cayley trees and demonstrated that hidden-state interactions can be successfully incorporated into the theory of Gibbs measures. These results motivate the study of more sophisticated hidden multi-layer systems involving different types of interactions at the hidden and observable levels.

Motivated by these developments, in the present paper we introduce a two-layer hidden Potts-observed SOS model on the Cayley tree. In the proposed framework, the hidden layer is represented by a three-state Potts configuration, whereas the observable layer is described by an SOS configuration generated through a local stochastic mechanism. This construction naturally combines ideas from hidden Markov models and Potts-SOS systems, providing a new class of Gibbsian graphical models with latent structure.

The main objective of this work is to establish a rigorous Gibbsian description of the proposed model. We derive compatibility conditions for Gibbs measures on Cayley trees of arbitrary order and investigate translation-invariant Gibbs measures. In particular, explicit forms of translation-invariant Gibbs measures are obtained for Cayley trees of orders one and two. Furthermore, we study the reconstruction and prediction of hidden states from the observable layer and identify parameter regions in which multiple Gibbs measures coexist.

The obtained results provide a bridge between the theory of Gibbs measures, Potts-SOS models, and hidden Markov structures. They also open new perspectives for the study of latent-variable systems on hierarchical graphs and their potential applications in probabilistic inference and machine learning.

1.1. The model

We now introduce the model under consideration. Let $\Gamma^k = (V, L)$ be a Cayley tree of order k, where V denotes the set of vertices and L denotes the set of edges. To each vertex $x \in V$ we assign a Potts-SOS spin $s(x)$ taking values in the finite set

$$\Phi = \{0, 1, 2\}.$$

A configuration of hidden spins is a mapping $s : V \to \Phi$, and the set of all such configurations is denoted by

$$\Omega = \Phi^V.$$

Similarly, $\sigma \in \Omega$ denotes the observable configuration.

The Hamiltonian $H(s, \sigma)$ of the hidden Markov model, which describes the energy associated with a pair of configurations (s, σ)–including both the Potts-SOS interactions and the dependence between the hidden Potts configuration $s \in \Omega$ and the observable SOS configuration $\sigma \in \Omega$–is defined as follows:

$$H(s, \sigma) = -J \sum_{\langle x,y \rangle} \big(\delta_{s(x)s(y)} - |\sigma(x) - \sigma(y)|\big) - \sum_{x \in V} p\big(s(x), \sigma(x)\big), \tag{1.1}$$

where $J \in \mathbb{R}$ is the interaction parameter, δ_{ij} is the Kronecker delta, and $p : \Phi \times \Phi \to \mathbb{R}$ is an external field depending on both hidden and observed states.

Remark 1.1. A more general formulation of (1.1) can be introduced by assigning separate coupling constants to the Potts and SOS interactions:

$$H(s,\sigma) = -J_1 \sum_{\langle x,y\rangle} \delta_{s(x)s(y)} + J_2 \sum_{\langle x,y\rangle} |\sigma(x) - \sigma(y)| - \sum_{x\in V} p\big(s(x),\sigma(x)\big), \tag{1.2}$$

where $J_1, J_2 \in \mathbb{R}$ are independent interaction parameters. Although the main results can be extended to this more general framework, the resulting compatibility equations become considerably more complicated. For this reason, in the present work we focus on the symmetric situation $J_1 = J_2 = J$, which already exhibits a rich and nontrivial behavior.

Remark 1.2. The hidden Potts interaction considered in (1.1) may also be replaced by more general interactions. In particular, one can study hidden models generated by Potts-SOS type Hamiltonians. Various aspects of Potts-SOS models on Cayley trees, including Gibbs measures, extremality questions, and ground-state structures, have been investigated in [14, 15, 18]. The development of hidden Markov counterparts of such models remains an interesting direction for future research.

2. The compatibility of measures

Define a finite-dimensional distribution of a probability measure μ_n in the $\Omega_n \times \Omega_n$ as

$$\mu_n(s_n,\sigma_n) = Z_n^{-1} \exp\left\{-\beta H_n(s_n,\sigma_n) + \sum_{x\in W_n} h_{s(x),\,\sigma(x),\,x}\right\}, \tag{2.1}$$

where $\beta = 1/T$, and $T > 0$ is the temperature, Z_n^{-1} is the normalizing factor,

$$\big\{h_x = (h_{0,0,x}, h_{0,1,x}, h_{0,2,x}, h_{1,0,x}, h_{1,1,x}, h_{1,2,x}, h_{2,0,x}, h_{2,1,x}, h_{2,2,x}) \in \mathbb{R}^9,\ x \in V\big\} \tag{2.2}$$

is a collection of vectors and

$$H_n(s_n,\sigma_n) = -J \sum_{\langle x,y\rangle\in L_n} \big(\delta_{s(x)s(y)} - |\sigma(x) - \sigma(y)|\big) - \sum_{x\in V_n} p(s(x),\sigma(x)).$$

Definition 2.1. We say that the probability distributions (2.1) are compatible if for all $n \geq 1$ and $s_{n-1} \in \Omega_{n-1},\ \sigma_{n-1} \in \Omega_{n-1}$:

$$\sum_{(w_n,\omega_n)\in\Phi^{W_n}\times\Phi^{W_n}} \mu_n\big(s_{n-1}\vee w_n,\ \sigma_{n-1}\vee\omega_n\big) = \mu_{n-1}(s_{n-1},\sigma_{n-1}), \tag{2.3}$$

where symbol $\vee$ denotes the concatenation (union) of the configurations.

In this case, by Kolmogorov's theorem [20] there exists a unique measure μ on the measurable space $(\Omega\times\Omega, \mathcal{F})$, where $\mathcal{F}$ is the sigma-algebra generated by the cylinder sets of $\Omega\times\Omega$, such that for all n and $s_n, \sigma_n \in \Omega_n$,

$$\mu\big(\{(s,\sigma)\mid_{V_n} = (s_n,\sigma_n)\}\big) = \mu_n(s_n,\sigma_n).$$

Such a measure is called a *splitting Gibbs measure* (SGM) corresponding to the Hamiltonian (1.1) and vector-valued functions (2.2).

Remark 2.1. In this paper, we exclusively study SGMs and thus omit the qualifier "splitting" for brevity.

The following theorem describes conditions on h_x guaranteeing compatibility of $\mu_n(s_n, \sigma_n)$.

Theorem 2.1. *The probability distributions $\mu_n(s_n, \sigma_n)$, $n \in \mathbb{N}$, defined by (2.1) are compatible if and only if, for any $x \in V \setminus \{x^0\}$, the following system of equations holds:*

$$u_{1,x} = a_{0,1} \prod_{y \in S(x)} \frac{1 + ru_{1,y} + u_{2,y} + r^{-1}u_{3,y} + u_{4,y} + r^{-1}u_{5,y} + r^{-1}u_{6,y} + u_{7,y} + r^{-1}u_{8,y}}{r + u_{1,y} + r^{-1}u_{2,y} + u_{3,y} + r^{-1}u_{4,y} + r^{-2}u_{5,y} + u_{6,y} + r^{-1}u_{7,y} + r^{-2}u_{8,y}}, \tag{2.4}$$

$$u_{2,x} = a_{0,2} \prod_{y \in S(x)} \frac{r^{-1} + u_{1,y} + ru_{2,y} + r^{-2}u_{3,y} + r^{-1}u_{4,y} + u_{5,y} + r^{-2}u_{6,y} + r^{-1}u_{7,y} + u_{8,y}}{r + u_{1,y} + r^{-1}u_{2,y} + u_{3,y} + r^{-1}u_{4,y} + r^{-2}u_{5,y} + u_{6,y} + r^{-1}u_{7,y} + r^{-2}u_{8,y}}, \tag{2.5}$$

$$u_{3,x} = a_{1,0} \prod_{y \in S(x)} \frac{1 + r^{-1}u_{1,y} + r^{-2}u_{2,y} + ru_{3,y} + u_{4,y} + r^{-1}u_{5,y} + u_{6,y} + r^{-1}u_{7,y} + r^{-2}u_{8,y}}{r + u_{1,y} + r^{-1}u_{2,y} + u_{3,y} + r^{-1}u_{4,y} + r^{-2}u_{5,y} + u_{6,y} + r^{-1}u_{7,y} + r^{-2}u_{8,y}}, \tag{2.6}$$

$$u_{4,x} = a_{1,1} \prod_{y \in S(x)} \frac{r^{-1} + u_{1,y} + r^{-1}u_{2,y} + u_{3,y} + ru_{4,y} + u_{5,y} + r^{-1}u_{6,y} + u_{7,y} + r^{-1}u_{8,y}}{r + u_{1,y} + r^{-1}u_{2,y} + u_{3,y} + r^{-1}u_{4,y} + r^{-2}u_{5,y} + u_{6,y} + r^{-1}u_{7,y} + r^{-2}u_{8,y}}, \tag{2.7}$$

$$u_{5,x} = a_{1,2} \prod_{y \in S(x)} \frac{r^{-2} + r^{-1}u_{1,y} + u_{2,y} + r^{-1}u_{3,y} + u_{4,y} + ru_{5,y} + r^{-2}u_{6,y} + r^{-1}u_{7,y} + u_{8,y}}{r + u_{1,y} + r^{-1}u_{2,y} + u_{3,y} + r^{-1}u_{4,y} + r^{-2}u_{5,y} + u_{6,y} + r^{-1}u_{7,y} + r^{-2}u_{8,y}}, \tag{2.8}$$

$$u_{6,x} = a_{2,0} \prod_{y \in S(x)} \frac{1 + r^{-1}u_{1,y} + r^{-2}u_{2,y} + u_{3,y} + r^{-1}u_{4,y} + r^{-2}u_{5,y} + ru_{6,y} + u_{7,y} + r^{-1}u_{8,y}}{r + u_{1,y} + r^{-1}u_{2,y} + u_{3,y} + r^{-1}u_{4,y} + r^{-2}u_{5,y} + u_{6,y} + r^{-1}u_{7,y} + r^{-2}u_{8,y}}, \tag{2.9}$$

$$u_{7,x} = a_{2,1} \prod_{y \in S(x)} \frac{r^{-1} + u_{1,y} + r^{-1}u_{2,y} + r^{-1}u_{3,y} + u_{4,y} + r^{-1}u_{5,y} + u_{6,y} + ru_{7,y} + u_{8,y}}{r + u_{1,y} + r^{-1}u_{2,y} + u_{3,y} + r^{-1}u_{4,y} + r^{-2}u_{5,y} + u_{6,y} + r^{-1}u_{7,y} + r^{-2}u_{8,y}}, \tag{2.10}$$

$$u_{8,x} = a_{2,2} \prod_{y \in S(x)} \frac{r^{-2} + r^{-1}u_{1,y} + u_{2,y} + r^{-2}u_{3,y} + r^{-1}u_{4,y} + u_{5,y} + r^{-1}u_{6,y} + u_{7,y} + ru_{8,y}}{r + u_{1,y} + r^{-1}u_{2,y} + u_{3,y} + r^{-1}u_{4,y} + r^{-2}u_{5,y} + u_{6,y} + r^{-1}u_{7,y} + r^{-2}u_{8,y}}, \tag{2.11}$$

where the quantities $u_{i,x}$, $i = 1, \dots, 8$, are defined by

$$u_{1,x} = a_{0,1}\, z_{0,1,x}, \quad u_{2,x} = a_{0,2}\, z_{0,2,x}, \quad u_{3,x} = a_{1,0}\, z_{1,0,x}, \quad u_{4,x} = a_{1,1}\, z_{1,1,x},$$

$$u_{5,x} = a_{1,2}\, z_{1,2,x}, \quad u_{6,x} = a_{2,0}\, z_{2,0,x}, \quad u_{7,x} = a_{2,1}\, z_{2,1,x}, \quad u_{8,x} = a_{2,2}\, z_{2,2,x}.$$

Here,

$$a_{\epsilon,\xi} = \exp\big(\beta\big(p(\epsilon,\xi) - p(0,0)\big)\big), \quad z_{\epsilon,\xi,x} = \exp\big(h_{\epsilon,\xi,x} - h_{0,0,x}\big),$$

for $\epsilon, \xi \in \{0, 1, 2\}$, and $r = \exp(J\beta)$. Moreover, $S(x)$ denotes the set of direct successors of x.

Proof. Using the same approach as in the proof of Theorem 1 in [7], one obtains the result. □

From Theorem 2.1, it follows that for any $u = \{u_{i,x},\ x \in V\}$ that satisfies the system of functional equations (2.4)–(2.11), there exists a unique SGM μ, and conversely. Therefore, the core problem of describing Gibbs measures reduces to solving the system of functional equations (2.4)–(2.11). However, solving this system is particularly challenging due to its non-linear nature, multidimensional structure, and the fact that the unknown functions are defined on a tree. Even the task of determining all constant functions (which are independent of the tree's vertices) is complex. In this paper, we present a class of such constant solutions and explore how the corresponding Gibbs measures can be applied in machine learning.

Our Theorem 2.1 ensures that the locally defined probability kernels derived from the Hamiltonian (1.1) correspond to a consistent global Gibbs measure. This guarantees the mathematical coherence of the model, stabilizes probabilistic inference procedures, and validates the correctness of methods used to estimate hidden states.

3. Translation-invariant Gibbs measures

In this section, we study translation-invariant Gibbs measures corresponding to solutions satisfying

$$u_{i,x} = u_i, \qquad x \in V. \tag{3.1}$$

Substituting condition (3.1) into equations (2.4)–(2.11), we obtain

$$u_1 = a_{0,1}\left(\frac{1 + ru_1 + u_2 + r^{-1}u_3 + u_4 + r^{-1}u_5 + r^{-1}u_6 + u_7 + r^{-1}u_8}{r + u_1 + r^{-1}u_2 + u_3 + r^{-1}u_4 + r^{-2}u_5 + u_6 + r^{-1}u_7 + r^{-2}u_8}\right)^k, \tag{3.2}$$

$$u_2 = a_{0,2}\left(\frac{r^{-1} + u_1 + ru_2 + r^{-2}u_3 + r^{-1}u_4 + u_5 + r^{-2}u_6 + r^{-1}u_7 + u_8}{r + u_1 + r^{-1}u_2 + u_3 + r^{-1}u_4 + r^{-2}u_5 + u_6 + r^{-1}u_7 + r^{-2}u_8}\right)^k, \tag{3.3}$$

$$u_3 = a_{1,0}\left(\frac{1 + r^{-1}u_1 + r^{-2}u_2 + ru_3 + u_4 + r^{-1}u_5 + u_6 + r^{-1}u_7 + r^{-2}u_8}{r + u_1 + r^{-1}u_2 + u_3 + r^{-1}u_4 + r^{-2}u_5 + u_6 + r^{-1}u_7 + r^{-2}u_8}\right)^k, \tag{3.4}$$

$$u_4 = a_{1,1}\left(\frac{r^{-1} + u_1 + r^{-1}u_2 + u_3 + ru_4 + u_5 + r^{-1}u_6 + u_7 + r^{-1}u_8}{r + u_1 + r^{-1}u_2 + u_3 + r^{-1}u_4 + r^{-2}u_5 + u_6 + r^{-1}u_7 + r^{-2}u_8}\right)^k, \tag{3.5}$$

$$u_5 = a_{1,2}\left(\frac{r^{-2} + r^{-1}u_1 + u_2 + r^{-1}u_3 + u_4 + ru_5 + r^{-2}u_6 + r^{-1}u_7 + u_8}{r + u_1 + r^{-1}u_2 + u_3 + r^{-1}u_4 + r^{-2}u_5 + u_6 + r^{-1}u_7 + r^{-2}u_8}\right)^k, \tag{3.6}$$

$$u_6 = a_{2,0}\left(\frac{1 + r^{-1}u_1 + r^{-2}u_2 + u_3 + r^{-1}u_4 + r^{-2}u_5 + ru_6 + u_7 + r^{-1}u_8}{r + u_1 + r^{-1}u_2 + u_3 + r^{-1}u_4 + r^{-2}u_5 + u_6 + r^{-1}u_7 + r^{-2}u_8}\right)^k, \tag{3.7}$$

$$u_7 = a_{2,1}\left(\frac{r^{-1} + u_1 + r^{-1}u_2 + r^{-1}u_3 + u_4 + r^{-1}u_5 + u_6 + ru_7 + u_8}{r + u_1 + r^{-1}u_2 + u_3 + r^{-1}u_4 + r^{-2}u_5 + u_6 + r^{-1}u_7 + r^{-2}u_8}\right)^k, \tag{3.8}$$

$$u_8 = a_{2,2}\left(\frac{r^{-2} + r^{-1}u_1 + u_2 + r^{-2}u_3 + r^{-1}u_4 + u_5 + r^{-1}u_6 + u_7 + ru_8}{r + u_1 + r^{-1}u_2 + u_3 + r^{-1}u_4 + r^{-2}u_5 + u_6 + r^{-1}u_7 + r^{-2}u_8}\right)^k. \tag{3.9}$$

3.1. Case: $a_{0,1} = a_{0,2} = a_{1,0} = a_{1,1} = a_{1,2} = a_{2,0} = a_{2,1} = a_{2,2} = 1$

Under this assumption, the system (3.2)–(3.9) can be represented as a fixed-point equation $F(t) = t$, where $t = (u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8) \in \mathbb{R}^8_+$ and the operator $F : \mathbb{R}^8_+ \to \mathbb{R}^8_+$ is defined by

$$\begin{cases} u'_1 = \left(\dfrac{1 + ru_1 + u_2 + r^{-1}u_3 + u_4 + r^{-1}u_5 + r^{-1}u_6 + u_7 + r^{-1}u_8}{r + u_1 + r^{-1}u_2 + u_3 + r^{-1}u_4 + r^{-2}u_5 + u_6 + r^{-1}u_7 + r^{-2}u_8}\right)^k, \\ u'_2 = \left(\dfrac{r^{-1} + u_1 + ru_2 + r^{-2}u_3 + r^{-1}u_4 + u_5 + r^{-2}u_6 + r^{-1}u_7 + u_8}{r + u_1 + r^{-1}u_2 + u_3 + r^{-1}u_4 + r^{-2}u_5 + u_6 + r^{-1}u_7 + r^{-2}u_8}\right)^k, \\ u'_3 = \left(\dfrac{1 + r^{-1}u_1 + r^{-2}u_2 + ru_3 + u_4 + r^{-1}u_5 + u_6 + r^{-1}u_7 + r^{-2}u_8}{r + u_1 + r^{-1}u_2 + u_3 + r^{-1}u_4 + r^{-2}u_5 + u_6 + r^{-1}u_7 + r^{-2}u_8}\right)^k, \\ u'_4 = \left(\dfrac{r^{-1} + u_1 + r^{-1}u_2 + u_3 + ru_4 + u_5 + r^{-1}u_6 + u_7 + r^{-1}u_8}{r + u_1 + r^{-1}u_2 + u_3 + r^{-1}u_4 + r^{-2}u_5 + u_6 + r^{-1}u_7 + r^{-2}u_8}\right)^k, \\ u'_5 = \left(\dfrac{r^{-2} + r^{-1}u_1 + u_2 + r^{-1}u_3 + u_4 + ru_5 + r^{-2}u_6 + r^{-1}u_7 + u_8}{r + u_1 + r^{-1}u_2 + u_3 + r^{-1}u_4 + r^{-2}u_5 + u_6 + r^{-1}u_7 + r^{-2}u_8}\right)^k, \\ u'_6 = \left(\dfrac{1 + r^{-1}u_1 + r^{-2}u_2 + u_3 + r^{-1}u_4 + r^{-2}u_5 + ru_6 + u_7 + r^{-1}u_8}{r + u_1 + r^{-1}u_2 + u_3 + r^{-1}u_4 + r^{-2}u_5 + u_6 + r^{-1}u_7 + r^{-2}u_8}\right)^k, \\ u'_7 = \left(\dfrac{r^{-1} + u_1 + r^{-1}u_2 + r^{-1}u_3 + u_4 + r^{-1}u_5 + u_6 + ru_7 + u_8}{r + u_1 + r^{-1}u_2 + u_3 + r^{-1}u_4 + r^{-2}u_5 + u_6 + r^{-1}u_7 + r^{-2}u_8}\right)^k, \\ u'_8 = \left(\dfrac{r^{-2} + r^{-1}u_1 + u_2 + r^{-2}u_3 + r^{-1}u_4 + u_5 + r^{-1}u_6 + u_7 + ru_8}{r + u_1 + r^{-1}u_2 + u_3 + r^{-1}u_4 + r^{-2}u_5 + u_6 + r^{-1}u_7 + r^{-2}u_8}\right)^k. \end{cases} \tag{3.10}$$

Consider the following set

$$I = \left\{ t \in \mathbb{R}^8_+ : u_1 = u_4 = u_7,\ u_2 = u_3 = u_5 = u_6 = u_8 = 1 \right\}.$$

The following statement follows directly from the definition of F.

Lemma 3.1. *The set I is invariant under the operator F, that is, $F(I) \subset I$.*

Next, we determine the fixed points of F restricted to the invariant set I. In this case, the fixed-point equation reduces to

$$u_1 = \left(\frac{2r + r^2u_1}{r^2 + 1 + ru_1}\right)^k. \tag{3.11}$$

The case $k = 1$

For $k = 1$, the system (3.11) takes the form

$$ru_1^2 + u_1 - 2r = 0.$$

Taking into account the condition $u_1 > 0$, we obtain the unique positive solution $u_1 = \frac{-1+\sqrt{8r^2+1}}{2r}$.
Consequently, for $k = 1$ and arbitrary $r > 0$, the system (3.2)–(3.13) admits the unique solution

$$(u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8) = (u_1, 1, 1, u_1, 1, 1, u_1, 1). \tag{3.12}$$

Combining Lemma 3.1 with (3.12), we get

Lemma 3.2. *Let $k = 1$. Then the operator F restricted to the invariant set I has a unique fixed point $(u_1, 1, 1, u_1, 1, 1, u_1, 1)$, for every $r > 0$.*

Applying Theorem 2.1, we obtain the following statement.

Theorem 3.1. *Let $k = 1$ and assume that $p(\epsilon, \xi) = p(0,0)$, $\forall \epsilon, \xi \in \Phi$. Then the Hamiltonian (1.1) admits at least one splitting Gibbs measure for all $r > 0$.*

Remark 3.1. Even in the special case $a_{0,1} = a_{0,2} = a_{1,0} = a_{1,1} = a_{1,2} = a_{2,0} = a_{2,1} = a_{2,2} = 1$, the complete analysis of the system (3.2)–(3.9) outside the invariant set I remains highly nontrivial.

Phase transitions in hidden Markov models correspond to qualitative changes in the behavior of the system caused by variations in model parameters or the structure of hidden states. Such transitions may appear through the loss of uniqueness of Gibbs measures, instability of reconstruction procedures, or the emergence of multiple equilibrium regimes.

Understanding these phenomena is important for identifying parameter regions where the model exhibits stable probabilistic behavior. In the next section, we investigate this issue through the non-uniqueness of Gibbs measures and the existence of multiple translation-invariant states.

The case $k = 2$

For $k = 2$, equation (3.11) reduces to the cubic equation

$$r^2 u_1^3 + \left(2r(r^2+1) - r^4\right)u_1^2 + \left((r^2+1)^2 - 4r^3\right)u_1 - 4r^2 = 0. \tag{3.13}$$

Analyzing equation (3.13), one can determine all its positive solutions. Combining these solutions with the invariance property established in Lemma 3.1, we obtain the following result.

Lemma 3.3. *Let $k = 2$. Then there exists a critical value $r_c \approx 7.0725$ such that the operator F, restricted to the invariant set I, exhibits the following behavior.*

(i) If $0 < r < r_c$, then F has a unique fixed point $\left(u_1^{(1)}(r), 1, 1, u_1^{(1)}(r), 1, 1, u_1^{(1)}(r), 1\right)$.

(ii) If $r = r_c$, then F has exactly two fixed points $\left(u_1^{(1)}(r), 1, 1, u_1^{(1)}(r), 1, 1, u_1^{(1)}(r), 1\right)$ and $\left(u_1^{(2)}(r), 1, 1, u_1^{(2)}(r), 1, 1, u_1^{(2)}(r), 1\right)$.

(iii) If $r > r_c$, then F has three fixed points $\left(u_1^{(1)}(r), 1, 1, u_1^{(1)}(r), 1, 1, u_1^{(1)}(r), 1\right)$, $\left(u_1^{(2)}(r), 1, 1, u_1^{(2)}(r), 1, 1, u_1^{(2)}(r), 1\right)$, and $\left(u_1^{(3)}(r), 1, 1, u_1^{(3)}(r), 1, 1, u_1^{(3)}(r), 1\right)$.

Here the functions $u_1^{(i)}(r)$, $i = 1, 2, 3$, are defined by

$$u_1^{(1)}(r) = \frac{M^2 + 4r(r^6 - 4r^5 + r^4 + 8r^3 + 2r^2 + 1) + 2M(r^3 - 2r^2 - 2)}{6Mr},$$

$$u_1^{(2,3)}(r) = \frac{-M^2 - 4(r^6 - 4r^5 + r^4 + 8r^3 + 2r^2 + 1) + 4M(r^3 - 2r^2 - 2)}{12Mr} \pm \frac{i\sqrt{3}\left(M^2 - 4(r^6 - 4r^5 + r^4 + 8r^3 + 2r^2 + 1)\right)}{12Mr},$$

where

$$M = \left(8r^9 - 48r^8 + 60r^7 + 104r^6 - 168r^5 + 24r^4 + 204r^3 + 24r^2 + 8 + 12r^2\sqrt{-\frac{3\left(r^{11} - 12r^{10} + 36r^9 + 12r^8 - 146r^7 + 180r^5 + 8r^4 - 71r^3 - 4r^2 - 4\right)}{r}}\right)^{1/3}.$$

Remark 3.2. The formulas for $u_1^{(2)}(r)$ and $u_1^{(3)}(r)$ are obtained from Cardano's representation of the roots of the cubic equation (3.13). Although these expressions involve complex quantities, the corresponding roots are real for all $r \geq r_c$. Furthermore, numerical and analytical verification shows that both roots are positive in their domain of existence. Hence, all three solutions $u_1^{(1)}(r)$, $u_1^{(2)}(r)$, and $u_1^{(3)}(r)$ correspond to admissible positive fixed points of the operator F.

Combining Lemmas 3.1 and 3.3 with Theorem 2.1, we arrive at the following classification of splitting Gibbs measures.

Theorem 3.2. *Let $k = 2$ and assume that $p(\epsilon, \xi) = p(0,0)$, $\forall, \epsilon, \xi \in \Phi$. Then the Hamiltonian (1.1) admits*

- *at least one splitting Gibbs measure μ_1 corresponding to the fixed point $\left(u_1^{(1)}(r), 1, 1, u_1^{(1)}(r), 1, 1, u_1^{(1)}(r), 1\right)$, for $0 < r < r_c$;*
- *at least two splitting Gibbs measures μ_1 and μ_2 corresponding to the fixed points $\left(u_1^{(1)}(r), 1, 1, u_1^{(1)}(r), 1, 1, u_1^{(1)}(r), 1\right)$ and $\left(u_1^{(2)}(r), 1, 1, u_1^{(2)}(r), 1, 1, u_1^{(2)}(r), 1\right)$, respectively, when $r = r_c$;*
- *at least three splitting Gibbs measures μ_1, μ_2, and μ_3 corresponding to the fixed points $\left(u_1^{(1)}(r), 1, 1, u_1^{(1)}(r), 1, 1, u_1^{(1)}(r), 1\right)$, $\left(u_1^{(2)}(r), 1, 1, u_1^{(2)}(r), 1, 1, u_1^{(2)}(r), 1\right)$, and $\left(u_1^{(3)}(r), 1, 1, u_1^{(3)}(r), 1, 1, u_1^{(3)}(r), 1\right)$, respectively, for $r > r_c$.*

4. Prediction of Hidden Potts States from Observable SOS Configurations

One of the central inference problems in the proposed two-layer hidden Potts-observed SOS model is the reconstruction of the hidden Potts configuration from the observed SOS layer. More precisely, given an observed configuration $\sigma = (\sigma(x))_{x \in V}$, our objective is to determine the conditional distribution of the hidden configuration $s = (s(x))_{x \in V}$ conditioned on the observed data.

Let $\mu_n(s_n, \sigma_n)$ denote the finite-volume Gibbs distribution corresponding to the Hamiltonian (1.1). The posterior distribution of the hidden Potts configuration is obtained from Bayes' formula:

$$\mu(s_n \mid \sigma_n) = \frac{1}{Z(\sigma_n)} \exp\Bigg(\beta J \sum_{\langle x,y\rangle \in L_n} \delta_{s(x)s(y)} + \sum_{x \in V_n} \beta p(s(x), \sigma(x)) + \sum_{x \in W_n} h_{s(x),\sigma(x),x}\Bigg) = \frac{1}{Z(\sigma_n)} \prod_{\langle x,y\rangle \in L_n} r^{\delta_{s(x)s(y)}} \prod_{x \in V_n} \exp\big(\beta p(s(x), \sigma(x))\big) \prod_{x \in W_n} z_{s(x),\sigma(x),x}, \tag{4.1}$$

where $Z(\sigma_n)$ is the normalization factor dependent on σ_n.

Formula (4.1) shows that, conditioned on the observed SOS configuration, the hidden Potts variables form an effective Gibbs field with modified local interactions determined by the observable layer. Consequently, the observed configuration acts as additional information that influences the probability distribution of hidden states.

Remark 4.1. For translation-invariant Gibbs measures obtained in the previous section, the boundary laws satisfy $z_{\epsilon,\xi,x} = z_{\epsilon,\xi}$, $\epsilon, \xi \in \Phi$, and therefore do not depend on the vertex x. In this case the posterior distribution (4.1) is completely determined by the translation-invariant solution of the corresponding consistency equations.

Recall that $z_{\epsilon,\xi} = \exp\big(h_{\epsilon,\xi}\big)$, $\epsilon, \xi \in \Phi$. Assuming without loss of generality that $h_{0,0} = 0$, we obtain

$$h_{\epsilon,\xi} = \log z_{\epsilon,\xi}, \quad \epsilon, \xi \in \Phi. \tag{4.2}$$

Table 1. Posterior probabilities of hidden Potts states for the observed configuration $\sigma_{\ell_0} = (0,0)$

s_{ℓ_0}	σ_{ℓ_0}	$\mu(s_{\ell_0} \mid \sigma_{\ell_0})$
$(0,0)$	$(0,0)$	$\dfrac{r}{r + a_{1,0} + a_{2,0} + (1 + ra_{1,0} + a_{2,0})a_{1,0}z_{1,0,y} + (1 + a_{1,0} + ra_{2,0})a_{2,0}z_{2,0,y}}$
$(0,1)$	$(0,0)$	$\dfrac{a_{1,0}z_{1,0,y}}{r + a_{1,0} + a_{2,0} + (1 + ra_{1,0} + a_{2,0})a_{1,0}z_{1,0,y} + (1 + a_{1,0} + ra_{2,0})a_{2,0}z_{2,0,y}}$
$(0,2)$	$(0,0)$	$\dfrac{a_{2,0}z_{2,0,y}}{r + a_{1,0} + a_{2,0} + (1 + ra_{1,0} + a_{2,0})a_{1,0}z_{1,0,y} + (1 + a_{1,0} + ra_{2,0})a_{2,0}z_{2,0,y}}$
$(1,0)$	$(0,0)$	$\dfrac{a_{1,0}}{r + a_{1,0} + a_{2,0} + (1 + ra_{1,0} + a_{2,0})a_{1,0}z_{1,0,y} + (1 + a_{1,0} + ra_{2,0})a_{2,0}z_{2,0,y}}$
$(1,1)$	$(0,0)$	$\dfrac{ra_{1,0}^2 z_{1,0,y}}{r + a_{1,0} + a_{2,0} + (1 + ra_{1,0} + a_{2,0})a_{1,0}z_{1,0,y} + (1 + a_{1,0} + ra_{2,0})a_{2,0}z_{2,0,y}}$
$(1,2)$	$(0,0)$	$\dfrac{a_{1,0}a_{2,0}z_{1,0,y}}{r + a_{1,0} + a_{2,0} + (1 + ra_{1,0} + a_{2,0})a_{1,0}z_{1,0,y} + (1 + a_{1,0} + ra_{2,0})a_{2,0}z_{2,0,y}}$
$(2,0)$	$(0,0)$	$\dfrac{a_{2,0}}{r + a_{1,0} + a_{2,0} + (1 + ra_{1,0} + a_{2,0})a_{1,0}z_{1,0,y} + (1 + a_{1,0} + ra_{2,0})a_{2,0}z_{2,0,y}}$
$(2,1)$	$(0,0)$	$\dfrac{a_{1,0}a_{2,0}z_{1,0,y}}{r + a_{1,0} + a_{2,0} + (1 + ra_{1,0} + a_{2,0})a_{1,0}z_{1,0,y} + (1 + a_{1,0} + ra_{2,0})a_{2,0}z_{2,0,y}}$
$(2,2)$	$(0,0)$	$\dfrac{ra_{2,0}^2 z_{2,0,y}}{r + a_{1,0} + a_{2,0} + (1 + ra_{1,0} + a_{2,0})a_{1,0}z_{1,0,y} + (1 + a_{1,0} + ra_{2,0})a_{2,0}z_{2,0,y}}$

The explicit translation-invariant Gibbs measures derived in Theorems 3.1 and 3.2 therefore provide explicit posterior distributions for the hidden Potts states. In particular, for every observed SOS configuration, the conditional probabilities $\mu_n(s_n \mid \sigma_n)$ can be computed directly from the corresponding Gibbs measure.

From the perspective of machine learning, this procedure can be interpreted as a latent-state prediction problem. The hidden Potts configuration represents unobserved classes or latent variables, while the SOS configuration corresponds to observed data. The posterior distribution $\mu_n(s_n \mid \sigma_n)$ therefore provides a probabilistic prediction of hidden states based on the available observations. Furthermore, the existence of multiple translation-invariant Gibbs measures indicates the presence of several competing inference regimes, which may be viewed as an analogue of phase transitions in latent-variable learning systems.

Remark 4.2. Formula (4.1) shows that the conditional distribution $\mu_n(s_n \mid \sigma_n)$ admits a factorized representation in terms of edge and vertex contributions. Consequently, the posterior distribution can be analyzed locally on an arbitrary edge of the Cayley tree.

Let $l_0 = \langle x, y \rangle$ be a fixed edge. For prescribed observable states $\sigma(x)$ and $\sigma(y)$, the posterior probability of the corresponding hidden Potts states is given by $\mu\big((s(x), s(y)) \mid (\sigma(x), \sigma(y))\big)$.

Using the factorized form of (4.1), we obtain

$$\mu(s_{l_0} \mid \sigma_{l_0}) = \frac{r^{\delta_{s(x)s(y)}} \exp(\beta p(s(x), \sigma(x)) \exp(\beta p(s(y), \sigma(y)) z_{s(y),\sigma(y),y}}{\sum_{\epsilon,\xi \in \{0,1,2\}} r^{\delta_{\epsilon\xi}} \exp(\beta p(\epsilon, \sigma(x)) \exp(\beta p(\xi, \sigma(y)) z_{\xi,\sigma(y),y}}.$$

Therefore, the problem of hidden-state prediction can be reduced to the evaluation of posterior probabilities on a single edge (see Table 1). Since the boundary law parameters $z_{\epsilon,\xi}$ are explicitly determined by the translation-invariant Gibbs measures obtained for Cayley trees of orders one and two, the corresponding edge posterior distributions can also be computed explicitly.

5. Conclusion

In this paper, we introduced a two-layer hidden Potts-observed SOS model on the Cayley tree and established its Gibbsian framework. The proposed model combines a hidden three-state Potts configuration with an observable SOS configuration, providing a natural extension of hidden Markov structures to multi-state spin systems with heterogeneous interactions.

For Cayley trees of arbitrary order, we derived the compatibility conditions for finite-volume Gibbs distributions and obtained the corresponding system of consistency equations. These equations provide a complete characterization of splitting Gibbs measures associated with the model.

Particular attention was devoted to translation-invariant Gibbs measures. For a specific class of model parameters, the consistency equations were reduced to a finite-dimensional fixed-point problem. Explicit translation-invariant Gibbs measures were obtained for Cayley trees of orders one and two. Assuming a homogeneous external field,

$$p(\epsilon, \xi) = p(0, 0), \quad \forall\, \epsilon, \xi \in \Phi,$$

we proved that for $k = 1$ the model admits a unique splitting Gibbs measure. For $k = 2$, a critical value r_c was found such that the model has a unique translation-invariant Gibbs measure for $0 < r < r_c$, while multiple translation-invariant Gibbs measures appear for $r \geq r_c$, signaling a phase transition.

We also investigated the inference aspect of the model. By deriving the posterior distribution of hidden Potts states conditioned on the observed SOS configuration, we showed that the hidden-state prediction problem can be reduced to the computation of local posterior probabilities on a single edge of the Cayley tree. The explicit forms of the translation-invariant Gibbs measures obtained in this paper allow these posterior probabilities to be calculated explicitly for Cayley trees of orders one and two.

The results presented here establish a rigorous connection between Gibbs measures, hidden Markov structures, and two-layer spin systems on hierarchical graphs. They provide a mathematical foundation for studying latent-variable models with spatial dependence and may serve as a basis for future investigations of periodic Gibbs measures, extremality properties, reconstruction problems, and more general hidden Potts-SOS models on Cayley trees.

Acknowledgements

The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Funding

There is no funding for this work.

Availability of data and materials

Not applicable.

Competing interests

The authors declare that they have no competing interests.

Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

References

[1] Abdullaev, L. U., Rozikov, U. A.: Gibbs Measures in Machine Learning. World Scientific, Singapore (2026). DOI: 10.1142/14350

[2] Akin, H. and Ulusoy, U.: *A new approach to studying the thermodynamic properties of the q-state Potts model on a Cayley tree.* Chaos, Solitons and Fractals (2023). DOI: 10.1016/j.chaos.2023.113811

[3] Baxter, R. J.: Exactly Solved Models in Statistical Mechanics. Academic Press, London/New York (1982).

[4] Botirov, G. I., Rozikov, U. A.: *On q-component models on the Cayley tree: the general case.* J. Stat. Mech. (2006). DOI: 10.1088/1742-5468/2006/10/P10006

[5] Botirov, G. I., Rozikov, U. A.: *Potts model with competing interactions on the Cayley tree: The contour method.* Theor. Math. Phys. (2007). DOI: 10.1007/s11232-007-0125-x

[6] Ferreira, L. S., Jorge, L. N., DaSilva, C. J., Caparica, A. A.: *An entropic approach to analyze phase transitions in the Potts model.* Physica A: Statistical Mechanics and its Applications. 674, 130694 (2025). DOI: 10.1016/j.physa.2025.130694

[7] Herrera, F., Rozikov, U.A., Velasco, M.V.: *Ising models with hidden Markov structure: applications to probabilistic inference in machine learning.* J. Stat. Mech. (2025). DOI: 10.1088/1742-5468/ade5f8

[8] Kurbanova, D. R., Magomedov, M. A., Dzhamaludinov, M. R., Ramazanov, M. K., Murtazaev, A. K.: *Phase diagram of the antiferromagnetic 3-state Potts model on a bcc lattice with competing interactions.* Physica A: Statistical Mechanics and its Applications. 676, 130870 (2025). DOI: 10.1016/j.physa.2025.130870

[9] Minlos, R. A.: Introduction to mathematical statistical physics. University lecture series. 1 (2000).

[10] Mukhamedov, F., Akin, H.: *Exact solution for the three-state asymmetric Potts model on a Cayley tree.* Chaos, Solitons and Fractals (2026). DOI: 10.1016/j.chaos.2026.118118

[11] Ostilli, M.: *Cayley Trees and Bethe Lattices: A concise analysis for mathematicians and physicists.* Physica A: Statistical Mechanics and its Applications. 391(12), 3417–3423 (2012). DOI: 10.1016/j.physa.2012.01.038

[12] Pirogov, S. A., Sinai, Ya. G.: Phase diagrams of classical lattice systems. Theoretical and Mathematical Physics (1975). DOI: 10.1007/BF01040127

[13] Preston, C. J.: Gibbs states on countable sets. Cambridge University Press, London (2011). DOI: 10.1017/CBO9780511897122

[14] Rahmatullaev, M. M., Rasulova, M. A.: *Extremality of translation-invariant Gibbs measures for the Potts-SOS model on the Cayley tree.* J. Stat. Mech. (2021). DOI: 10.1088/1742-5468/ac08ff

[15] Rahmatullaev, M. M., Rasulova, M. A.: *Ground States and Gibbs Measures for the Potts-SOS Model with an External Field on the Cayley Tree.* Lobachevskii Journal of Mathematics (2024). DOI: 10.1134/S1995080224010451

[16] Rahmatullaev, M. M., Rasulova, M. A., *Potts models featuring hidden Markov structure.* Available at SSRN: https://ssrn.com/abstract=6819388 or http://dx.doi.org/10.2139/ssrn.6819388

[17] Rasulova, M. A.: *Ground states for the Potts model with an external field.* Reports on Mathematical Physics (2024). DOI: 10.1016/S0034-4877(24)00082-X

[18] Rasulova, M. A.: *On the Relationship between Configurations and Limiting Gibbs Measures for the Potts-SOS Model on the Cayley Tree.* Journal of Mathematical Sciences (2026). DOI: 10.1007/s10958-026-08419-x

[19] Rozikov, U.A.: *A Contour method on Cayley trees.* J. Stat. Phys. (2008). DOI: 10.1007/s10955-007-9455-1

[20] Rozikov, U. A.: Gibbs measures on Cayley trees. World Scientific, Singapore (2013). DOI: 10.1142/8841

[21] Rozikov, U. A.: Gibbs Measures in Biology and Physics: The Potts Model. World Scientific, Singapore (2023). DOI: 10.1142/12694

[22] Rozikov, U. A., Rakhmatullaev, M. M., Khakimov, R. M.: *Periodic Gibbs measures for the Potts model in translation-invariant and periodic external fields on the Cayley tree.* Theoretical and Mathematical Physics (2022). DOI: 10.1134/S004057792201010X

[23] Sinai, Ya. G.: Theory of phase transitions: Rigorous Results. Pergamon, Oxford (1982).

Affiliations

Rahmatullaev M.M.
Address: V.I.Romanovskiy Institute of Mathematics, Dept. of Namangan, 100174, Tashkent-Uzbekistan.
e-mail: mrahmatullaev@rambler.ru
ORCID ID: https://orcid.org/0000-0003-2987-7714

Rasulova M.A.
Address: V.I.Romanovskiy Institute of Mathematics, Dept. of Namangan, 100174, Tashkent-Uzbekistan.
e-mail: m_rasulova_a@rambler.ru
ORCID ID: https://orcid.org/0000-0002-2905-7591

Uzbekistan Journal of Mathematics and Computer Science
Volume-2, Issue-3-special, Page 60–70 (2026)
DOI: https://doi.org/10.56143/ujmcs.v2i3s.6
Research article

Inverse Scattering Transform for the Fifth-Order Fractional KdV Equation with a Self-Consistent Source

A.B. Babajonov

Abstract

In this paper, we constract and integrate the fifth-order Riesz fractional KdV (fRfKdV) equation with a self-consistent source in the class of rapidly decreasing functions via the inverse scattering transform(IST) method. We show that fifth-order Riesz fractional KdV equation with self-consistent sources is also an important theoretical model, since it is a completely integrable system. For one soliton cases, explicit formulas for the solutions of the problem under consideration are derived as examples to simulate their spatial structures and analyze their structural properties by selecting different values of fractional orders. The results indicate that both the source and the fractional order significantly influence the velocity of soliton propagation, while no energy dissipation occurs through out the entire motion process. This behavior corresponds to a super-dispersive transport mechanism, which is an experimentally testable prediction of the present theory. Furthermore, the analysis reveals that the absolute value of wave velocity becomes larger as ϵ increases. This study contributes valuable insights into nonlinear dynamics and soliton behavior, improving the understanding of multifaceted wave interactions in nonlinear fractional equations.

Keywords: *Riesz fractional Korteweg-de Vries equation, self-consistent source, inverse scattering transform method, Sturm-Liouville operator.*

AMS Subject Classification (2020): *Primary: 37K10 ; Secondary: 37K15; 35R11.*

1. Introduction

The fifth-order KdV equation, sometimes referred to as a Kawahara-type equation, arises as a model for plasma waves, capillary–gravity waves, and various other dispersive phenomena [1]-[8].

Our goal is to constract and integrate the fifth-order fractional KdV equation with a self-consistent source in the class of rapidly decreasing functions via the inverse scattering transform (IST) method.

The fractional equations often predict physically measurable quantities that follow power laws. For example, in anomalous diffusion, the mean squared displacement is related to time by a power law $t^{\alpha}, \alpha > 0$[9, 10, 11, 12]. Similarly, for integrable soliton equations, fractional generalizations predict anomalous dispersion where the speed of localized solitonic waves are related to their amplitude by a power law[13].

In 2022, Ablowitz, Been, and Carr established for the first time that the Riesz fractional KdV equation[14]

$$q_t + \gamma(L^A)q_x = 0$$

Received : *28–april–2026,* ***Accepted :*** *31–july–2026*
*** Corresponding author***

where

$$\gamma(L^A) = -4L^A|4L^A|^{\varepsilon}, \qquad L^A = -\frac{1}{4}\partial_x^2 - q + \frac{1}{2}q_x \int_x^{\infty} dy$$

admits an inverse scattering formulation within the class of rapidly decaying functions. Following the work of Ablowitz, Been, and Carr, several nonlinear evolution equations involving the Riesz fractional derivative were also integrated by means of the direct and inverse scattering problem methods of the scattering theory [15, 17, 18, 19, 20, 21, 22].

In this paper, we study the integration of fifth-order fractional KdV equation with a self-consistent source by the inverse scattering method. We show that fRfKdV equation with a self-consistent source is also an important theoretical model, since it is a completely integrable system. For the one soliton case, explicit formulas for the solutions of the problem under consideration are derived as examples to simulate their spatial structures and analyze their structural properties by selecting different values of fractional orders.

The concept of self-consistent sources in integrable nonlinear evolution equations was first introduced by V. K. Melnikov, who showed that such equations can retain their integrability while describing richer physical and mathematical structures[23]. Physically, the sources appear in solitary waves with non-constant velocity and lead to a variety of dynamics of physical models. They have important applications in plasma physics, hydrodynamics, solid-state physics, etc.[24, 25, 26, 27, 28]. Recently, numerous studies have been devoted to the investigation of nonlinear evolution equations with a sources, motivated by both physical and mathematical considerations in[29]-[47].

2. Statement of the Problem

In the present paper, we consider the following fifth-order Riesz fractional Korteweg-de Vries equation with a self-consistent source

$$\begin{cases} u_t + \Omega(L^*)u_x = 4\sum\limits_{m=1}^{N} \dfrac{\partial}{\partial x}\left(|\varphi_m|^2\right), \\ -\varphi_m'' + u(x,t)\varphi_m = \lambda_m \varphi_m, \quad m = 1, ..., N \end{cases} \tag{2.1}$$

under initial condition

$$u(x,0) = u_0(x), \qquad x \in \mathbb{R}. \tag{2.2}$$

Here, the operator $\Omega(L^*)$:

$$\Omega(L^*) = 16(L^*)^2|4L^*|^{\varepsilon}, \qquad L^* = -\frac{1}{4}\partial_x^2 - u + \frac{1}{2}u_x \int_x^{\infty} dy \tag{2.3}$$

is constructed from the anomalous dispersion relation

$$\Omega(k^2) = 16k^4|2k|^{2\epsilon}, \tag{2.4}$$

of the linearization part of Equation (2.1)

$$u_t + |-\partial_x^2|^{\varepsilon} u_{xxxxx} = 0,$$

replacing the spectral variable k^2 by the operator L^* in (2.4) yields precisely the definition of $\Omega(L^*)$ introduced in (2.3). Here $|-\partial_x^2|^{\epsilon}$, $\epsilon \in (0,1)$ is the Riesz fractional derivative [48] defined as

$$\left(-\partial_x^2\right)^{\epsilon} u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{u}(k,t)|k|^{2\epsilon} e^{ikx}\, dk,$$

$$\hat{u}(k,t) = \int_{-\infty}^{\infty} u(x,t)e^{-ikx}dx.$$

The explicit form of the fractional power $|4L^*|^{\varepsilon}$, $\varepsilon \in (0,1)$ will be given in section 5.

In (2.1), the functions $\varphi_m = \varphi_m(x,t)$, $m = 1,...,N$, are eigenfunctions of the time-dependent operator

$$\mathcal{L}(t) := -\frac{d^2}{dx^2} + u(x,t), \quad x \in \mathbb{R}, \ t > 0,$$

associated with the eigenvalues $\lambda_m(t) = -\nu_m^2(t)$, $m = 1,...,N$, which satisfy the following normalization condition

$$\int_{-\infty}^{\infty} |\varphi_m^2(x,t)|\,dx = \beta_m(t), \qquad m = 1,...,N$$

where $\beta_m(t)$ are given positive continuous functions.

The real-valued initial function $u_0(x)$ is assumed to satisfy:

1. $$\int_{-\infty}^{\infty} (1+|x|)|u_0(x)|\,dx < \infty;$$

2. The operator
$$\mathcal{L}(0) := -\frac{d^2}{dx^2} + u_0(x), \quad x \in \mathbb{R},$$
has exactly N simple negative eigenvalues $\lambda_j(0) < 0$, $j = 1,\dots,N$.

In the problem (2.1)-(2.2), the functions $u(x,t)$ and $\varphi_m(x,t)$, $m = 1,...,N$, are unknown.

The main aim of this work is to derive representations for the solutions

$$\{u(x,t),\ \varphi_1(x,t),\ \varphi_2(x,t),...,\varphi_N(x,t)\}$$

of the problem (2.1)-(2.2) with the use of the inverse scattering method for the operator $\mathcal{L}(t)$.

Note. In (2.3), if we set $\epsilon = 0$, then we obtain the classical fifth-order KdV equation with a self-consistent source:

$$u_t + u_{xxxxx} + 10uu_{xxx} + 20u_xu_{xx} + 30u^2u_x = G(x,t),$$

where $G(x,t) = 4\sum_{m=1}^{N} \frac{\partial}{\partial x}\left(|\varphi_m|^2\right)$.

3. Preliminaries

In this section, we give the basic facts about the scattering theory of the Sturm-Liouville operator $\mathcal{L}(t)$ [49, 50].

We consider the Sturm-Liouville scattering problem

$$\mathcal{L}(t)y \equiv -y'' + u(x,t)y = k^2y, \qquad |x| < \infty, \ \forall t \geq 0 \tag{3.1}$$

where k^2 is the spectral parameter and $u(x,t)$ is real function satisfying condition

$$\int_{-\infty}^{\infty} (1+|x|)|u(x,t)|\,dx < \infty, \ \forall t \geq 0. \tag{3.2}$$

We denote by $f = f(x,k,t)$ and $g = g(x,k,t)$ the Jost solutions of (3.1) with the asymptotics

$$f(x,k,t) = e^{ikx} + o(1), \quad x \to \infty; \qquad g(x,k,t) = e^{-ikx} + o(1), \quad x \to -\infty, \quad (\operatorname{Im} k = 0). \tag{3.3}$$

Under condition (3.2), such solutions exist and are uniquely determined by asymptotics (3.3). The solutions $f(x,k,t)$ and $g(x,k,t)$ satisfy the representations

$$f(x,k,t) = e^{ikx} + \int_x^\infty K^+(x,s;t)e^{iks}\,ds, \qquad g(x,k,t) = e^{-ikx} + \int_{-\infty}^x K^-(x,s;t)e^{-iks}\,ds, \tag{3.4}$$

where kernels $K^+(x,s;t)$ and $K^-(x,s;t)$ are real functions related to the potential $u(x,t)$ by the relations

$$u(x,t) = \mp 2\frac{d}{dx}K^\pm(x,x;t).$$

With respect to the variable k, the Jost solutions $f(x,k,t)$ and $g(x,k,t)$ continue analytically into the upper half-plane $\operatorname{Im} k > 0$.

For real k, pairs of functions $\{f(x,k,t), f(x,-k,t)\}$ and $\{g(x,k,t), g(x,-k,t)\}$ are pairs of linearly independent solutions of equation (3.1), therefore

$$f(x,k,t) = b(k,t)g(x,k,t) + a(k,t)g(x,-k,t),$$

$$g(x,k,t) = -b(-k,t)f(x,k,t) + a(k,t)f(x,-k,t).$$

The functions

$$r^+(k,t) = -\frac{b(-k,t)}{a(k,t)}, \qquad r^-(k,t) = \frac{b(k,t)}{a(k,t)}$$

are called reflection coefficients (right and left, respectively). The coefficients $a = a(k,t)$ and $b = b(k,t)$ have the following properties (see [49]).

A. For real $k \neq 0$

$$a(-k,t) = \overline{a(k,t)}, \quad b(-k,t) = \overline{b(k,t)}, \quad |a(k,t)|^2 = 1 + |b(k,t)|^2;$$

$$a(k,t) = -\frac{1}{2ik}W\{f(x,k,t), g(x,k,t)\}, \qquad b(k,t) = \frac{1}{2ik}W\{f(x,k,t), g(x,-k,t)\};$$

$$a(k,t) = 1 + O\left(\frac{1}{k}\right), \qquad b(k,t) = O\left(\frac{1}{k}\right), \qquad k \to \pm\infty;$$

where

$$W\{f(x,k,t), g(x,k,t)\} \equiv f(x,k,t)g'(x,k,t) - f'(x,k,t)g(x,k,t);$$

B. With respect to the variable k, the function $a(k,t)$ extends analytically to the half-plane $\operatorname{Im} k > 0$ and there it has a finite number of zeros $k_n(t) = i\nu_n(t)$, $(\nu_n(t) > 0,\ \forall t \geq 0)$, $n = 1,2,\ldots,N$, these zeros are simple, and $\lambda_n(t) = -\nu_n^2(t)$ is an eigenvalue of the operator $\mathcal{L}(t)$. In addition, the following relation holds

$$g(x,k_n(t),t) = \chi_n(t)f(x,k_n(t),t), \qquad n = 1,2,\ldots,N. \tag{3.5}$$

The set

$$\{r^+(k,t), k \in \mathbb{R};\ \nu_1(t), \nu_2(t), \ldots, \nu_N(t);\ \chi_1(t), \chi_2(t), \ldots, \chi_N(t)\} \tag{3.6}$$

is called the scattering data for problem (3.1). The direct scattering problem is to determine the scattering data from the potential $u(x,t)$, and the inverse problem is to reconstruct from the scattering data the potential

$u(x,t)$ of Eq.(3.1). The kernel $K^+(x,y)$ in representation (3.4) is a solution to the Gelfand-Levitan-Marchenko (GLM) integral equation

$$K^+(x,y;t) + F^+(x+y;t) + \int_x^\infty K^+(x,z;t)F^+(z+y;t)\,dz = 0, \qquad (y > x), \tag{3.7}$$

where

$$F^+(x,t) = \sum_{j=1}^{N} \alpha_j^+(t)e^{-\nu_j(t)x} + \frac{1}{2\pi}\int_{-\infty}^{\infty} r^+(k,t)e^{ikx}\,dk, \tag{3.8}$$

$$\alpha_j^+(t) = \frac{\chi_j(t)}{i\,\dfrac{da(z,t)}{dz}\Big|_{z=i\nu_j(t)}}, \tag{3.9}$$

and $a(z,t)$ is the analytic continuation of the function $a(k,t)$ to the upper half-plane $\operatorname{Im} k > 0$.

Finally, the potential is reconstructed from the GLM kernel via

$$u(x,t) = -2\frac{\partial}{\partial x}K^+(x,x;t). \tag{3.10}$$

It is worthy to remark that the functions

$$h_n(x,t) = \left(\frac{da(k,t)}{dk}\Big|_{k=i\nu_n(t)}\right)^{-1} \frac{d}{dk}[g(x,k;t) - \chi_n(t)f(x,k;t)]\Big|_{k=i\nu_n(t)}, \quad n = 1,2,...,N \tag{3.11}$$

are solutions of the equations (3.1). For $\operatorname{Im} k > 0$, using (3.3), we obtain the following asymptotics

$$h_n(x,t) \to e^{\nu_n(t)x}, \quad x \to +\infty, \tag{3.12}$$

$$h_n(x,t) \to -\chi_n(t)e^{-\nu_n(t)x}, \quad x \to -\infty. \tag{3.13}$$

From asymptotics (3.3), (3.12) and (3.13) we have

$$W\{h_n(x,t),\ f(x,k_n(t);t)\} = -2\nu_n(t),$$
$$W\{h_n(x,t),\ g(x,k_n(t);t)\} = -2\nu_n(t)\chi_n(t). \tag{3.14}$$

4. The time dependence of the scattering data

Following the procedure in cf. Ref. [14] the time dependence of the scattering data is given by

$$\frac{dr^+(k,t)}{dt} = 32ik^5|4k^2|^\epsilon r^+(k,t), \tag{4.1}$$

$$\frac{d\chi_n(t)}{dt} = \left(32ik_n^5|4k_n^2|^\epsilon + 2\beta_n(t)\right)\chi_n(t), \tag{4.2}$$

$$\frac{d\nu_n(t)}{dt} = 0, \qquad n = 1,\dots,N. \tag{4.3}$$

Remark 1. Taking into account (3.9) and (4.2), it is easy to show that

$$\frac{d\alpha_n^+(t)}{dt} = \left(32ik_n^5|4k_n^2|^\epsilon + 2\beta_n(t)\right)\alpha_n^+(t).$$

Remark 2. The obtained results completely determine the time evolution of scattering data, which allows solving problem (2.1)–(2.2) by the following algorithm. Assume that $u_0(x)$ is given.

1. For given functions $u_0(x)$, find the scattering data

$$\{r^+(k,0), k \in \mathbb{R};\ \nu_1(0), \nu_2(0), \ldots, \nu_N(0);\ \chi_1(0), \chi_2(0), \ldots, \chi_N(0)\}$$

for $\mathcal{L}(0)$.

2. From (4.1)–(4.3), obtain the time evolution of the scattering data

$$\{r^+(k,t), k \in \mathbb{R};\ \nu_1(t), \nu_2(t), \ldots, \nu_N(t);\ \chi_1(t), \chi_2(t), \ldots, \chi_N(t)\}$$

for $\mathcal{L}(t)$.

3. With this scattering data, uniquely determine the function $F^+(x,t)$ from relation (3.8).

4. Substituting $F^+(x,t)$ in Gelfand-Levitan-Marchenko integral equations (3.7) and solving this system, obtain $K^+(x,y,t)$.

5. Further, from (3.10) find the solution $u(x,t)$.

5. Explicit Form of the fKdV equation with a self-consistent source

As shown in [51], the eigenfunctions of L^* are any of the three functions: $\{\partial_x g^2, \partial_x f^2, \partial_x(gf)\}$ which can be represent generically as Ψ^* with eigenvalue $\lambda = k^2$. Here g and f are Jost solutions of (3.1). Then, we have

$$\Omega(L^*)\Psi^* = \Omega(k^2)\Psi^*.$$

The set of eigenfunctions Ψ^* is complete [52]. Hence, any function $h(x,t)$ that is sufficiently regular with respect to x admits the expansion

$$h(x,t) = \int\limits_{\Gamma_\infty} \frac{\tau^2(k,t)}{4\pi i k} dk \int\limits_{-\infty}^{\infty} \mathcal{G}(x,y,k;t) h(y,t) dy,$$

where $\Gamma_\infty = \lim_{R\to\infty} \Gamma_R$ with Γ_R the semicircular contour in the upper half plane evaluated from $k = -R$ to $k = R$. τ is the transmission coefficient defined by the relation

$$f(x,k;t)\tau(k,t) = g(x,-k;t) + r^+(k,t)g(x,k;t),$$

and

$$\mathcal{G}(x,y,k;t) = \partial_x\big(f^2(x,k;t)g^2(y,k;t) - g^2(x,k;t)f^2(y,k;t)\big). \tag{14}$$

The operator $\Omega(L^*)$ acting on an arbitrary function $h(x,t)$ may be represented by the spectral expansion

$$\Omega(L^*)h(x,t) = \int\limits_{\Gamma_\infty} \Omega(k^2) \frac{\tau^2(k,t)}{4\pi i k} dk \int\limits_{-\infty}^{\infty} \mathcal{G}(x,y,k;t) h(y,t) dy. \tag{5.1}$$

Hence, equation (5.1) provides an explicit representation of fKdV with self-consistent source, i.e. equation (5.1) with $h(x,t) = u_x(x,t)$ and $\Omega(L^*) = 16(L^*)^2|4L^*|^\varepsilon$ which may be written as

$$u_t + \int\limits_{\Gamma_\infty} |4k^2|^\epsilon \frac{\tau^2(k,t)}{4\pi i k} dk \int\limits_{-\infty}^{\infty} \mathcal{G}(x,y,k;t)\left(u_{yyyyy} + 10uu_{yyy} + 20u_y u_{yy} + 30u^2 u_y\right) dy = G(x,t). \tag{5.2}$$

If we set $\epsilon = 0$ in (5.2) then we recover fifth-order KdV equation with a self-consistent source:

$$u_t + u_{xxxxx} + 10uu_{xxx} + 20u_xu_{xx} + 30u^2u_x = G(x,t).$$

The expression (5.1) can be evaluated using contour integration to give a representation of $\Omega(L^*)$ in terms of integrals along the real line and a sum over discrete values along the imaginary axis:

$$\Omega(L^*)u_x(x,t) = \iint_{\mathbb{R}^2} \Omega\left(k^2\right) \frac{\tau^2(k,t)}{4\pi ik} \mathcal{G}_c(x,y,k;t)\, u_y(y,t)\, dy\, dk + \sum_{j=1}^{N} \Omega(-\nu_j^2) \int_{-\infty}^{\infty} \mathcal{G}_{d,j}(x,y,t) u_y(y,t) dy. \tag{5.3}$$

Here the continuous contribution $\mathcal{G}_c$ is defined by

$$\mathcal{G}_c(x,y,k;t) = \partial_x\left(f^2(x,k;t)g^2(y,k;t) - g^2(x,k;t)f^2(y,k;t)\right)$$

and the discrete contribution, which comes from the poles of $\tau(k,t)$ at $k_j(t) = i\nu_j(t)$, $j = 1,2,\ldots,N$, is given by

$$\mathcal{G}_{d,j}(x,y,t) = i\eta_j(t)\frac{\partial}{\partial x}\left[f^2(y,k;t)f(x,k;t)\partial_k g(x,k;t) - f^2(y,k;t)g(x,k;t)\partial_k f(x,k;t)\right]_{k=k_j(t)} -$$

$$-i\eta_j(t)\frac{\partial}{\partial x}\left[f^2(x,k;t)g(y,k;t)\partial_k f(y,k;t) - f^2(x,k;t)g(y,k;t)\partial_k f(y,k;t)\right]_{k=k_j(t)}, \tag{5.4}$$

where

$$\eta_j(t) = \frac{\chi_j(t)}{\nu_j(t)(\partial_k a(k_j(t)))^2}. \tag{5.5}$$

6. Example

We consider the problem (2.1)–(2.2) for the one soliton initial condition

$$u(x,0) = -\frac{2}{ch^2x},$$

with $\beta_1(t) = 5t^4$. In this case, $G(x,t) = 4\frac{\partial}{\partial x}\left(\varphi_1^2(x,t)\right)$ and the solution has the form

$$u(x,t) = -\frac{2}{ch^2\left(x - 4^{2+\varepsilon}t - t^5\right)}. \tag{6.1}$$

7. Graphical representation of soliton solution of the equation (2.1)

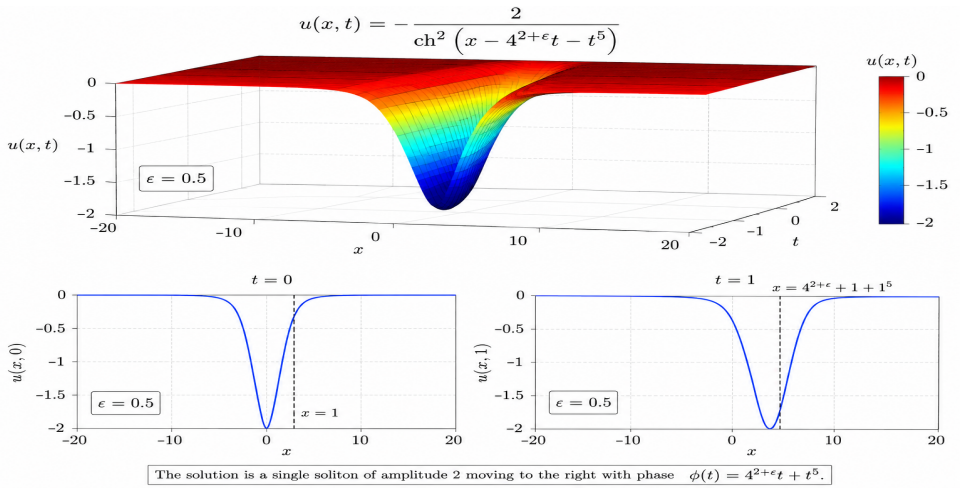


Figure 1. One soliton solution of the fRfKdV equation with a self-consistent source for $\varepsilon = \frac{1}{2}$, $\beta_1(t) = 5t^4$.

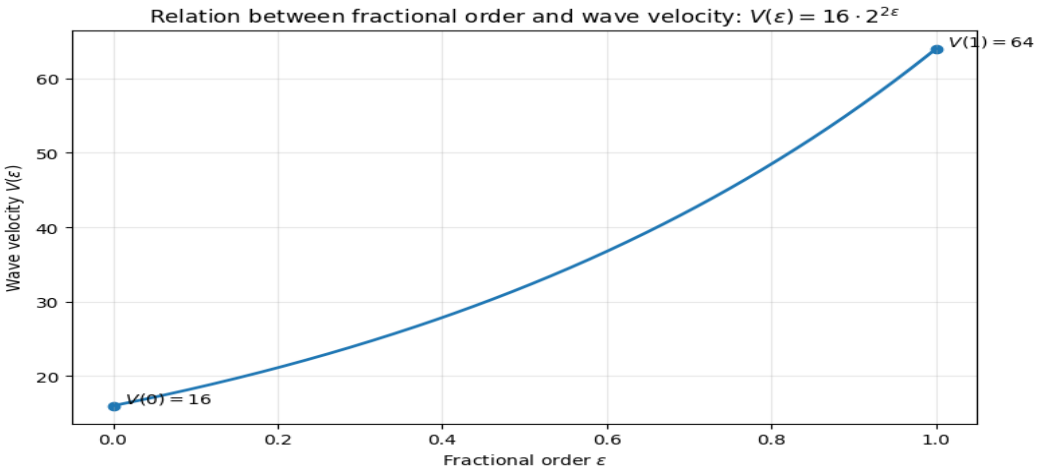


Figure 2. Relationship between the fractional derivative order $\varepsilon \in (0, 1)$ and the wave velocity $v(\varepsilon, t) = 4^{2+\varepsilon} + 5t^4$ for the one soliton case.

Figure 2 indicates that the soliton velocity consists of a fractional-order-dependent constant term and a time-dependent contribution. The analysis shows that as the fractional derivative order ϵ increases, the absolute value of the wave velocity also increases. From a physical point of view, this result implies that the fractional derivative enhances the effective propagation speed of the soliton. Moreover, the presence of the time-dependent term $5t^4$ leads to a non-uniform motion of the soliton, meaning that the soliton accelerates as time evolves. This behavior reflects the influence of memory effects and nonlocal properties introduced by the fractional derivative, as well as the contribution of the self-consistent source to the soliton dynamics.

8. Conclusion

The ITS method has been successfully used to obtain the soliton solutions of the fifth-order Riesz fractional KdV equation with a self-consistent source. We have determined the time evolution of the scattering data

for the Sturm-Liouville problem associated with the solution of the fifth-order Riesz fractional KdV equation with a self-consistent source. Then, using the solution of the inverse scattering problem with respect to the time-dependent scattering data, we recover the desired solution of the fifth-order Riesz fractional KdV equation with a self-consistent source. For the one soliton case, explicit formulas for the solution of the problem under consideration are obtained. Finally, we showed that the one-soliton solutions of the fifth-order Riesz fractional KdV equation with a self-consistent source have power law relationships between the solitons amplitude and velocity. The self-consistent source further modifies the energy balance of the system by continuously interacting with the wave field, thereby accelerating the soliton dynamics.

theorem

References

[1] Drazin, P.G.; Johnson, R.S. *Solitons: An Introduction*; Cambridge University Press: New York, NY, USA, 1989.

[2] Kakutani, T.; Ono, H. *Weak non-linear hydromagnetic waves in a cold collision free plasma.* J. Phys. Soc. Jpn. **26**, 1305–1318 (1969). doi:10.1143/JPSJ.26.1305

[3] Iqbal, N.; Akgul, A.; Shah, R.; Bariq, A.; Mossa Al-Sawalha, M.; Ali, A. *On Solutions of Fractional-Order Gas Dynamics Equation by Effective Techniques.* J. Funct. Spaces. 1–14 (2022). doi:10.1155/2022/2147805

[4] El-Tantawy, S.A.; Salas Alvaro, H.; Alharthi, M.R. *Novel analytical cnoidal and solitary wave solutions of the Extended Kawahara equation.* Chaos Solitons Fractals, **147**, 110965 (2021). doi:10.1016/j.chaos.2021.110965

[5] El-Tantawy, S.A.; Salas Alvaro, H.; Alharthi, M.R. *On the dissipative extended Kawahara solitons and cnoidal waves in a collisional plasma: Novel analytical and numerical solutions.* Phys. Fluids, **33**, 106101 (2021). doi:10.1063/5.0065659

[6] Wazwaz, A.-M. *New solitary wave solutions to the Kuramoto–Sivashinsky and the Kawahara equations.* Appl. Math. Comput. **182**, 1642–1650 (2006). doi:10.1016/j.amc.2006.05.002

[7] Wazwaz, A.-M. *Two-mode fifth-order KdV equations: Necessary conditions for multiple-soliton solutions to exist.* Nonlinear Dyn. **87**, 1685–1691 (2017). doi:10.1007/s11071-016-3174-z

[8] Goswami, A.; Singh, J.; Kumar, D. *Numerical simulation of fifth order KdV equations occurring in magneto-acoustic waves.* Ain Shams Eng. J. **9**, 2265–2273 (2018). doi:10.1016/j.asej.2017.03.004

[9] E. Koscielny-Bunde, A. Bunde, S. Havlin, H. E. Roman, Y. Goldreich, and H. Schellnhuber, *Indication of a universal persistence law governing atmospheric variability,* Physical Review Letters, **81**, 729–732 (1998). https://doi.org/10.1103/physrevlett.81.729

[10] M. F. Shlesinger, B. J. West, and J. Klafter, *Levy dynamics of enhanced diffusion: application to turbulence,* Physical Review Letters, **58**, 1100–1103 (1987). https://doi.org/10.1103/physrevlett.58.1100

[11] W. Wang, A. G. Cherstvy, A. V. Chechkin, S. Thapa, F. Seno, X. Liu, and R. Metzler, *Fractional Brownian motion with random diffusivity: emerging residual nonergodicity below the correlation time,* Journal of Physics A: Mathematical and Theoretical, **53**, 474001 (2020). https://doi.org/10.1088/1751-8121/aba467

[12] R. Metzler and J. Klafter, *The random walk's guide to anomalous diffusion: a fractional dynamics approach,* Physics Reports, **339**, 1–77 (2000). https://doi.org/10.1016/s0370-1573(00)00070-3

[13] B. J. West, P. Grigolini, R. Metzler, and T. F. Nonnenmacher, *Fractional diffusion and Levy stable processes,* Physical Review E, **55**, 99–106 (1997). https://doi.org/10.1103/physreve.55.99

[14] M. J. Ablowitz, J. B. Been, and L. D. Carr, *Fractional Integrable Nonlinear Soliton Equations,* Physical Review Letters, **128**, 184101 (2022). https://doi.org/10.1103/PhysRevLett.128.184101

[15] M. J. Ablowitz, J. B. Been, and L. D. Carr, *Fractional integrable and related discrete nonlinear Schrodinger equations,* Physics Letters A, **452**, 128459 (2022). https://doi.org/10.1016/j.physleta.2022.128459

[16] M. J. Ablowitz, J. B. Been, and L. D. Carr, *Integrable fractional modified Korteweg–de Vries, sine-Gordon, and sinh-Gordon equations,* Journal of Physics A: Mathematical and Theoretical, **55**, 384010 (2022). https://doi.org/10.1088/1751-8121/ac8844

[17] M. Zhang, W. Weng, and Z. Yan, *Interactions of fractional N-solitons with anomalous dispersions for the integrable combined fractional higher-order mKdV hierarchy,* Physica D: Nonlinear Phenomena, **444**, 133614 (2022). https://doi.org/10.1016/j.physd.2022.133614

[18] W. Weng, M. Zhang, G. Zhang, and Z. Yan, *Dynamics of fractional N-soliton solutions with anomalous dispersions of integrable fractional higher-order nonlinear Schrodinger equations,* Chaos: An Interdisciplinary Journal of Nonlinear Science, **32**, 123110 (2022). https://doi.org/10.1063/5.0101921

[19] L. An, L. Ling, and X. Zhang, *Inverse scattering transform for the integrable fractional derivative nonlinear Schrodinger equation,* Physica D: Nonlinear Phenomena, **458**, 133888 (2023). https://doi.org/10.1016/j.physd.2023.133888

[20] L. An, L. Ling, and X. Zhang, *Nondegenerate solitons in the integrable fractional coupled Hirota equation,* Physics Letters A, **460**, 128629 (2023). https://doi.org/10.1016/j.physleta.2023.128629

[21] S. Zhang, H. Li, and B. Xu, *Inverse scattering integrability and fractional soliton solutions of a variable-coefficient fractional-order KdV-type equation,* Fractal and Fractional, **8**, 520 (2024). https://doi.org/10.3390/fractalfract8090520

[22] B. Babajanov and F. Abdikarimov, *Integration of the fractional modified Korteweg–de Vries–sine-Gordon equation by the inverse scattering method,* Results in Applied Mathematics, **26**, 100586 (2025). https://doi.org/10.1016/j.rinam.2025.100586

[23] V. Melnikov, *Exact solutions of the Korteweg–de Vries equation with a self-consistent source,* Physics Letters A, **128**, 488–492 (1988). https://doi.org/10.1016/0375-9601(88)90881-x

[24] V. Melnikov, *Integration method of the Korteweg–de Vries equation with a self-consistent source,* Physics Letters A, **133**, 493–496 (1988). https://doi.org/10.1016/0375-9601(88)90522-1
[25] V. K. Melnikov, *Integration of the Korteweg–de Vries equation with a source,* Inverse Problems, **6**, 233–246 (1990). https://doi.org/10.1088/0266-5611/6/2/007
[26] V. K. Melnikov, *Integration of the nonlinear Schroedinger equation with a self-consistent source,* Communications in Mathematical Physics, **137**, 359–381 (1991). https://doi.org/10.1007/bf02431884
[27] J. Leon and A. Latifi, *Solution of an initial-boundary value problem for coupled nonlinear waves,* Journal of Physics A: Mathematical and General, **23**, 1385–1403 (1990). https://doi.org/10.1088/0305-4470/23/8/013
[28] C. Claude, A. Latifi, and J. Leon, *Nonlinear resonant scattering and plasma instability: an integrable model,* Journal of Mathematical Physics, **32**, 3321–3330 (1991). https://doi.org/10.1063/1.529443
[29] D. Zhang and D. Chen, *The N-soliton solutions of the sine-Gordon equation with self-consistent sources,* Physica A: Statistical Mechanics and Its Applications, **321**, 467–481 (2003). https://doi.org/10.1016/s0378-4371(02)01742-9
[30] Y. Zeng, Y. Shao, and W. Xue, *Negaton and positon solutions of the soliton equation with self-consistent sources,* Journal of Physics A: Mathematical and General, **36**, 5035–5043 (2003). https://doi.org/10.1088/0305-4470/36/18/308
[31] P. G. Grinevich and I. A. Taimanov, *Spectral conservation laws for periodic nonlinear equations of the Melnikov type,* Translations of the American Mathematical Society, **224**, 125–138 (2008). https://doi.org/10.1090/trans2/224/05
[32] B. Babajanov and F. Abdikarimov, *New exact soliton and periodic wave solutions of the nonlinear fractional evolution equations with additional term,* Partial Differential Equations in Applied Mathematics, **8**, 100567 (2023). https://doi.org/10.1016/j.padiff.2023.100567
[33] B. A. Babajanov, A. K. Babadjanova, and A. S. Azamatov, *Integration of the differential-difference sine-Gordon equation with a self-consistent source,* Theoretical and Mathematical Physics, **210**, 327–336 (2022). https://doi.org/10.1134/s0040577922030035
[34] A. Baev, *Direct and inverse problems for Korteweg–de Vries family equations in the Fermi–Pasta–Ulam–Tsingou model for an inhomogeneous structure of lattice,* Journal of Inverse and Ill-Posed Problems, **33**, 665–681 (2025). https://doi.org/10.1515/jiip-2025-0018
[35] A. V. Baev, *On an inverse problem for the KdV equation with variable coefficient,* Mathematical Notes, **106**, 837–841 (2019). https://doi.org/10.1134/s0001434619110166
[36] B. A. Babajanov, A. S. Azamatov, and R. B. Atajanova, *Integration of the Kaup–Boussinesq system with time-dependent coefficients,* Theoretical and Mathematical Physics, **216**, 961–972 (2023). https://doi.org/10.1134/s004057792307005x
[37] B. A. Babajanov and D. O. Atajonov, *Integration of the generalized Camassa–Holm equation in the class of periodic functions,* Theoretical and Mathematical Physics, **223**, 624–635 (2025). https://doi.org/10.1134/s0040577925040075
[38] A. B. Khasanov and M. M. Matyakubov, *Integration of the nonlinear Korteweg–de Vries equation with an additional term,* Theoretical and Mathematical Physics, **203**, 596–607 (2020). https://doi.org/10.1134/s0040577920050037
[39] G. Urazboev, I. Baltaeva, and O. Ismoilov, *Integration of the negative order Korteweg–de Vries equation by the inverse scattering method,* Vestnik Udmurtskogo Universiteta. Matematika. Mekhanika. Komp'yuternye Nauki, **33**, 523–533 (2023). https://doi.org/10.35634/vm230309
[40] G. T. Bekova, G. N. Shaikhova, K. R. Yesmakhanova, and R. Myrzakulov, *Lax representation and soliton solutions for the (2+1)-dimensional two-component complex modified Korteweg–de Vries equations,* Journal of Physics: Conference Series, **804**, 012004 (2017). https://doi.org/10.1088/1742-6596/804/1/012004
[41] M. Zhassybayeva, K. Yesmakhanova, and Z. Myrzakulova, *Modified Fokas–Lenells equation: self-consistent sources and soliton solutions of the spin and (2+1)-dimensional models,* Symmetry, **17**, 1961 (2025). https://doi.org/10.3390/sym17111961
[42] Q. Li, H. Huang, and Q. Duan, *Solutions of three nonlocal equations with self-consistent sources by the inverse scattering transform and reductions,* Theoretical and Mathematical Physics, **222**, 198–210 (2025). https://doi.org/10.1134/s0040577925020023
[43] U. A. Hoitmetov, *Integration of the loaded KdV equation with a self-consistent source of integral type in the class of rapidly decreasing complex-valued functions,* Siberian Advances in Mathematics, **32**, 102–114 (2022). https://doi.org/10.1134/s1055134422020043
[44] U. B. Muminov and A. B. Khasanov, *Integration of a defocusing nonlinear Schrodinger equation with additional terms,* Theoretical and Mathematical Physics, **211**, 514–531 (2022). https://doi.org/10.1134/s0040577922040067
[45] A. B. Khasanov, K. N. Normurodov, and U. O. Khudaerov, *Integrating the modified Korteweg–de Vries–sine-Gordon equation in the class of periodic infinite-gap functions,* Theoretical and Mathematical Physics, **214**, 170–182 (2023). https://doi.org/10.1134/s0040577923020022
[46] A. B. Khasanov and A. B. Yakhshimuratov, *The Korteweg–de Vries equation with a self-consistent source in the class of periodic functions,* Theoretical and Mathematical Physics, **164**, 1008–1015 (2010). https://doi.org/10.1007/s11232-010-0081-8
[47] A. B. Khasanov and A. A. Reyimberganov, *On the Hirota equation with a self-consistent source,* Theoretical and Mathematical Physics, **221**, 1852–1866 (2024). https://doi.org/10.1134/s0040577924110059
[48] M. Riesz, *L'integrale de Riemann–Liouville et le probleme de Cauchy, Acta Mathematica* **81**, 1–222 (1949). https://doi.org/10.1007/bf02395016
[49] B. M. Levitan, Inverse Sturm–Liouville Problems, (Nauka, Moscow, 1984).
[50] M. J. Ablowitz, Nonlinear Dispersive Waves, (Cambridge University Press, Cambridge, 2011). https://doi.org/10.1017/cbo9780511998324
[51] M. J. Ablowitz, D. J. Kaup, A. C. Newell, and H. Segur, *The inverse scattering transform–Fourier analysis for nonlinear problems, Studies in Applied Mathematics* **53**, 249–315 (1974). https://doi.org/10.1002/sapm1974534249
[52] R. L. Sachs, *Completeness of derivatives of squared Schrodinger eigenfunctions and explicit solutions of the linearized KdV equation,* SIAM Journal on Mathematical Analysis, **14**, 674–683 (1983). https://doi.org/10.1137/0514051

Affiliations

Babajonov Alisher
Address: Urgench State University named after Abu Rayhon Beruni, Department of Applied Mathematics and Mathematical Physics, 220100, Urgench-Uzbekistan.
e-mail: alisherbabajanav@gmail.com
ORCID ID: https://orcid.org/0009-0007-9591-2756

Uzbekistan Journal of Mathematics and Computer Science
Volume-2, Issue-3-special, Page 71–86 (2026)
DOI: https://doi.org/10.56143/ujmcs.v2i3s.7
Research article

The Cauchy problem for a nonhomogeneous time-fractional beam vibration equation involving the Gerasimov–Caputo operator

Sh.T. Karimov * **R.A. Mamajonova**

Abstract

In this paper, a nonhomogeneous initial value problem posed in a half-plane for a fourth-order differential equation involving a time-fractional derivative is investigated. The problem is divided into three parts. First, the solution of the corresponding problem with homogeneous initial conditions is obtained. Subsequently, using the obtained fundamental solution, an explicit expression for the solution of the nonhomogeneous initial value problem is derived.

Keywords: *Cauchy problem; beam vibration equation; Schrödinger equation; Mittag–Leffler function; Wright function; Gerasimov–Caputo fractional derivative; Riemann–Liouville fractional integral.*

AMS Subject Classification (2020): *Primary: 35R11; Secondary: 26A33; 33E12; 35A08; 74K10.*

1. Introduction

The study of fractional differential equations has experienced significant growth in recent years due to its broad range of theoretical and practical applications. Unlike classical differential equations based on integer-order derivatives, fractional models naturally incorporate memory-dependent behavior and spatial or temporal nonlocality, allowing for a more realistic description of many complex phenomena [1, 2, 3]. Consequently, differential equations involving fractional derivatives are now widely used in various fields, including materials science, viscoelasticity, heat and mass diffusion, mechanics, and engineering applications [4, 5].

In mechanical systems, particularly in problems related to the vibrations of rods, beams, and plates, the analysis of oscillatory processes plays an important role in structural mechanics, stability analysis, and engineering design. Such vibration phenomena are often modeled by fourth-order partial differential equations [6, 7]. Among these classes of equations, beam vibration equations occupy a special place because of their significant theoretical interest and wide range of practical applications.

It is well known that the vibration of a dynamic beam is governed by the following equation:

$$\rho\frac{\partial^2 u(x,t)}{\partial t^2} + EI\frac{\partial^4 u(x,t)}{\partial x^4} = f(x,t), \tag{1.1}$$

where $u(x,t)$ denotes the transverse displacement of the beam at position x and time t, ρ is the linear mass density representing the mass per unit length, and EI is the flexural rigidity of the beam, with E being Young's modulus and I the second moment of area of the cross-section. The function $f(x,t)$ represents the external force acting on the beam.

Received : *28–april–2026,* ***Accepted :*** *31–july–2026*
*** Corresponding author***

Introducing appropriate notation, equation (1.1) can be rewritten in the following form:

$$\frac{\partial^2 u\,(x,t)}{\partial t^2} + a^2 \frac{\partial^4 u\,(x,t)}{\partial x^4} = F\,(x,t)\,.$$

Various initial, mixed, and inverse problems for equations of this type, as well as for equations obtained from them through suitable transformations, have been investigated in numerous studies [8, 9, 10, 11, 12, 13]. In addition, the existence and qualitative properties of solutions to the Cauchy problem and initial–boundary value problems in the case $F(x,t) = 0$ were analyzed in [14, 15].

Furthermore, increasing attention has recently been devoted to beam vibration equations involving fractional-order operators, such as

$$^{C}D_{0t}^{\alpha} V(x,t) + \frac{\partial^4 V(x,t)}{\partial x^4} = g(x,t), \tag{1.2}$$

where $1 < \alpha < 2$, and $^{C}D_{0t}^{\alpha}$ denotes the Gerasimov–Caputo fractional derivative, defined by

$$^{C}D_{0t}^{\alpha} f(x,t) = \frac{1}{\Gamma(2-\alpha)} \int_0^t (t-\tau)^{1-\alpha} f''(x,\tau)\, d\tau, \qquad 1 < \alpha < 2$$

When $\alpha = 2$, equation (1.2) reduces to the classical beam vibration equation.

Significant contributions to this rapidly developing research area can be found in [13, 16, 17] and the references therein. In [18, 19], various inverse problems for the fractional beam vibration equation were investigated based on the the considered direct problem. In [20], the authors investigated an initial–boundary value problem for a multidimensional fractional beam equation and established its solution by applying the method of separation of variables. The authors of [21] developed a fractional Euler–Bernoulli beam theory and successfully explained the experimentally observed stiffness enhancement in microbeams through comparisons with experimental data. Fractional beam vibration models were also employed in [5, 22] to describe nonlocal effects, memory properties, and material damage processes in a mathematically rigorous framework. In [23], several initial and boundary value problems for the beam vibration equation were solved by exploiting its connections with Brownian motion and the heat equation. Moreover, the fractional beam vibration equation was analyzed, revealing deep relationships among stochastic processes, anomalous diffusion, and fractional dynamics. Finally, in [27], a half-line initial value problem for the homogeneous fractional beam vibration equation was studied.

In this paper, we investigate the problem of finding a solution $V(x,t) \in M$ of the fractional differential equation (1.2) in the domain

$$\Omega = \{(x,t) : x \in \mathbb{R},\ t > 0\},$$

subject to the initial conditions

$$V(x,0) = \varphi(x), \qquad V_t(x,0) = \psi(x), \tag{1.3}$$

where $1 < \alpha < 2$, and $\varphi(x)$, $\psi(x)$, and $g(x,t)$ are given sufficiently smooth functions. Here, M denotes the class of functions that are α-times differentiable with respect to t and four times differentiable with respect to x in Ω, and are continuously differentiable once with respect to t and twice with respect to x in a neighborhood of the boundary of $\overline{\Omega} = (x,t) : x \in \mathbb{R}, ; t \geq 0$.

In addition, the following decay conditions at infinity are imposed:

$$\lim_{|x|\to\infty} V(x,t) = 0, \qquad \lim_{|x|\to\infty} V_{xx}(x,t) = 0. \tag{1.4}$$

To facilitate the subsequent analysis, we first recall several properties of the operators and special functions that will be used throughout the paper.

2. Preliminary functions and operators

Definition 2.1. Let $\alpha > 0$ and let $f(t)$ be an integrable function. The Riemann–Liouville fractional integral of order α is defined by

$$I_{0t}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau.$$

Definition 2.2. For $\alpha > 0$, $n-1 \leq \alpha < n$ and $f \in C^n(0,t)$, the Riemann-Liouville fractional derivative is defined as follows:

$${}^{RL}D_{0t}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-\tau)^{n-\alpha-1} f(\tau)\, d\tau$$

Definition 2.3. For $\alpha > 0$, $n-1 < \alpha \leq n$ and $f \in C^n(0,t)$, the Gerasimov-Caputo fractional derivative is defined as follows:

$${}^{C}D_{0t}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau)\, d\tau$$

Lemma 2.1. *If $\alpha \geq 0$ and f is a continuous function, then*

$${}^{C}D_{0,t}^{\alpha} I_{0,t}^{\alpha} f(t) = f(t) \tag{2.1}$$

$${}^{RL}D_{0,t}^{\alpha} I_{0,t}^{\alpha} f(t) = f(t).$$

Lemma 2.2. *Let $\alpha > 0$, $n \notin \mathbb{N}$, $n = [\alpha] + 1$ and $f \in C^m[a,b]$. Then ${}^{C}D_{at}^{\alpha} f \in C[a,b]$ and ${}^{C}D_{at}^{\alpha} f(a) = 0$ holds.*

Lemma 2.3. *Let $0 < \alpha < 1$, $f \in A^1[a,b]$ and $f(0) = 0$. Then*

$$I_{0t}^{\alpha}\, {}^{C}D_{0t}^{\alpha} f(t) = f(t), \tag{2.2}$$

$$I_{0t}^{\alpha}\, {}^{RL}D_{0t}^{\alpha} f(t) = f(t), \tag{2.3}$$

$${}^{RL}D_{0,t}^{\alpha} f(t) = {}^{C}D_{0,t}^{\alpha} f(t). \tag{2.4}$$

Lemma 2.4. *Let $a < b$ and $n \in \mathbb{N}$, and assume that $f \in C^n[a,b]$. Moreover, let $\alpha, \beta > 0$ be such that there exists $l \in \mathbb{N}$, $l \leq n$, and $\alpha, \alpha+\beta \in [l-1, l]$. Then the following property holds:*

$${}^{C}D_{at}^{\alpha}\, {}^{C}D_{at}^{\beta} f(t) = {}^{C}D_{at}^{\alpha+\beta} f(t) \tag{2.5}$$

The proofs of the above lemmas can be found in [24].

Lemma 2.5. *([27]) Let $\alpha \in (1,2)$, $f \in C^2(a,b) \cap C^1[a,b]$, and $f'(a) = 0$. Then the following equality holds:*

$${}^{C}D_{at}^{\alpha} f(t) = {}^{C}D_{at}^{\alpha/2}\, {}^{C}D_{at}^{\alpha/2} f(t). \tag{2.6}$$

Lemma 2.6. *([29]) Let $u(t) \in C[a,b]$, $t \in (a,b)$, and $\alpha \in (0,1)$. Then*

$${}^{RL}D_{0t}^{\alpha} f^2(t) \geq 2 f(t)\, {}^{RL}D_{0t}^{\alpha} f(t), \quad f(t) \in C[0,\infty). \tag{2.7}$$

Now we present the special functions that will be used in this work and the relations between them.

Definition 2.4. The one-parameter Mittag–Leffler function is defined as follows:

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \alpha \in \mathbb{C}, \operatorname{Re}\alpha > 0, z \in \mathbb{C}. \tag{2.8}$$

The two-parameter Mittag–Leffler function is defined as follows:

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \alpha, \beta \in \mathbb{C}, \operatorname{Re}\alpha > 0, \operatorname{Re}\beta > 0, z \in \mathbb{C} \tag{2.9}$$

For the Mittag–Leffler function, the following estimate holds:

Lemma 2.7. *[1] If $\alpha < 2$, β is an arbitrary real number, and μ satisfies the condition $\pi\alpha/2 < \mu < \min\pi, \pi\alpha$, and C is a real constant, then*

$$|E_{\alpha,\beta}(z)| \le \frac{C}{1+|z|},$$
$$\mu \le |\arg(z)| \le \pi, \quad |z| \ge 0.$$

Definition 2.5. The Wright function is defined as follows:

$$W_{\lambda,\mu}(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!\Gamma(\lambda n + \mu)}, \lambda > -1, \mu \in \mathbb{C}, z \in \mathbb{C} \tag{2.10}$$

Definition 2.6. Also, an auxiliary function related to the Wright function was introduced in [3] as follows:

$$M_\nu(z) = W_{-\nu,1-\nu}(-z), 0 < \nu < 1$$

or

$$M_\nu(z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\Gamma(-\nu n + 1 - \nu)} = \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{(-z)^n}{n!} \Gamma(\nu n + \nu) \sin(\pi\nu(n+1)) \tag{2.11}$$

Definition 2.7. In [30], the following Wright-type function is introduced:

$$e_{\alpha,\beta}^{\mu,\delta}(x) = \sum_{n=0}^{+\infty} \frac{x^n}{\Gamma(\alpha n + \mu)\Gamma(\delta - \beta n)}. \tag{2.12}$$

Between (2.10) and(2.12), the following relation holds:

$$W_{-\beta,\delta}(x) = e_{1,\beta}^{1,\delta}(x).$$

Lemma 2.8. *For (2.10), the following properties hold [28]:*

$$\frac{d^n}{dt^n}\left(t^{\mu-1} W_{\lambda,\mu}\left(-ct^\lambda\right)\right) = t^{\mu-n-1} W_{\lambda,\mu-n}\left(-ct^\lambda\right), \tag{2.13}$$

$${}^C D_{0t}^\alpha \left(t^{\mu-1} W_{\lambda,\mu}\left(-ct^\lambda\right)\right) = t^{\mu-\alpha-1} W_{\lambda,\mu-\alpha}\left(-ct^\lambda\right), \tag{2.14}$$

$$\frac{d}{dt} W_{\lambda,\mu}(t) = W_{\lambda,\mu+\lambda}(t) :: \tag{2.15}$$

$$|W_{\lambda,\mu}(t)| \le C \exp\left(-\nu |t|^{\frac{1}{1-\delta}}\right), \quad |t| \to \infty$$

where $C = C(\lambda,\mu,\nu)$, $\lambda \in (0,1)$, $\varepsilon \in \mathbb{R}$, and $\nu < (1-\lambda)^{\frac{\lambda}{1-\lambda}} \cos\frac{\pi - |\arg t|}{1-\lambda}$; $\frac{1+\lambda}{2}\pi < |\arg t| \le \pi$.

$$\int_0^{+\infty} W_{\lambda,\mu}(\delta t)\, dx = -\frac{1}{\lambda\Gamma(\lambda+\varepsilon)}, \quad \frac{1+\lambda}{2}\pi < |\arg\lambda| \le \pi. \tag{2.16}$$

Furthermore, between (2.8) and (2.11), the following relation holds:

$$\mathcal{F}\left[\frac{1}{2}M_{\nu}\left(|x|\right)\right] = E_{2\nu}\left(-s^2\right) \tag{2.17}$$

where $\mathcal{F}$ denotes the Fourier transform and is defined as

$$\mathcal{F}\left(f\left(x\right)\right) = \int_{-\infty}^{\infty} e^{-isx} f\left(x\right) dx.$$

Detailed properties of (2.8), (2.9), and (2.10), as well as the proof of , are given in [17]. For (2.12), the following lemma is provided in [30].

Lemma 2.9. *If* $\pi \gg |\arg z| > \pi(\alpha+\beta)/2 + \varepsilon$, $\varepsilon > 0$, $k = 0, 1, 2, \ldots$, *then as* $z \to \infty$ *the following limit relations hold:*

$$\lim_{|z|\to\infty} e^{\mu,\delta}_{\alpha,\beta}(z) = 0,$$

$$\lim_{|z|\to\infty}, ze^{\mu,\delta}_{\alpha,\beta}(z) = -\frac{1}{\Gamma(\mu-\alpha)\Gamma(\delta+\beta)},$$

$$\lim_{|z|\to\infty}, z^2 e^{\mu,-\beta-k}_{\alpha,\beta}(z) = \frac{1}{\Gamma(-k-\alpha)\Gamma(\delta+2\beta)},$$

$$\lim_{|z|\to\infty}, z^2 e^{\mu,-\beta-k}_{\alpha,\beta}(z) = \frac{1}{\Gamma(\mu-2\alpha)\Gamma(-k+\beta)}.$$

In [3], formulas (F.33) and (F.34) for $M_{\nu}(x)$ are generalized to $W_{\lambda,\mu}\left(x\right)$ as follows:

$$\int_0^{\infty} x^{\delta} W_{\lambda,\mu}\left(-x\right) dx = \frac{\Gamma\left(\delta+1\right)}{2\pi i} \int_{Ha} \frac{e^{\sigma}}{\sigma^{\mu-\lambda(\delta+1)}} = \frac{\Gamma\left(\delta+1\right)}{\Gamma\left(\mu-\lambda\left(\delta+1\right)\right)}.$$

$$\mathcal{F}\left[\frac{1}{2}W_{\lambda,\mu}\left(|x|\right)\right] = E_{-2\lambda,\mu-\lambda}\left(-\kappa^2\right) \tag{2.18}$$

3. Main results

Since equation (1.2) is linear, we represent the solution of the problem (1.2)–(1.4) in the following form

$$V(x,t) = V_1(x,t) + V_2(x,t) + V_3(x,t), \quad x \in \mathbb{R},\ t > 0$$

where $V_1(x,t)$ and $V_2(x,t)$ are respectively solutions of the equation

$${}^{C}D^{\alpha}_{0t}V\left(x,t\right) + V_{xxxx}\left(x,t\right) = 0 \tag{3.1}$$

satisfying the conditions

$$V_1(x,0) = \varphi(x), V_{1t}(x,0) = 0, x \in \mathbb{R}. \tag{3.2}$$

$$V_2(x,0) = 0, V_{2t}(x,0) = \psi(x), x \in \mathbb{R}. \tag{3.3}$$

and $V_3(x,t)$ is the solution of equation (1.3) satisfying the conditions

$$V_3(x,0) = 0, V_{3t}(x,0) = 0, x \in \mathbb{R}. \tag{3.4}$$

If the solution $V_1(x,t)$ of equation (3.1) satisfying the initial conditions (3.2) is known, then the result of the following lemma is used to construct $V_2(x,t)$.

Lemma 3.1. *If $V_1(x,t)$ is a solution of the problem (3.1)-(3.2), then*

$$V_2(x,t) = \int_0^t V_1(x,s)\,ds \tag{3.5}$$

is a solution of equation (3.1) satisfying the following initial conditions:

$$V_2(x,0) = 0, V_{2t}(x,0) = \varphi(x).$$

Proof. First, differentiating (3.5) with respect to t yields

$$V_{2t}(x,t) = V_1(x,t),$$

then applying the $(\alpha-1)$-order Gerasimov–Caputo derivative gives

$${}^C D_{0t}^{\alpha-1}\left(\frac{\partial V_2(x,t)}{\partial t}\right) = {}^C D_{0t}^{\alpha-1}(V_1(x,t)) \tag{3.6}$$

Using Lemma 2.4 and relation (2.4), and taking condition (3.2) into account, we can rewrite (3.6) as

$${}^C D_{0t}^{\alpha} V_2(x,t) = \int_0^t {}^C D_{0t}^{\alpha} V_1(x,s)\,ds. \tag{3.7}$$

Now differentiating both sides of (3.5) four times with respect to x, we obtain

$$V_{2xxxx}(x,t) = \int_0^t V_{1xxxx}(x,s)\,ds. \tag{3.8}$$

Adding (3.7) and (3.8), we obtain

$${}^C D_{0t}^{\alpha} V_2(x,t) + V_{2xxxx}(x,t) = \int_0^t {}^C D_{0t}^{\alpha} V_1(x,s) + V_{1xxxx}(x,s)\,ds \tag{3.9}$$

From (3.9), since the right-hand side of the equality is equal to zero, it follows that the left-hand side is also zero, and hence in (3.5) it follows that the function $V_2(x,t)$ also satisfies equation (3.1). Now we verify the initial conditions:

$$V_2(x,0) = \lim_{t\to 0} \int_0^t V_1(x,s)\,ds = 0$$

and

$$V_{2t}(x,0) = V_1(x,0) = \varphi(x).$$

If we replace $\varphi(x)$ with $\psi(x)$, then we obtain the solution $V_2(x,t)$ of equation (3.1) satisfying the following initial conditions:

$$V_2(x,0) = 0, V_{2t}(x,0) = \psi(x).$$

□

Lemma 3.2. *If $V_3(x,t)$ is a solution of problem (1.2)-(3.4), then*

$$V_3(x,t) = \int_0^t w(x,t-s,s)\,ds \tag{3.10}$$

is a solution of the equation

$$^C D_{0t}^{\alpha} w(x,t,s) + w_{xxxx}(x,s) = 0$$

satisfying the initial conditions

$$w(x,0,s) = 0, \quad w_t(x,0,s) = \lim_{t\to 0} {}^C D_{0s}^{2-\alpha} g(x,s)$$

and is therefore a solution of the problem satisfying the initial conditions.

Proof. *Differentiating (3.10) with respect to t gives*

$$V_{3t}(x,t) = \int_0^t w_t(x,t-s,s)\,ds + w(x,0,t) - 0 = \int_0^t w_t(x,t-s,s)\,ds. \tag{3.11}$$

Now applying the $\alpha-1$-order derivative with respect to t to the obtained equality, we have

$$\begin{aligned}
^C D_{0t}^{\alpha-1} V_{3t}(x,t) &= {}^C D_{0t}^{\alpha} V_3(x,t) = {}^C D_{0t}^{\alpha-1} \int_0^t w_t(x,t-s,s)\,ds \\
&= I_{0t}^{2-\alpha} \frac{\partial}{\partial t} \int_0^t w_t(x,t-s,s)\,ds = I_{0t}^{2-\alpha} \left(\int_0^t w_{tt}(x,t-s,s)\,ds + w_t(x,0,t) \right) \\
&= \int_0^t {}^C D_{0t}^{\alpha} w(x,t-s,s)\,ds + I_{0t}^{2-\alpha} w_t(x,0,t) - \int_0^t w_{xxxx}(x,t-s,s)\,ds + f(x,t) \\
&= -\frac{\partial^4}{\partial x^4} \int_0^t w(x,t-s,s)\,ds + f(x,t) = -V_{3xxxx}(x,t) + f(x,t)
\end{aligned}$$

$$^C D_{0t}^{\alpha} v(x,t) = -v_{xxxx}(x,t) + f(x,t)$$

Taking the limit $t \to 0$ in (3.10) and (3.11), we obtain that the initial conditions (3.4) are satisfied.

Now we present the following theorem for solving problem (3.1)-(3.2).

Theorem 3.1. *[27] The function $V_1(x,t) \in M$ is a solution of problem (3.1)-(3.2) if and only if*

$$U(x,t) = V_1(x,t) + iI_{0t}^{\alpha/2} V_{1xx}(x,t) \tag{3.12}$$

is a solution of the equation

$$^C D_{0t}^{\alpha/2} U(x,t) - iU_{xx}(x,t) = 0, (x,t) \in \Omega \tag{3.13}$$

satisfying the initial condition

$$U(x,0) = \varphi(x), x \in \mathbb{R} \tag{3.14}$$

where i is the imaginary unit.

Proof. *Necessity.* Assume that $V_1(x,t) \in M$ is a solution of problem (3.1)-(3.2). We show that the function $U(x,t)$ defined in (3.12) is a solution of problem (3.13)-(3.14). From (3.12), applying the Gerasimov–Caputo derivative of order $\alpha/2$ with respect to t yields

$$ {}^{C}D_{0t}^{\alpha/2} U(x,t) = {}^{C}D_{0t}^{\alpha/2} V_1(x,t) + i\,{}^{C}D_{0t}^{\alpha/2} I_{0t}^{\alpha/2} V_{1xx}(x,t). $$

Using (2.1), we obtain

$$ {}^{C}D_{0t}^{\alpha/2} U(x,t) = {}^{C}D_{0t}^{\alpha/2} V_1(x,t) + iV_{1xx}(x,t). \tag{3.15} $$

Taking the second-order derivative of (3.12) with respect to x and multiplying by i, we obtain

$$ iU_{xx}(x,t) = iV_{xx}(x,t) - I_{0t}^{\alpha/2} V_{xxxx}(x,t) \tag{3.16} $$

Subtracting (3.16) from (3.15), we arrive at

$$ {}^{C}D_{0t}^{\alpha/2} U(x,t) - iU_{xx}(x,t) = {}^{C}D_{0t}^{\alpha/2} V_1(x,t) + I_{0t}^{\alpha/2} V_{1xxxx}(x,t) \tag{3.17} $$

By conditions (3.2), Lemma 2.2, and relations (2.2),(2.6), we obtain

$$ {}^{C}D_{0t}^{\alpha/2} V_1(x,t) = I_{0t}^{\alpha/2\,C} D_{0t}^{\alpha} V_1(x,t) \tag{3.18} $$

Taking into account (3.18), we can rewrite (3.17) as

$$ {}^{C}D_{0t}^{\alpha/2} U(x,t) - iU_{xx}(x,t) = I_{0t}^{\alpha/2} \left({}^{C}D_{0t}^{\alpha} V_1(x,t) + V_{1xxxx}(x,t) \right). \tag{3.19} $$

Since $V_1(x,t)$ is a solution of equation (3.1), it follows from (3.19) that $U(x,t)$ is a solution of equation (3.13).

From $V_1(x,t) \in M$ and relation (3.12), at $t = 0$ we obtain

$$ U(x,0) = V(x,0) \tag{3.20} $$

which implies the initial condition (3.14).

2. Sufficiency. Let the function $U(x,t)$ defined by (3.12) be a solution of problem (3.13)-(3.14). We now show that $V_1 \in M$ is a solution of problem (3.1)-(3.2).

First, we verify the initial condition (3.2). From (3.20), it follows that $V_1(x,0) = \varphi(x)$. From (3.13) and (3.15), we obtain ${}^{C}D_{0t}^{\alpha/2} V_1(x,t) = i\frac{\partial^2}{\partial x^2}(U - V_1)$, and applying the $(1-\alpha/2)$-order Gerasimov–Caputo derivative, we get

$$ {}^{C}D_{0t}^{1-\alpha/2\,C} D_{0t}^{\alpha/2} V_1(x,t) = i\,{}^{C}D_{0t}^{1-\alpha/2} (U_{xx} - V_{1xx}), $$

From (2.5), we obtain $V_{1t}(x,t) = i\,{}^{C}D_{0t}^{1-\alpha/2}(U_{xx} - V_{1xx})$, and letting $t \to 0$, we get $V_{1t}(x,t) = 0$. Hence, from (3.19), it follows that $V_1(x,t)$ is a solution of problem (3.1)-(3.2). Thus, Theorem 2 is proved. □

From this theorem, the following corollary follows.

Remark 3.1. Assume that $U \in M$ is a solution of problem (3.13)–(3.14), where $\varphi(x)$ is a real-valued function. Then the function $V_1(x,t) = \operatorname{Re} U(x,t)$ is a solution of problem (3.1)–(3.2). Here, $\operatorname{Re} U(x,t)$ denotes the real part of the function $U(x,t)$.

3.1. Uniqueness of the solution

Theorem 3.2. *The solution of problem (1.2)–(1.4) is unique.* ***Proof.*** *It is sufficient to show that the problem*

$$ {}^{C}D_{0t}^{\alpha} V(x,t) + V_{xxxx}(x,t) = 0, \quad V(x,0) = 0, \quad V_t = 0 $$

admits only the trivial solution $V(x,t) \equiv 0$. *The uniqueness of the solution to the present problem is an immediate consequence of Theorem 1 together with the uniqueness result established for problem (3.13)–(3.14). Therefore, it is enough to prove uniqueness for (3.13),(3.14).*

Assume that $U_1(x,t)$ *and* $U_2(x,t)$ *are two solutions of problem (3.13)-(3.14), and define*

$$U(x,t) = U_1(x,t) - U_2(x,t).$$

Then $U(x,t)$ *satisfies equation (3.13) and the homogeneous initial condition*

$$U(x,0) = 0, \quad x \in \mathbb{R}, \ t > 0. \tag{3.21}$$

Now we apply the Fourier transform with respect to the variable x*:*

$$W(s,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-isx} U(x,t)\, dx \tag{3.22}$$

and obtain from (3.13) the following equation:

$${}^{C}D_{0t}^{\alpha/2} W(s,t) + is^2 W(s,t) = 0.$$

Let $W(s,t) = a(s,t) + ib(s,t)$. *Substituting this into (3.22), we obtain*

$${}^{C}D_{0t}^{\alpha/2} \left(a(s,t) + ib(s,t)\right) + is^2 \left(a(s,t) + ib(s,t)\right) = 0,$$

$${}^{C}D_{0t}^{\alpha/2} a(s,t) - s^2 b(s,t) + i\left({}^{C}D_{0t}^{\alpha/2} b(s,t) + s^2 a(s,t)\right) = 0.$$

Separating real and imaginary parts, we obtain the system

$${}^{C}D_{0t}^{\alpha/2} a(s,t) - s^2 b(s,t) = 0 \ {}^{C}D_{0t}^{\alpha/2} b(s,t) + s^2 a(s,t) = 0$$

Multiplying the first equation by $a(s,t)$ *and the second by* $b(s,t)$ *and adding, we obtain*

$$a(s,t)\, {}^{C}D_{0t}^{\alpha/2} a(s,t) + b(s,t)\, {}^{C}D_{0t}^{\alpha/2} b(s,t) = 0. \tag{3.23}$$

From condition (3.21) we obtain

$$W(s,0) = a(s,0) + ib(s,0) = 0, \quad a(s,0) = 0, \quad b(s,0) = 0.$$

Taking into account (2.4) and (2.7), relation (3.23) can be written as

$${}^{RL}D_{0t}^{\alpha/2} a^2(s,t) + {}^{RL}D_{0t}^{\alpha/2} b^2(s,t) \le 0.$$

Applying a fractional derivative of order $\alpha/2$ *to both sides and using relation (2.3), we obtain*

$$a^2(s,t) + b^2(s,t) \le 0. \tag{3.24}$$

From (3.24) it follows that $a(s,t) = 0$ *and* $b(s,t) = 0$. *Hence* $W(s,t) \equiv 0$. *Since* $U(x,t) \in C_{x,t}^{2,1}\left(\overline{\Omega}\right)$, *it follows that* $U(x,t) \equiv 0$ *for* $t > 0$. □

To solve problems (3.1)-(3.2), we use Remark 1. Then, we arrive at the following problem for the one-dimensional time-fractional Schrödinger equation with respect to the initial condition (3.13):

$${}^{C}D_{0t}^{\alpha/2} U(x,t) - iU_{xx}(x,t) = 0.$$

The solution of this problem was obtained in [25] in the following form:

$$U(x,t) = \int_{-\infty}^{\infty} \varphi(\xi)\, g_\alpha(x-\xi,t)\, d\xi,$$

where

$$g_\alpha(x-\xi,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(x-\xi)} E_{\frac{\alpha}{2},1}\left(-ik^2 t^{\frac{\alpha}{2}}\right) dk.$$

Since the Mittag–Leffler function $E\alpha/2,1\left(-ik^2t^{\frac{\alpha}{2}}\right)$ is an even function, the above expression can be rewritten as

$$g_\alpha(x-\xi,t) = \frac{1}{\pi}\int_{0}^{\infty} \cos\left(k\,(x-\xi)\right) E_{\frac{\alpha}{2},1}\left(-ik^2 t^{\frac{\alpha}{2}}\right) dk.$$

Using relation (2.17), we express the solution in terms of the Wright auxiliary function:

$$g_\alpha(x-\xi,t) = \frac{1}{2}\frac{1}{\sqrt{i\pi}\,t^{\frac{\alpha}{4}}} M_{\frac{\alpha}{4}}\left(\frac{|x-\xi|}{\sqrt{i}t^{\frac{\alpha}{4}}}\right)$$

From Result 1, the solution of problem (3.1)-(3.2) is given by

$$V_1(x,t) = \int_{-\infty}^{\infty} \varphi(\xi)\, G_{1\alpha}(x-\xi,t)\, d\xi, \tag{3.25}$$

where

$$G_{1\alpha}(x-\xi,t) = Re\left(g_{1\alpha}(x-\xi,t)\right) = \frac{1}{\pi}\int_{0}^{\infty} \cos\left(k\,(x-\xi)\right) E_{\alpha,1}\left(-k^4 t^\alpha\right) dk,$$

or equivalently,

$$G_{1\alpha}(x-\xi,t) = \frac{1}{2t^{\frac{\alpha}{4}}}\sum_{n=0}^{\infty} \frac{\left(-\frac{|x-\xi|}{t^{\frac{\alpha}{4}}}\right)^n \cos\left(\frac{\pi}{4}n+\frac{\pi}{4}\right)}{n!\Gamma\left(-\frac{\alpha}{4}n+1-\frac{\alpha}{4}\right)}. \tag{3.26}$$

3.2. Existence of the solution.

We can write (3.25) in following form:

$$V_1(x,t) = \frac{1}{2}\int_{0}^{\infty}\left[\varphi\left(x-\eta t^{\alpha/4}\right)+\varphi\left(x+\eta t^{\alpha/4}\right)\right]\sum_{n=0}^{\infty}\frac{(-\eta)^n\cos\left(\frac{\pi}{4}n+\frac{\pi}{4}\right)}{n!\Gamma\left(-\frac{\alpha}{4}n+1-\frac{\alpha}{4}\right)}d\eta$$

If the function $\varphi(x)$ is bounded on $(-\infty,+\infty)$, then the improper integral in (3.25) converges uniformly in the domain $\overline{\Omega}$.

$$V_1(x,0) = \lim_{t\to 0}\frac{1}{2}\left[\int_{0}^{\infty}\left[\varphi\left(x-\eta t^{\alpha/4}\right)+\varphi\left(x+\eta t^{\alpha/4}\right)\right]\sum_{n=0}^{\infty}\frac{(-\eta)^n\cos\left(\frac{\pi}{4}n+\frac{\pi}{4}\right)}{n!\Gamma\left(-\frac{\alpha}{4}n+1-\frac{\alpha}{4}\right)}d\eta\right] =$$

$$= \varphi(x)\,\mathrm{Re}\left(\int_{0}^{\infty} M(-\zeta)\,d\zeta\right) = \varphi(x)$$

Now we show that $V_{1t}(x,0)=0$. For this purpose, we express the solution in terms of the Wright function and, using relation (2.13), take the derivative and integrating by parts 4 times, we obtain:

$$\frac{\partial}{\partial t}V_1(x,t)=V_t(x,0)\lim_{\xi\to x}t^{\alpha/2-1}\varphi''(\xi)\,\mathrm{Re}\left(W_{-\alpha/4,3\alpha/4}\left(-\frac{x-\xi}{\sqrt{i}}t^{-\alpha/4}\right)\right)$$

$$+\frac{t^{\alpha-1}}{2}\int_0^\infty\left[\varphi^{(IV)}\left(x+\eta t^{\alpha/4}\right)+\varphi^{(IV)}\left(x-\eta t^{\alpha/4}\right)\right]\mathrm{Re}\left(\sqrt{i}W_{-\alpha/4,\alpha}\left(-\frac{\eta}{\sqrt{i}}\right)\right)d\eta$$

Taking into account the relations in Lemma 2.9:

$$V_{1t}(x,0)=\lim_{t\to0}\lim_{\xi\to x}t^{\alpha/2-1}\varphi''(\xi)\,\mathrm{Re}\left(W_{-\alpha/4,3\alpha/4}\left(-\frac{x-\xi}{\sqrt{i}}t^{-\alpha/4}\right)\right)$$

$$+\lim_{t\to0}\frac{t^{\alpha-1}}{2}\int_0^\infty\left[\varphi^{(IV)}\left(x+\eta t^{\alpha/4}\right)+\varphi^{(IV)}\left(x-\eta t^{\alpha/4}\right)\right]\mathrm{Re}\left(\sqrt{i}W_{-\alpha/4,\alpha}\left(-\frac{\eta}{\sqrt{i}}\right)\right)d\eta=0$$

Theorem 3.3. *Let $\varphi^{(k)}(x)\in C(\mathbb{R})$ and $\left|\varphi^{(k)}(x)\right|<\infty$, $k=0,1,2,3,4$. Then the function defined by the following representation is a solution of problem (3.1)-(3.2).*

$$V_1(x,t)=\int_{-\infty}^{\infty}\varphi(\xi)\,G_{1\alpha}(x-\xi,t)\,d\xi,$$

where

$$G_{1\alpha}(x-\xi,t)=\frac{t^{-\alpha/4}}{2}\sum_{n=0}^{+\infty}\frac{\left(-\frac{|x-\xi|}{t^{\alpha/4}}\right)^n\cos\left(\frac{n+1}{4}\pi\right)}{\Gamma\left(-\frac{\alpha}{4}n+1-\frac{\alpha}{4}\right)}.$$

3.3. Justification of the solution

Using (2.14) and (2.13), we take the fractional derivative of order α with respect to t and the fourth-order derivative with respect to x of $V_1(x,t)$, respectively, and obtain

$${}^{C}D_{0t}^{\alpha}V_1(x,t)=\frac{t^{-5\alpha/4}}{\sqrt{i}}\int_{-\infty}^{+\infty}\varphi(\xi)\,\mathrm{Re}\left(\frac{1}{\sqrt{i}}W_{-\alpha/4,1-5\alpha/4}\left(-\frac{|x-\xi|}{\sqrt{i}}t^{-\alpha/4}\right)\right)$$

$$\frac{\partial^4}{\partial x^4}V(x,t)=-\frac{t^{-5\alpha/4}}{2}\int_{-\infty}^{+\infty}\varphi(\xi)\,\mathrm{Re}\left(\frac{1}{\sqrt{i}}W_{-\alpha/4,1-5\alpha/4}\left(-\frac{|x-\xi|}{\sqrt{i}}t^{-\alpha/4}\right)\right)d\xi$$

and we have

$${}^{C}D_{0t}^{\alpha}V_1(x,t)+\frac{\partial^4}{\partial x^4}V_1(x,t)=0$$

Lemma 3.3. *The function*

$$G_{1\alpha}(x-\xi,t)=\frac{t^{-\alpha/4}}{2}\sum_{n=0}^{+\infty}\frac{\left(-\frac{|x-\xi|}{t^{\alpha/4}}\right)^n\cos\left(\frac{n+1}{4}\pi\right)}{\Gamma\left(-\frac{\alpha}{4}n+1-\frac{\alpha}{4}\right)}.$$

possesses the following properties:

(a) $G_{1\alpha} \in C^{\infty}(\Omega)$*;*
(b) $G_{1\alpha}$ *is a solution of equation (3.1) in the domain* Ω*;*
(c) $\lim_{t\to 0} G_{1\alpha}(x-\xi,t) = 0$*;*
(d) $\int_{-\infty}^{+\infty} G_{1\alpha}(x-\xi,t)\,d\xi = 1$.

Using the solution $V_1(x,t)$ and applying Lemma 3.1, we construct the function $V_2(x,t)$ satisfying problem (3.1)-(3.3). From (3.5), we obtain

$$V_2(x,t) = \int_0^t V_1(x,s)\,ds = \int_0^t \int_{-\infty}^{+\infty} \psi(\xi)\, G_{1\alpha}(x-\xi,s)\,d\xi\,ds.$$

Interchanging the order of integration yields

$$V_2(x,t) = \int_{-\infty}^{+\infty} \psi(\xi) G_{2\alpha}(x-\xi,t)\,d\xi, \tag{3.27}$$

where

$$G_{2\alpha}(x-\xi,t) = \int_0^t G_{1\alpha}(x-\xi,s)\,ds = \frac{1}{\pi}\int_0^{\infty} \cos\big(k(x-\xi)\big)\left(\int_0^t E_{\alpha,1}\left(-k^4 s^{\alpha}\right)\,ds\right)dk.$$

According to formula (2.1.53) in [2], the above integral can be written as

$$G_{2\alpha}(x-\xi,t) = \frac{t}{\pi}\int_0^{\infty} E_{\alpha,2}\left(-k^4 t^{\alpha}\right)\cos\big(k(x-\xi)\big)\,dk.$$

Using relation (2.18) with $\lambda = -\alpha/4, \mu = 2$ and

$$\mathcal{F}^{-1}\left[E_{\alpha/2,2}\left(-\kappa^2\right)\right] \to W_{-\alpha/4,2-\alpha/4}(-x),$$

we obtain

$$G_{2\alpha}(x-\xi,t) = \frac{t^{1-\alpha/4}}{2}\sum_{n=0}^{\infty} \frac{\left(-\dfrac{|x-\xi|}{t^{\alpha/4}}\right)^n \cos\left(\dfrac{(n+1)\pi}{4}\right)}{n!\,\Gamma\left(-\dfrac{\alpha n}{4}+2-\dfrac{\alpha}{4}\right)}. \tag{3.28}$$

Theorem 3.4. *If* $\psi(x) \in C(\mathbb{R})$ *and* $|\psi(x)| < \infty$*, then the following function is a solution of problem (3.1)-(3.3):*

$$V_2(x,t) = \int_{-\infty}^{\infty} \psi(\xi)\, G_{2\alpha}(x-\xi,t)\,d\xi,$$

where

$$G_{2\alpha}(x-\xi,t) = \frac{t^{1-\alpha/4}}{2}\sum_{n=0}^{+\infty} \frac{\left(-\frac{|x-\xi|}{t^{\alpha/4}}\right)^n \cos\left(\frac{n+1}{4}\pi\right)}{\Gamma\left(-\frac{\alpha}{4}n+2-\frac{\alpha}{4}\right)}.$$

The justification of the solution of problem (3.1)–(3.3) is carried out similarly to that of $V_1(x,t)$.

Lemma 3.4.

$$G_{2\alpha}(x-\xi,t) = \frac{t^{1-\alpha/4}}{2}\sum_{n=0}^{+\infty} \frac{\left(-\frac{|x-\xi|}{t^{\alpha/4}}\right)^n \cos\left(\frac{n+1}{4}\pi\right)}{\Gamma\left(-\frac{\alpha}{4}n+2-\frac{\alpha}{4}\right)}$$

has the following properties:

a) $G_{2\alpha} \in C^{\infty}(\Omega)$,
b) $G_{2\alpha}$ *is a solution of equation (3.1) in the domain* Ω,
c) $\lim\limits_{t\to 0} G_{2\alpha}(x-\xi,t) = 0$,
d) $\frac{\partial}{\partial t} G_{2\alpha}(x-\xi,t) = G_{1\alpha}(x-\xi,t)$,
e) $\int\limits_{-\infty}^{+\infty} G_{2\alpha}(x-\xi,t)\,d\xi = 1$.

Now, in order to construct the solution $V_3(x,t)$, we use Lemma 3.2. The solution of the function $w(x,t,s)$ appearing there is also determined in the form of (3.27):

$$w(x,t,s) = \int\limits_{-\infty}^{+\infty} {}^{C}D_{0t}^{2-\alpha} g(x,s)\, G_{2\alpha}(x-\xi,t)\,d\xi$$

Hence, by means of (3.10), we obtain the solution $V_3(x,t)$:

$$V_3(x,t) = \int\limits_0^t w(x,t-s,s)\,ds = \int\limits_0^t \int\limits_{-\infty}^{+\infty} {}^{C}D_{0S}^{2-\alpha} g(\xi,s)\, G_{2\alpha}(x-\xi,t-s)\,d\xi ds \tag{3.29}$$

Taking into account the following fractional integration by parts formula from [28],

$$\int_{\eta}^{y} h(t) D_{\eta t}^{-\delta} g(t)\,dt = \int_{\eta}^{y} g(t) D_{yt}^{-\delta} h(t)\,dt, \quad \delta > 0;$$

expression (3.29) can be written in the form

$$V_3(x,t) = \int\limits_0^t \int\limits_{-\infty}^{+\infty} g(\xi,s)\, {}^{C}D_{t-,s}^{2-\alpha} G_{2\alpha}(x-\xi,t-s)\,d\xi ds. \tag{3.30}$$

where

$$G_{3\alpha}(x-\xi,t-s) == \frac{(t-s)^{\frac{3\alpha}{4}-1}}{2} \sum_{n=0}^{\infty} \frac{\left(-\frac{|x-\xi|}{\sqrt{i(t-s)^{\alpha/2}}}\right) \cos\left(\frac{n+1}{4}\pi\right)}{\Gamma\left(-\frac{\alpha}{4}n + \frac{3\alpha}{4}\right)} \tag{3.31}$$

Theorem 3.5. *If* $g(x,t) \in C\left(\overline{\Omega}\right)$ *and* $|g(x,t)| < \infty$, *then the function defined by (3.7) satisfies problem (1.2)-(3.4):*

$$V_3(x,t) = \int\limits_0^t \int\limits_{-\infty}^{+\infty} g(\xi,s)\, {}^{C}D_{t-,s}^{2-\alpha} G_{2\alpha}(x-\xi,t-s)\,d\xi ds.$$

where

$$G_{3\alpha}(x-\xi,t-s) = \frac{(t-s)^{3\alpha/4-1}}{2} \sum_{n=0}^{+\infty} \frac{\left(-\frac{|x-\xi|}{(t-s)^{\alpha/4}}\right)^{n} \cos\left(\frac{n+1}{4}\pi\right)}{\Gamma\left(-\frac{\alpha}{4}n + \frac{3\alpha}{4}\right)}.$$

The justification of the solution of problem (1.2)-(3.4) is carried out similarly to that of $V_1(x,t)$.

Lemma 3.5.

$$G_{3\alpha}(x-\xi,t-s) = \frac{t^{3\alpha/4-1}}{2} \sum_{n=0}^{+\infty} \frac{\left(-\frac{|x-\xi|}{t^{\alpha/4}}\right)^{n} \cos\left(\frac{n+1}{4}\pi\right)}{\Gamma\left(-\frac{\alpha}{4}n + \frac{3\alpha}{4}\right)}$$

has the following properties:

a) $G_{3\alpha} \in C^{\infty}(\Omega)$,
b) $G_{3\alpha}$ *is a solution of equation (1.2) in the domain* Ω,
c) $\lim\limits_{t\to 0} G_{3\alpha}(x-\xi,t) = 0$,
d) $\int\limits_{-\infty}^{+\infty} G_{3\alpha}(x-\xi,t)\,d\xi = 1$,
e) ${}^{C}D_{0t}^{2-\alpha} G_{2\alpha}(x-\xi,t) = G_{3\alpha}(x-\xi,t)$,
f) $G_{3\alpha}(x-\xi,t)$ *is the fundamental solution of equation (1.2).*

Using (3.25), (3.27), and (3.30), the solution of problem (1.2)-(1.5) can be written in the form

$$V(x,t) = V_1(x,t) + V_2(x,t) + V_3(x,t)$$
$$= \int\limits_{-\infty}^{+\infty} \left[\varphi(\xi)\, G_{1\alpha}(x-\xi,t) + \psi(\xi)\, G_{2\alpha}(x-\xi,t) + \int\limits_{0}^{t} g(\xi,s)\, G_{3\alpha}(x-\xi,t-s)\,ds \right] d\xi$$

where $G_{1\alpha}(x-\xi,t)$, $G_{2\alpha}(x-\xi,t)$, and $G_{3\alpha}(x-\xi,t-s)$ are defined by (3.26), (3.28), and (3.31), respectively.

4. Conclusion

In this paper, the Cauchy problem for a fourth-order partial differential equation involving a time-fractional derivative was investigated in the half-plane. This equation can be regarded as a mathematical model of vibrations of a homogeneous beam. Despite the mathematical complexity of the model, the original Cauchy problem can be converted, by means of two consecutive order-reduction procedures, into an equivalent Cauchy problem for the homogeneous Schrödinger equation. This transformation enables the construction of an explicit solution formula for the time-fractional beam vibration equation through the solution of the resulting auxiliary problem.

The obtained results demonstrate the existence of a solution for arbitrary functions specified in the initial conditions. The proposed approach differs methodologically from previously known investigations and provides a theoretical foundation for further studies of fractional-order beam models and related higher-order evolution equations.

Acknowledgements

The authors have no acknowledgements to declare.

Funding

The authors received no financial support for this research.

Availability of data and materials

No datasets were generated or analyzed during the current study.
Not applicable.

Competing interests

The authors declare that they have no competing interests.

The authors declare that they have no competing interests.

Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript. All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

References

[1] Podlubny, I. *Fractional Differential Equations.* Academic Press, San Diego, 1999.
[2] Kilbas, A. A., Srivastava, H. M., Trujillo, J. J. *Theory and Applications of Fractional Differential Equations.* Elsevier, 2006.
[3] Mainardi, F. *Fractional Calculus and Waves in Linear Viscoelasticity*. Imperial College Press, London, 2010.
[4] Caputo, M., Fabrizio, M. *Damage and Fatigue Described by a Fractional Derivative Model.* Journal of Computational Physics, **293** (2015): 400–408.
[5] Rahimi, Z., Sumelka, W., Yang, X.-J. *A New Fractional Nonlocal Model and its Application in Free Vibration of Timoshenko and Euler–Bernoulli Beams. The European Physical Journal Plus,* **132**(11): 479, 2017.
[6] Tikhonov A.N.; *Samarskii A.A. Equation of Mathematical Physics.* Nauka, Moscow. 1966. p. 724.
[7] Krylov A.N. *Ship Oscillations.* Leningrad, Moscow. 1936. p. 442.
[8] Karimov, Sh. T., Sitnik, S. M., Tulasheva, Y. *THE CAUCHY PROBLEM FOR THE DEGENERATE BEAM VIBRATION EQUATION.* Journal of Mathematical Sciences, **277**(3) (2023): 356–370.
[9] Karimov, Sh. T. *The Cauchy problem for the degenerated partial differential equation of the high even order.* Siberian Electronic Mathematical Reports, **15** (2018): 853-862.
[10] Sh.T. Karimov, Sh. A. Oripov. *On a Method for Constructing the Riemann Function for Partial Differential Equations with a Singular Bessel Operator.* Lobachevskii Journal of Mathematics. 2020, **41**(6), pp. 1087–1093.
[11] Sh.T. Karimov. *On One Method for the Solution of an Analog of the Cauchy Problem for a Polycaloric Equation with Singular Bessel Operator.* Ukrainian Mathematical Journal, 2018, **69**(10), pp. 1593–1606.
[12] Urinov A.K.; Azizov M.S. *Degenerate High Even-Order Partial Differential Equation with the Bessel Operator.* Lobachevskii J. Math. 2024. **Volume 45,** pp. 864-874.
[13] Pengbin Feng, E.T.Karimov. *Inverse source problems for time-fractional mixed parabolic-hyperbolic type equations.* Journal of Inverse and Ill-posed problems. **23** (4), 2015, pp.339-353.
[14] Sabitov, K. B. *Cauchy Problem for the Beam Vibration Equation.* Differential Equations, **53**(5) (2017): 663–673.
[15] Sabitov K.B. *A remark on the theory of initial-boundary value problems for the equation of rods and beams.* Differential Equations. 2017. Volume 53, pp. 86-98.
[16] Timoshenko, S., Gere, J. *Theory of Elastic Stability. McGraw–Hill,* New York, 1961.
[17] Graff, K. F. *Wave Motion in Elastic Solids.* Dover Publications, New York, 1991.
[18] Durdiev U.D. *Inverse Problem of Determining the Unknown Coefficient in the Beam Vibration Equation in an Infinite Domain. Differential Equations.* 2023. Volume 59, pp. 462-472.
[19] Durdiev U.D. and Rahmonov A.A. *Inverse problem for a fourth-order differential equation with the fractional Caputo operator.* Russian Mathematics, 2024, No9, 22–33.
[20] Sh. G. Kasimov, U. S. Madrakhimov. *Initial–Boundary Value Problem for the Beam Vibration Equation in the Multidimensional Case*, Differential Equations, 2019, **55**(10), pp. 1336–1348.
[21] Sumelka, W., Błaszczyk, T., Liebold, C. *Fractional Euler–Bernoulli Beams: Theory, Numerical Study and Experimental Validation.* European Journal of Mechanics – A/Solids, **54** (2015): 243–251.
[22] Tekin, I., Mehraliyev, Y., Ismailov, M. *Existence and Uniqueness of an Inverse Problem for Nonlinear Klein–Gordon-Type Equations.* Mathematical Methods in the Applied Sciences, **42**(10) (2019): 3550–3564.
[23] Orsingher Enzo, D'ovidio Mirko. *Vibrations and fractional vibrations of rods, plates and Fresnel pseudo-processes,* Journal of Statistical Physics, **145** (2011) 143–174.
[24] Diethelm, K. *The Analysis of Fractional Differential Equations: An Application-Oriented Exposition Using Differential Operators of Caputo Type,* Springer, Berlin, Heidelberg, 2010.
[25] B. N. Narahari Achar, Bradley T. Yale, and John W. Hanneken: *Time Fractional Schrodinger Equation Revisited, Advances in Mathematical Physics,* **2013** (1), 1-11.
[26] Gradshteyn, I. S., Ryzhik, I. M. *Table of Integrals, Series, and Products. 7th Edition,* Academic Press, San Diego, 2007.
[27] Karimov Sh., Mamajonova R. *The Cauchy problem for a beam vibration equation involving the Caputo time-fractional operator.* Bull. Inst. Math., 2025, **8**(6), pp. 154-160.
[28] Pskhu A.V. *Fundamental solutions and Cauchy problems for an odd-order partial differential equation with Differential Equations*. Electronic Journal of **2019**(21), P. 1—13.
[29] Ibrahim, RW. Jalab, HA. *Differential Equations, Analytic and numerical solutions for systems of fractional Schrödinger equation*. Journal of Inequalities and Applications. 2015 Vol **2015**(23), pp. 1-11.
[30] Pskhu, A. V. *Uravneniya v chastnykh proizvodnykh drobnogo poryadka Partial differential equations of fractional order*. Nauka Moskva, 2005, 201 p.

Affiliations

Sh.T. Karimov
Address: Fergana State University, Dept. of Applied Mathematics and Informatics, 150100, Fergana, Uzbekistan.
e-mail: shaxkarimov@gmail.com
ORCID ID: https://orcid.org/0000-0001-8325-998X

R.A. Mamajonova
Address: Fergana State University, Dept. of Applied Mathematics and Informatics, 150100, Fergana, Uzbekistan.
e-mail: mamajonovarobiyakhon@gmail.com
ORCID ID: https://orcid.org/0009-0004-5714-9997

Uzbekistan Journal of Mathematics and Computer Science
Volume-2, Issue-3-special, Page 87–101 (2026)
DOI: https://doi.org/10.56143/ujmcs.v2i3s.8
Research article

Initial and Initial-Boundary Value Problems for Multi-Parameter Integral-Differential Equations

I.U. Khaydarov * Sh.M. Khasanov

Abstract

This paper investigates initial and initial–boundary value problems for a class of multi-parameter integral-differential equations. In particular, we formulate the Cauchy problem for an ordinary differential equation involving a multi-parameter integral-differential operator. The problem is equivalently transformed into a Volterra integral equation of the second kind. By applying the method of successive approximations, we derive an explicit representation of the solution. As an application of the newly established results, we study direct and inverse source problems for a partial differential equation that incorporates the same multi-parameter integral-differential operator in the time variable. An example is presented to illustrate the theoretical findings.

Keywords: *Multi-parameter integral-differential operator; Cauchy problem; Inverse source problem; Mittag-Leffler function; sub-diffusion equation.*

AMS Subject Classification (2020): *Primary: 26A33 ; Secondary: 34A08; 35R11.*

1. Introduction

Integral–differential operators are an essential mathematical tool for modeling complex phenomena across a wide spectrum of applied sciences, including physics, biology, economics, nanotechnology, and continuum mechanics. Their ability to simultaneously incorporate local dynamics and nonlocal interactions has led to successful applications in viscoelastic processes [1], nanotechnology [2], fractional dynamics [3], continuum mechanics [4], economic and financial modeling [5], as well as biological systems [6]. This broad applicability has stimulated growing interest in the development and analysis of new classes of integral–differential operators.

Comprehensive theoretical foundations of integral–differential operators, together with their connections to ordinary and partial differential equations and related applications, are presented in the monographs [7], [8]. In many modeling frameworks, integral–differential operators naturally arise in the description of complex processes characterized by memory effects and nonlocal behavior, particularly in fractal or heterogeneous media.

Motivated by these features, numerous generalized operator constructions have been proposed in recent years. In particular, various types of integral–differential operators have been introduced and systematically investigated in [8], [9], [10], [11], [12], [13], [22], [14], and [16], where their analytical properties and applications to ordinary and partial differential equations were explored. These studies demonstrate that integral–differential operators provide a flexible analytical framework for capturing long-range interactions and hereditary effects in mathematical models.

Received : *28–april–2026,* ***Accepted :*** *31–july–2026*
**** Corresponding author***

Despite these advances, many existing operator families remain limited in their capacity to represent multiple interacting parameters that commonly appear in realistic systems. This observation motivates the development of more general operator structures that offer greater parametric freedom while preserving analytical tractability.

In this paper, we introduce a new multi-parameter integral–differential operator that extends several known operators as special cases. The proposed construction allows the incorporation of multiple structural characteristics within a unified framework. We investigate its fundamental properties and derive new operational relations, thereby establishing a foundation for future applications to generalized differential equations and related mathematical models.

2. Useful results

Theorem 2.1.[11] *If* $\alpha > 0,\ p > 0,$ *and* $f(x) \in L_1(a,b)$, then

$$ {}_K D_{a+}^{\alpha,p}\, {}_K I_{a+}^{\alpha,p} f(x) = f(x). \tag{2.1}$$

Here,

$$ {}_K I_{a+}^{\alpha,p} f(x) = \frac{p^{1-\alpha}}{\Gamma(\alpha)} \int_a^x \frac{t^{p-1} f(t)}{(x^p - t^p)^{1-\alpha}} dt, $$

and

$$ {}_K D_{a+}^{\alpha,p} f(x) = \frac{p^{\alpha-n+1}}{\Gamma(n-\alpha)} \left(x^{1-p} \frac{d}{dx} \right)^n \int_a^x \frac{t^{p-1} f(t)}{(x^p - t^p)^{\alpha-n+1}} dt. $$

Theorem 2.2.[11] *If* $n-1 < \alpha < n, n = [a]+1, n \in N, p > 0,$ ${}_K I_{a+}^{n-\alpha,p} f(x) \in AC_{\delta_p}^n [a,b]$ *then*

$$ {}_K I_{a+}^{\alpha,p}\, {}_K D_{a+}^{\alpha,p} f(x) = f(x) - \sum_{j=1}^{n} \frac{\left(\frac{x^p - a^p}{p}\right)^{\alpha-j}}{\Gamma(\alpha-j+1)} \left[\lim_{x \to a+} \left(x^{1-p} \frac{d}{dx} \right)^{n-j} {}_K I_{a+}^{n-\alpha,p} f(x) \right], \tag{2.2}$$

$$ {}_K I_{b-}^{\alpha,p}\, {}_K D_{b-}^{\alpha,p} f(x) = f(x) - \sum_{j=1}^{n} \frac{(-1)^{n-j} \left(\frac{b^p - x^p}{p}\right)^{\alpha-j}}{\Gamma(\alpha-j+1)} \left[\lim_{x \to b-} \left(x^{1-p} \frac{d}{dx} \right)^{n-j} {}_K I_{b-}^{n-\alpha,p} f(x) \right]. \tag{2.3}$$

Here,

$$ {}_K I_{b-}^{\alpha,p} f(x) = \frac{p^{1-\alpha}}{\Gamma(\alpha)} \int_x^b \frac{t^{p-1} f(t)}{(t^p - x^p)^{1-\alpha}} dt, $$

and

$$ {}_K D_{b-}^{\alpha,p} f(x) = \frac{p^{\alpha-n+1}}{\Gamma(n-\alpha)} \left(-x^{1-p} \frac{d}{dx} \right)^n \int_x^b \frac{t^{p-1} f(t)}{(t^p - x^p)^{\alpha-n+1}} dt. $$

Theorem 2.3.[21] *If* $f(t) \in C^1[0.T]$ *obtains the maximum (minimum) at the point* $t_0 \in (0,T)$ *and* $\alpha \in (0,1)$, *then the following inequalities hold:*

$$ {}_K^C D_{0+}^{\alpha,p} f(t_0) \geq (\leqslant) \frac{p^{\alpha} {t_0}^{-\alpha p}}{\Gamma(1-\alpha)} [f(t_0) - f(0)] \geq 0\, (\leqslant 0). \tag{2.4}$$

Here

$$ {}^{C}_{K}D^{\alpha,p}_{0+}f(x) = \frac{p^{\alpha}}{\Gamma(1-\alpha)}\int\limits_{0}^{x}\frac{t^{p-1}}{(x^{p}-t^{p})^{\alpha}}\left(t^{1-p}\frac{d}{dt}\right)f(t)\,dt. $$

3. Analytical approach: ODE

In this section, we will use an analytical approach to obtain an explicit solution for the Cauchy problem.

3.1. New Multi-Parameter Integral-Differential operators.

Let us define the multi-parameter integral-differential operators, then study some properties of them.

Definition 3.1. Let $f(x) \in L_1(a,b)$, ${}_{K}I^{(1-\mu)(n-\beta),p}_{a+}f(x) \in AC^{n}_{\delta_p}[a,b]$, $n-1<\alpha\le n$, $n-1<\beta\le n$. The operators

$$ \mathbb{M}^{(\alpha,\beta)\mu}_{a+}f(x) = {}_{K}I^{\mu(n-\alpha),p}_{a+}\left(x^{1-p}\frac{d}{dx}\right)^{n}{}_{K}I^{(1-\mu)(n-\beta),p}_{a+}f(x), $$

$$ \mathbb{M}^{(\alpha,\beta)\mu}_{b-}f(x) = {}_{K}I^{\mu(n-\alpha),p}_{b-}\left(-x^{1-p}\frac{d}{dx}\right)^{n}{}_{K}I^{(1-\mu)(n-\beta),p}_{b-}f(x) $$

are called the left and right-sided multi-parameter integral-differential operators. Here $p\in\mathbb{R}^{+}$, α and β are orders of the operators and μ $(0\le\mu\le 1)$ is type of the operators. Also,

$$ AC^{n}_{\delta_p}[a,b] = \left\{f:[a,b]\to R \text{ and } \delta^{n-1}_{p}f(x)\in AC[a,b]\ ,\ \delta_p = x^{1-p}\frac{d}{dx}\right\}, $$

$$ AC[a,b] = \left\{f:[a,b]\to\mathbb{R}:\ f(x)=c+\int\limits_{a}^{x}\varphi(t)dt,\ \varphi(t)\in L_1(a,b)\right\}, $$

$$ L_p(a,b) = \left\{\|f\|_p = \left(\int\limits_{a}^{b}|f(t)|^{p}dt\right)^{\frac{1}{p}};\ \|f\|_{\infty} = esssup_{a\le x\le b}|f(x)|;1\le p\le+\infty;\ -\infty\le a,b\le+\infty\right\}, $$

and $AC[a,b]$ is a class of absolutely continuous functions and $L_p(a,b)$ is a class of Lebesgue measurable functions.

Let us consider properties of the proposed operators, which will be useful in the next sections.

Lemma 3.1. *If* $p\in\mathbb{R}^{+}$, $n-1<\alpha,\beta\le n$, $\gamma=\beta+\mu(n-\beta)$, $\sigma=\beta+\mu(\alpha-\beta)$, $0\le\mu\le 1$ *then*

$$ \mathbb{M}^{(\alpha,\beta)\mu}_{a+}f(x) = {}_{K}I^{\gamma-\sigma,p}_{a+}{}_{K}D^{\gamma,p}_{a+}f(x), $$

where $n-1<\gamma\le n$, $n-1<\sigma\le n$.

Proof.

$$ \mathbb{M}^{(\alpha,\beta)\mu}_{a+}f(x) = {}_{K}I^{\mu(n-\alpha),p}_{a+}\left(x^{1-p}\frac{d}{dx}\right)^{n}{}_{K}I^{(1-\mu)(n-\beta),p}_{a+}f(x) = $$

$$ {}_{K}I^{\beta+\mu n-\mu\beta-\beta-\mu\alpha+\mu\beta,p}_{a+}\left(x^{1-p}\frac{d}{dx}\right)^{n}{}_{K}I^{n-\beta-\mu(n-\beta),p}_{a+}f(x) = $$

$$ {}_{K}I^{\beta+\mu(n-\beta)-(\beta+\mu(\alpha-\beta)),p}_{a+}\left(x^{1-p}\frac{d}{dx}\right)^{n}{}_{K}I^{n-\gamma,p}_{a+}f(x) = {}_{K}I^{\gamma-\sigma,p}_{a+}{}_{K}D^{\gamma,p}_{a+}f(x). $$

Lemma 3.2. *If* $f(x) \in L_1(a,b)$, $p \in \mathbb{R}^+$, $0 < \alpha, \beta < 1$, $\gamma = \beta + \mu(1-\beta)$, $\sigma = \beta + \mu(\alpha - \beta)$, $0 \le \mu \le 1$, then

$$\mathbb{M}_{a+}^{(\alpha,\beta)\mu} f(x) = {}_K D_{a+}^{\sigma,p} \left(f(x) - \frac{{}_K I_{a+}^{1-\gamma,p} f(a)}{\Gamma(\gamma)} \left(\frac{x^p - t^p}{p} \right)^{\gamma-1} \right),$$

where ${}_K I_{a+}^{1-\gamma} f(a) = {}_K I_{a+}^{1-\gamma} f(x)\Big|_{x=a}$.

Proof. Using Lemma 3.1, we make the following evaluations:

$$\mathbb{M}_{a+}^{(\alpha,\beta)\mu} f(x) = {}_K I_{a+}^{\mu(1-\alpha),p} \left(x^{1-p} \frac{d}{dx} \right) {}_K I_{a+}^{(1-\mu)(1-\beta),p} f(x) = {}_K I_{a+}^{\gamma-\sigma,p} {}_K D_{a+}^{\gamma,p} f(x) =$$
$${}_K I_{a+}^{\gamma-\sigma,p} \left(x^{1-p} \frac{d}{dx} \right) {}_K I_{a+}^{1-\gamma,p} f(x) = \left(x^{1-p} \frac{d}{dx} \right) {}_K I_{a+}^{1,p} {}_K I_{a+}^{\gamma-\sigma,p} \left(x^{1-p} \frac{d}{dx} \right) {}_K I_{a+}^{1-\gamma,p} f(x) =$$
$$\left(x^{1-p} \frac{d}{dx} \right) {}_K I_{a+}^{\gamma-\sigma,p} {}_K I_{a+}^{1,p} \left(x^{1-p} \frac{d}{dx} \right) {}_K I_{a+}^{1-\gamma,p} f(x) =$$
$$\left(x^{1-p} \frac{d}{dx} \right) {}_K I_{a+}^{\gamma-\sigma,p} \left[{}_K I_{a+}^{1-\gamma,p} f(x) - {}_K I_{a+}^{1-\gamma,p} f(a) \right] =$$
$$\left(x^{1-p} \frac{d}{dx} \right) {}_K I_{a+}^{1-\sigma,p} f(x) - \left(x^{1-p} \frac{d}{dx} \right) {}_K I_{a+}^{\gamma-\sigma,p} \left[{}_K I_{a+}^{1-\gamma,p} f(a) \right] =$$
$$\left(x^{1-p} \frac{d}{dx} \right) {}_K I_{a+}^{1-\sigma,p} \left(f(x) - \frac{{}_K I_{a+}^{1-\gamma,p} f(a)}{\Gamma(1+\gamma-\sigma)} {}_K I_{a+}^{\sigma-1,p} \left(\frac{x^p - t^p}{p} \right)^{\gamma-\sigma} \right) =$$
$${}_K D_{a+}^{\sigma,p} \left(f(x) - \frac{{}_K I_{a+}^{1-\gamma,p} f(a)}{\Gamma(\gamma)} \left(\frac{x^p - t^p}{p} \right)^{\gamma-1} \right).$$

Lemma 3.3. *If* $0 < \gamma \le 1$, $0 < \sigma \le 1$, $0 \le \mu < 1$, $\gamma = \beta + \mu(1-\beta)$, $\sigma = \beta + \mu(\alpha - \beta)$, $f(x) \in L_1(a,b)$, ${}_K I_{a+}^{(1-\mu)(1-\beta),p} f(x) \in AC_{\delta_p}^n [a,b]$, *then*

$$\mathbb{M}_{a+}^{(\alpha,\beta)\mu} f(x) = {}_K^C D_{a+}^{\sigma,p} f(x) + \frac{f(a)}{\Gamma(1-\sigma)} \left(\frac{x^p - a^p}{p} \right)^{-\sigma}.$$

Proof. According to Definition 3.1, we have

$$\mathbb{M}_{a+}^{(\alpha,\beta)\mu} f(x) = {}_K I_{a+}^{\mu(1-\alpha),p} \left(x^{1-p} \frac{d}{dx} \right) {}_K I_{a+}^{(1-\mu)(1-\beta),p} f(x) = {}_K I_{a+}^{\mu(1-\alpha),p} \varphi(x),$$

where

$$\varphi(x) = \left(x^{1-p} \frac{d}{dx} \right) {}_K I_{a+}^{(1-\mu)(1-\beta),p} f(x).$$

We will simplify this item using Definition 3.1:

$$\varphi(x)=\left(x^{1-p}\frac{d}{dx}\right){}_{K}I_{a+}^{(1-\mu)(1-\beta),p}f\left(x\right)=\left(x^{1-p}\frac{d}{dx}\right){}_{K}I_{a+}^{1-\gamma,p}f\left(x\right)=$$

$$\left(x^{1-p}\frac{d}{dx}\right)\frac{p^{\gamma}}{\Gamma\left(1-\gamma\right)}\int\limits_{a}^{x}\left(x^{p}-t^{p}\right)^{-\gamma}t^{p-1}f(t)dt=$$

$$\left(x^{1-p}\frac{d}{dx}\right)\frac{-p^{\gamma-1}}{\Gamma\left(1-\gamma\right)}\int\limits_{a}^{x}\left(x^{p}-t^{p}\right)^{-\gamma}f(t)d\left(x^{p}-t^{p}\right)=$$

$$\left(x^{1-p}\frac{d}{dx}\right)\frac{-p^{\gamma-1}}{\Gamma\left(1-\gamma\right)}\left[\frac{\left(x^{p}-t^{p}\right)^{1-\gamma}}{1-\gamma}f(t)\Big|_{a}^{x}-\int\limits_{a}^{x}\frac{\left(x^{p}-t^{p}\right)^{1-\gamma}}{1-\gamma}f'(t)dt\right]=$$

$$\left(x^{1-p}\frac{d}{dx}\right)\frac{p^{\gamma-1}}{\Gamma\left(1-\gamma\right)}\left[\frac{\left(x^{p}-a^{p}\right)^{1-\gamma}}{1-\gamma}f(a)+\int\limits_{a}^{x}\frac{\left(x^{p}-t^{p}\right)^{1-\gamma}}{1-\gamma}f'(t)dt\right]=$$

$$\frac{f(a)}{\Gamma\left(1-\gamma\right)}\left(\frac{x^{p}-a^{p}}{p}\right)^{-\gamma}+\frac{1}{\Gamma\left(1-\gamma\right)}\int\limits_{a}^{x}\left(\frac{x^{p}-a^{p}}{p}\right)^{-\gamma}f'(t)dt\;=$$

$$\frac{f(a)}{\Gamma\left(1-\gamma\right)}\left(\frac{x^{p}-a^{p}}{p}\right)^{-\gamma}+\frac{1}{\Gamma\left(1-\gamma\right)}\int\limits_{a}^{x}\left(\frac{x^{p}-a^{p}}{p}\right)^{-\gamma}t^{p-1}\left(t^{1-p}\frac{d}{dt}\right)f(t)dt=$$

$${}_{K}I_{a+}^{1-\gamma,p}\left(x^{1-p}\frac{d}{dx}\right)f(x)+\frac{f(a)}{\Gamma\left(1-\gamma\right)}\left(\frac{x^{p}-a^{p}}{p}\right)^{-\gamma}.$$

Hence,

$$\mathbb{M}_{a+}^{(\alpha,\beta)\mu}f(x)=$$

$${}_{K}I_{a+}^{\mu(1-\alpha),p}\left[{}_{K}I_{a+}^{1-\gamma,p}\left(x^{1-p}\frac{d}{dx}\right)f(x)+\frac{f(a)}{\Gamma\left(1-\gamma\right)}\left(\frac{x^{p}-a^{p}}{p}\right)^{-\gamma}\right]=$$

$${}_{K}I_{a+}^{\mu-\mu\alpha+1-\gamma,p}\left(x^{1-p}\frac{d}{dx}\right)f(x)+\frac{f(a)}{\Gamma\left(1-\gamma\right)}{}_{K}I_{a+}^{\mu(1-\alpha),p}\left(\frac{x^{p}-a^{p}}{p}\right)^{-\gamma}=$$

$${}_{K}I_{a+}^{\mu-\mu\alpha+1-\gamma,p}\left(x^{1-p}\frac{d}{dx}\right)f(x)+\frac{f(a)}{\Gamma\left(1+\mu-\mu\alpha-\gamma\right)}\left(\frac{x^{p}-a^{p}}{p}\right)^{\mu-\mu\alpha-\gamma}=$$

$${}_{K}I_{a+}^{1-\sigma,p}\left(x^{1-p}\frac{d}{dx}\right)f(x)+\frac{f(a)}{\Gamma\left(1-\sigma\right)}\left(\frac{x^{p}-a^{p}}{p}\right)^{-\sigma}={}_{K}^{C}D_{a+}^{\sigma,p}f(x)+\frac{f(a)\left(\frac{x^{p}-a^{p}}{p}\right)^{-\sigma}}{\Gamma\left(1-\sigma\right)}.$$

Lemma 3.4. *If* ${}_{K}I_{a+}^{n-\gamma,p}f(x)\in AC_{\delta_{p}}^{n}[a,b],n-1<\alpha,\beta<n$ *and* $0\leq\mu\leq1$ *then*

$${}_{K}I_{a+}^{\sigma,p}\,\mathbb{M}_{a+}^{(\alpha,\beta)\mu}f(x)=f(x)-\sum_{j=1}^{n}\frac{C_{n-j}}{\Gamma(\gamma-j+1)}\left(\frac{x^{p}-a^{p}}{p}\right)^{\gamma-j},$$

where $\gamma=\beta+\mu\left(n-\beta\right),\ \sigma=\beta+\mu\left(\alpha-\beta\right),\ C_{n-j}=\lim\limits_{x\to a+}\left(x^{1-p}\frac{d}{dx}\right)^{n-j}I_{a+}^{n-\gamma,p}f(x).$

Proof. Applying Lemma 3.1, we obtain

$$ {}_K I_{a+}^{\sigma,p}\, \mathbb{M}_{a+}^{(\alpha,\beta)\mu} f(x) = {}_K I_{a+}^{\sigma,p}\, {}_K I_{a+}^{\gamma-\sigma,p}\, {}_K D_{a+}^{\gamma,p} f(x) = {}_K I_{a+}^{\gamma,p}\, {}_K D_{a+}^{\gamma,p} f(x). \tag{3.1} $$

Since ${}_K I_{a+}^{n-\gamma,p} f(x) \in AC_{\delta_p}^n [a,b]$, and according to (2.2),(3.1), one can easily get

$$ {}_K I_{a+}^{\gamma,p}\, {}_K D_{a+}^{\gamma,p} f(x) = f(x) - \sum_{j=1}^{n} \frac{1}{\Gamma(\gamma-j+1)} \left(\frac{x^p}{p}\right)^{\gamma-j} \left[\lim_{x\to a+} \left(x^{1-p}\frac{d}{dx}\right)^{n-j} I_{a+}^{n-\gamma,p} f(x)\right] = $$

$$ = f(x) - \sum_{j=1}^{n} \frac{C_{n-j}}{\Gamma(\gamma-j+1)} \left(\frac{x^p}{p}\right)^{\gamma-j}. $$

3.2. A Cauchy problem.

Let us consider the equation

$$ \mathbb{M}_{0+}^{(\alpha,\beta)\mu} u(t) - \lambda u(t) = f(t)\,,\ t \in (0,b), \tag{3.2} $$

with the following initial conditions:

$$ \lim_{x\to 0+} \left(t^{1-p}\frac{d}{dt}\right)^{n-k} {}_K I_{0+}^{n-\beta-\mu(n-\beta),p} u(t) = A_k\,,\ t \in [0,b],\ k = \overline{1,n}\,. \tag{3.3} $$

Here $n-1 < \alpha, \beta < n,\ 0 \le \mu \le 1\,,\ n \in \mathbb{N},\ p > 0,\ A_j\,, \lambda, p \in \mathbb{R}$ and $f(t)$ is a given function.

Lemma 3.5. The Cauchy problem (3.2)-(3.3) is equivalent to the following integral equation:

$$ u(t) - \lambda\, {}_K I_{0+}^{\sigma,p} u(t) = {}_K I_{0+}^{\sigma,p} f(t) + \sum_{j=1}^{n} \frac{A_j}{\Gamma(\gamma-j+1)} \left(\frac{t^p}{p}\right)^{\gamma-j}. \tag{3.4} $$

Here $\gamma = \beta + \mu\,(n-\beta)\,,\ \sigma = \beta + \mu\,(\alpha-\beta)\,.$

Proof. First, we rewrite (3.2) using Lemma 3.1, then we apply the ${}_K I_{0+}^{\sigma,p}$ integral operator to both sides of the obtained equality. In a final step, we will use Lemma 3.4. Let us show these processes.

$$ {}_K I_{0+}^{\sigma,p}\, {}_K I_{0+}^{\gamma-\sigma,p}\, {}_K D_{0+}^{\gamma,p} u(t) - \lambda {}_K I_{0+}^{\sigma,p} u(t) = {}_K I_{0+}^{\sigma,p} f(t) \to $$

$$ {}_K I_{0+}^{\gamma,p}\, {}_K D_{0+}^{\gamma,p} u(t) - \lambda {}_K I_{0+}^{\sigma,p} u(t) = {}_K I_{0+}^{\sigma,p} f(t) \to $$

$$ u(t) - \sum_{j=1}^{n} \frac{1}{\Gamma(\gamma-j+1)} \left(\frac{t^p}{p}\right)^{\gamma-j} \left[\lim_{x\to 0+} \left(t^{1-p}\frac{d}{dt}\right)^{n-j} I_{0+}^{n-\gamma,p} u(t)\right] = $$

$$ \lambda {}_K I_{0+}^{\sigma,p} u(t) + {}_K I_{0+}^{\sigma,p} f(t) $$

or

$$ u(t) - \lambda {}_K I_{0+}^{\sigma,p} u(t) = {}_K I_{0+}^{\sigma,p} f(t) + \sum_{j=1}^{n} \frac{\left(\frac{t^p}{p}\right)^{\gamma-j}}{\Gamma(\gamma-j+1)} \left[\lim_{x\to 0+} \left(t^{1-p}\frac{d}{dt}\right)^{n-j} I_{0+}^{n-\gamma,p} u(t)\right]. $$

Considering initial conditions (3.3), we obtain the integral equation (3.4).

Lemma 3.6. If $p > 0$ and ${}_K I_{0+}^{\sigma,p} f(t)$ is summable, then the integral equation (3.4) has a unique solution.

Proof. Let us rewrite (3.4) as the second kind Volterra integral equation:

$$u(t) = \lambda \int_0^t K(t,s)u(s)ds + g(t), \tag{3.5}$$

where

$$g(t) = {}_K I_{0+}^{\sigma,p} f(t) + \sum_{j=1}^{n} \frac{A_j}{\Gamma(\gamma - j + 1)} \left(\frac{t^p}{p}\right)^{\gamma - j}, \quad K(t,s) = \frac{\left(\frac{t^p - s^p}{p}\right)^{\sigma-1}}{\Gamma(\sigma) s^{1-p}}.$$

We know that $n-1 < \sigma < n$. If $p > 1$ the kernel of (3.5) integral equation is a continuous on $[0, b]$ and it has the estimate

$$|K(t,s)| = \left| \frac{1}{\Gamma(\sigma)} \left(\frac{t^p - s^p}{p}\right)^{\sigma-1} s^{p-1} \right| \le \frac{M_1}{\Gamma(\sigma)}, \; M_1 = \max_{0 \leqslant s \leqslant t \leqslant b} \left(\frac{b^p - s^p}{p}\right)^{\sigma-1} s^{p-1}.$$

If $p > 0$, the kernel of (3.5) integral equation has a weak singularity and can be estimated as follows:

$$|K(t,s)| = \left| \frac{\left(\frac{t^p - s^p}{p}\right)^{\sigma-1}}{\Gamma(\sigma) s^{1-p}} \right| \le \frac{\left(\frac{b^p - s^p}{p}\right)^{\sigma-1}}{\Gamma(\sigma) s^{1-p}} = \frac{M}{s^{1-p}}, \quad M = \max_{0 \leqslant s \leqslant t \leqslant b} \frac{1}{\Gamma(\sigma)} \left(\frac{b^p - s^p}{p}\right)^{\sigma-1}.$$

If ${}_K I_{0+}^{\sigma,p} f(t)$ is summable, then $g(t)$ will be also summable. Therefore, according to the general theory of the Volterra integral equations [18], there exists a unique solution of (3.5).

Now we find the solution of (3.4) integral equation by using the method of successive approximation.

$$u_0(t) = \sum_{j=1}^{n} \frac{A_j}{\Gamma(\gamma - j + 1)} \left(\frac{t^p}{p}\right)^{\gamma - j},$$

$$u_k(x) = u_0(t) + \lambda {}_K I_{0+}^{\sigma,p} u_{k-1}(t) + {}_K I_{0+}^{\sigma,p} f(t),$$

$$u_1(x) = u_0(t) + \lambda {}_K I_{0+}^{\sigma,p} u_0(t) + {}_K I_{0+}^{\sigma,p} f(t) = \sum_{k=1}^{2} \lambda^{(k-1)} {}_K I_{0+}^{(k-1)\sigma,p} u_0(t) + \sum_{k=1}^{1} {}_K I^{k\sigma,p}_{0+} f(t),$$

$$u_2(x) = u_0(t) + \lambda {}_K I_{0+}^{\sigma,p} u_0(t) + \lambda^2 {}_K I_{0+}^{2\sigma,p} u_0(t) + \lambda {}_K I_{0+}^{2\sigma,p} f(t) + {}_K I_{0+}^{\sigma,p} f(t) =$$

$$\sum_{k=1}^{3} \lambda^{(k-1)} {}_K I_{0+}^{(k-1)\sigma,p} u_0(t) + \sum_{k=1}^{2} \lambda^{k-1} {}_K I^{k\sigma,p}_{0+} f(t),$$

$$u_3(x) = \sum_{k=1}^{4} \lambda^{(k-1)} {}_K I_{0+}^{(k-1)\sigma,p} u_0(t) + \sum_{k=1}^{3} \lambda^{k-1} {}_K I^{k\sigma,p}_{0+} f(t).$$

If we continue this process h times, we get

$$u_h(x) = \sum_{k=1}^{h+1} \lambda^{(k-1)} {}_K I_{0+}^{(k-1)\sigma,p} u_0(t) + \sum_{k=1}^{h} \lambda^{k-1} {}_K I^{k\sigma,p}_{0+} f(t). \tag{3.6}$$

We calculate each term on the right side of the (3.6) equation. At first, we calculate the first term of the equation as follows:

$$\sum_{k=1}^{h+1}\lambda^{(k-1)}{}_K I_{0+}^{(k-1)\sigma,p}u_0(t)=\sum_{k=1}^{h+1}\lambda^{(k-1)}{}_K I_{0+}^{(k-1)\sigma,p}\left(\sum_{j=1}^{n}\frac{A_j}{\Gamma(\gamma-j+1)}\left(\frac{t^p}{p}\right)^{\gamma-j}\right)=$$

$$\sum_{j=1}^{n}\frac{A_j}{\Gamma(\gamma-j+1)}\sum_{k=1}^{h+1}\lambda^{(k-1)}{}_K I_{0+}^{(k-1)\sigma,p}\left(\frac{t^p}{p}\right)^{\gamma-j}=$$

$$\sum_{j=1}^{n}\frac{A_j}{\Gamma(\gamma-j+1)}\sum_{k=1}^{h+1}\frac{\lambda^{(k-1)}p^{1-(k-1)\sigma}}{\Gamma\left((k-1)\sigma\right)}\int_0^t\frac{s^{p-1}}{\left(t^p-s^p\right)^{1-(k-1)\sigma}}\left(\frac{s^p}{p}\right)^{\gamma-j}ds=$$

$$\sum_{j=1}^{n}\frac{A_j}{\Gamma(\gamma-j+1)}\sum_{k=1}^{h+1}\frac{\lambda^{(k-1)}p^{1-(k-1)\sigma}p^{j-\gamma}}{\Gamma\left((k-1)\sigma\right)}\int_0^t\frac{s^{p-1}s^{p(\gamma-j)}ds}{\left(t^p-s^p\right)^{1-(k-1)\sigma}}.$$

We continue the calculation by changing the variable of the last integral, by the following remarks $\left|y=\frac{s^p}{t^p},\ s^{p-1}ds=p^{-1}t^p dy, s^p=t^p y\right|$ and obtain

$$\sum_{j=1}^{n}\frac{A_j}{\Gamma(\gamma-j+1)}\sum_{k=1}^{h+1}\frac{\lambda^{(k-1)}p^{1-(k-1)\sigma}p^{j-\gamma}}{\Gamma\left((k-1)\sigma\right)}\int_0^1\frac{(t^p y)^{(\gamma-j)}p^{-1}t^p dy}{\left(t^p-t^p y\right)^{1-(k-1)\sigma}}=$$

$$\sum_{j=1}^{n}\frac{A_j}{\Gamma(\gamma-j+1)}\sum_{k=1}^{h+1}\frac{\lambda^{(k-1)}}{\Gamma\left((k-1)\sigma\right)}\left(\frac{t^p}{p}\right)^{(k-1)\sigma+\gamma-j}\int_0^1 y^{\gamma-j+1-1}(1-y)^{(k-1)\sigma-1}dy=$$

$$\sum_{j=1}^{n}\frac{A_j}{\Gamma(\gamma-j+1)}\sum_{k=1}^{h+1}\frac{\lambda^{(k-1)}}{\Gamma\left((k-1)\sigma\right)}\frac{\Gamma(\gamma-j+1)\Gamma\left((k-1)\sigma\right)}{\Gamma\left((k-1)\sigma+\gamma-j+1\right)}\left(\frac{t^p}{p}\right)^{(k-1)\sigma+\gamma-j}=$$

$$\sum_{j=1}^{n}A_j\sum_{k=1}^{h+1}\frac{\lambda^{(k-1)}}{\Gamma\left((k-1)\sigma+\gamma-j+1\right)}\left(\frac{t^p}{p}\right)^{(k-1)\sigma+\gamma-j}.$$

Now we compute the second term of the equation (3.6).

$$\sum_{k=1}^{h}\lambda^{k-1}{}_K I_{0+}^{k\sigma,p}f(t)=\sum_{k=1}^{h}\frac{\lambda^{k-1}p^{1-k\sigma}}{\Gamma(k\sigma)}\int_0^t\frac{s^{p-1}f(s)}{\left(t^p-s^p\right)^{1-k\sigma}}ds=$$

$$\int_0^t\sum_{k=1}^{h}\frac{\lambda^{k-1}}{\Gamma(k\sigma)}\left(\frac{t^p-s^p}{p}\right)^{k\sigma-1}s^{p-1}f(s)ds.$$

If we rewrite each term on the right side of equation (3.6) with the appropriate form, we have

$$u_h(x)=\sum_{j=1}^{n}A_j\sum_{k=1}^{h+1}\frac{\lambda^{(k-1)}}{\Gamma\left((k-1)\sigma+\gamma-j+1\right)}\left(\frac{t^p}{p}\right)^{(k-1)\sigma+\gamma-j}+$$

$$\int_0^t\sum_{k=1}^{h}\frac{\lambda^{k-1}}{\Gamma(k\sigma)}\left(\frac{t^p-s^p}{p}\right)^{k\sigma-1}s^{p-1}f(s)ds. \tag{3.7}$$

If we compute $\lim\limits_{h\to+\infty} u_h(t)$ by using (3.7) and after that we simplify the obtained function, then we get $u(t)$ which is the solution of the integral equation (3.4):

$$u(t)=\sum_{j=1}^{n}A_j\left(\frac{t^p}{p}\right)^{\beta+\mu(n-\beta)-j}E_{\beta+\mu(\alpha-\beta),\beta+\mu(n-\beta)+1-j}\left[\lambda\left(\frac{t^p}{p}\right)^{\beta+\mu(\alpha-\beta)}\right]+$$

$$\int_0^t s^{p-1}\left(\frac{t^p-s^p}{p}\right)^{\beta+\mu(\alpha-\beta)-1}E_{\beta+\mu(\alpha-\beta),\beta+\mu(\alpha-\beta)}\left[\lambda\left(\frac{t^p-s^p}{p}\right)^{\beta+\mu(\alpha-\beta)}\right]f(s)ds. \tag{3.8}$$

Theorem 3.1 *Let $n-1<\alpha,\beta<n$, $0\le\mu\le 1$, $n\in\mathbb{N}$, $p>0$. If $f(t)\in C^1[0,b]$ and ${}_KI_{0+}^{\sigma,p}f(t)$ is a summable function, then there exists a unique solution for* (3.2)-(3.3).

Proof. According to Lemma 3.5, the problem (3.2) - (3.3) is equivalent to the integral equation (3.4). Then, based on Lemma 3.6, one can state that (3.8) is a unique solution to the problem (3.2) - (3.3).

3.3. Solvability of the Cauchy problem in different classes of functions.

In this subsection, we provide a statement on the solvability of the Cauchy problem (3.2)-(3.3) in different classes of functions. Namely, the following statement holds:

Theorem 3.3 *If $f(s)\in C[0,b]$, $n-1<\gamma,\sigma<n$, then a unique solution of the problem* (3.2)-(3.3) *exists and $\left(\frac{t^p}{p}\right)^{n-\gamma}u(t)\in C[0,b]$.*

Proof. The proof of this statement can be deduced easily if one considers the continuity of the Mittag-Leffler function.

Let us investigate problem (3.2)-(3.3) in other class of functions.

Definition 3.2[22] $L_1^p(0,b)$ class of functions is defined as

$$L_1^p(0,b)=\left\{u:(0,b)\to R,\ \ u(t)t^{p-1}\in L_1(0,b),\ \ u(t)t^{p-1}=g'(t),\ \ g(t)\in AC^1_{\delta_p}(0,b]\right\},$$

$$\|f(t)\|_{L_1^p(0,b)}=\int_0^b\left|t^{p-1}f(t)\right|dt\,,\quad(0<p<+\infty)\,.$$

If $p=1$ then $L_1^p(0,b)=L_1(0,b)$.

We know from [11], if $f(t)\in AC^n_{\delta_p}(a,b]$, then the following equalities are valid:

$$f(t)=\sum_{k=0}^{n-1}C_k\left(\frac{t^p}{p}\right)^k+\frac{1}{\Gamma(n)}\int_0^t\left(\frac{t^p-\tau^p}{p}\right)^{n-1}\varphi(\tau)d\tau,$$

$$C_k=\lim_{t\to0+}\frac{1}{k!}\left(t^{1-p}\frac{d}{dt}\right)^k f(t)\ ,\ \varphi(\tau)\in L_1^p(0,b).$$

Theorem 3.4 *If $n-1<\alpha,\beta<n$ and $f(t)\in L_1^p(0,b)$, then $u(t)\in L_1^p(0,b)$.*

Proof. The proof of this theorem can be easily obtained in an analytical way based on the definition of the function and function (3.8), which is the solution of the Cauchy problem (3.2)-(3.3).

Theorem 3.5 *If $n-1<\alpha,\beta<n$ and $f(t)\in C^1[0,b]$, then for $u(t)$ which the solution of Cauchy problem, the following ${}_KI_{0+}^{(1-\mu)(n-\beta),}$ $u(t)\in AC^n_{\delta_p}(0,b]$ and $u(t)\in L_1^p(0,b)$ holds true.*

Proof. The proof of this theorem can be easily obtained analytically based on the definitions of the functions.

3.4. A priori estimates.

In this subsection, we present a priori estimates.

Lemma 3.7. *If* $0 < \alpha, \beta < 1$, $g(a) = 0$, $0 \le \mu \le 1$ *and* $g(t)$ *is an absolutely continuous function, then we have this inequality*

$$g(t)\,\mathbb{M}_{a+}^{(\alpha,\beta)\mu,p} g(t) \ge \frac{1}{2}\,\mathbb{M}_{a+}^{(\alpha,\beta)\mu,p} g^2(t)\,.$$

Proof. If we apply $g(a) = 0$ (condition of Lemma 3.7), from Lemma 3.3 we get

$$\mathbb{M}_{a+}^{(\alpha,\beta)\mu,p} g(t) = {}_K^C D_{a+}^{\sigma,p} g(t)\,. \tag{3.9}$$

If we show that

$$g(t)\mathbb{M}_{a+}^{(\alpha,\beta)\mu,p} g(t) - \frac{1}{2}\mathbb{M}_{a+}^{(\alpha,\beta)\mu,p} g^2(t) \ \ge 0\,, \tag{3.10}$$

is true, then it is enough to prove Lemma 3.7.

Let us show that (3.10) is true, by using (3.9):

$$g(t){}_K^C D_{a+}^{\sigma,p} g(t) - \frac{1}{2}{}_K^C D_{a+}^{\sigma,p} g^2(t) = g(t)\frac{p^\sigma}{\Gamma(1-\sigma)}\int_a^t \frac{s^{p-1}}{(t^p-s^p)^\sigma}\left(s^{1-p}\frac{d}{ds}\right) g(s)ds -$$

$$\frac{1}{2}\frac{p^\sigma}{\Gamma(1-\sigma)}\int_a^t \frac{s^{p-1}}{(t^p-s^p)^\sigma}\left(s^{1-p}\frac{d}{ds}\right) g^2(s)ds =$$

$$g(t)\frac{p^\sigma}{\Gamma(1-\sigma)}\int_a^t \frac{g'(s)}{(t^p-s^p)^\sigma}ds - \frac{1}{2}\frac{p^\sigma}{\Gamma(1-\sigma)}\int_a^t \frac{2g(s)g'(s)}{(t^p-s^p)^\sigma}ds =$$

$$\frac{p^\sigma}{\Gamma(1-\sigma)}\int_a^t \frac{g'(s)\,[g(t)-g(s)]}{(t^p-s^p)^\sigma}ds = \frac{p^\sigma}{\Gamma(1-\sigma)}\int_a^t \frac{g'(s)}{(t^p-s^p)^\sigma}ds\int_s^t g'(\eta)\,d\eta =$$

$$\frac{p^\sigma}{\Gamma(1-\sigma)}\int_a^t g'(\eta)\,d\eta\int_0^\eta \frac{g'(s)}{(t^p-s^p)^\sigma}ds =$$

$$\frac{p^\sigma}{\Gamma(1-\sigma)}\int_a^t (t^p-\eta^p)^\sigma\frac{g'(\eta)}{(t^p-\eta^p)^\sigma}d\eta\int_a^\eta \frac{g'(s)}{(t^p-s^p)^\sigma}ds =$$

$$\frac{p^\sigma}{2\Gamma(1-\sigma)}\int_a^t (t^p-\eta^p)^\sigma\frac{d}{d\eta}\left(\int_a^\eta \frac{g'(s)}{(t^p-s^p)^\sigma}ds\right)^2 d\eta =$$

$$\frac{p^\sigma}{2\Gamma(1-\sigma)}\left[(t^p-\eta^p)^\sigma\left(\int_a^\eta \frac{g'(s)}{(t^p-s^p)^\sigma}ds\right)^2\Bigg|_{\eta=a}^{\eta=t} -\right.$$

$$\left.\int_a^t (-p)\,\eta^{p-1}\sigma(t^p-\eta^p)^{\sigma-1}\left(\int_a^\eta \frac{g'(s)}{(t^p-s^p)^\sigma}ds\right)^2 d\eta\right] =$$

$$\frac{p^{\sigma+1}\sigma}{2\Gamma(1-\sigma)}\int_a^t \frac{\eta^{p-1}}{(t^p-\eta^p)^{1-\sigma}}\left(\int_a^\eta \frac{g'(s)}{(t^p-s^p)^\sigma}ds\right)^2 d\eta \geqslant 0.$$

Theorem 3.6. *If $f(t) \in C^1[0.T]$, $\alpha \in (0,1)$, $\beta \in (0,1)$, and function $f(t)$ gets its maximum (minimum) value at the point $t_0 \in (0,T)$, then following inequalities are hold:*

$$\mathbb{M}_{0+}^{(\alpha,\beta)\mu} f(t_0) \geq (\leqslant) \frac{p^\sigma {t_0}^{-\sigma p}}{\Gamma(1-\sigma)} f(t_0) \ . \tag{3.11}$$

Furthermore, if $f(t_0) \geq 0 (\leqslant 0)$, then we obtain $\mathbb{M}_{0+}^{(\alpha,\beta)\mu} f(t_0) \geq 0 (\leqslant 0)$.
Here $\sigma = \beta + \mu(\alpha-\beta)$ and $0 < \sigma < 1$, $0 \leq \mu < 1$.

Proof. Using the result of Lemma 3.3 and Theorem 2.3, if the $f(t)$ function gets its maximum value at the point $t_0 \in (0,T)$, then we can obtain

$$\mathbb{M}_{0+}^{(\alpha,\beta)\mu} f(t_0) = {}_K^C D_{0+}^{\sigma,p} f(t_0) + \frac{p^\sigma {t_0}^{-\sigma p}}{\Gamma(1-\sigma)} f(0) \geq \frac{p^\sigma {t_0}^{-\sigma p}}{\Gamma(1-\sigma)} [f(t_0) - f(0)] + \frac{p^\sigma {t_0}^{-\sigma p}}{\Gamma(1-\sigma)} f(0) =$$
$$\frac{p^\sigma {t_0}^{-\sigma p}}{\Gamma(1-\sigma)} f(t_0).$$

As a result, we can write the following inequality:

$$\mathbb{M}_{0+}^{(\alpha,\beta)\mu} f(t_0) \geq \frac{p^\sigma {t_0}^{-\sigma p}}{\Gamma(1-\sigma)} f(t_0). \tag{3.12}$$

Based on (3.12), it is easy to know that if $f(t_0) \geq 0$, then $\mathbb{M}_{0+}^{(\alpha,\beta)\mu} f(t_0) \geq 0$ is true.
The second part of Theorem 3.6 can be proved similarly.

4. Analytical approach: PDE

4.1. A direct problem.

Let us consider the following PDE

$$\mathbb{M}_{0+}^{(\alpha,\beta)\mu} u(t,x) - u_{xx}(t,x) = f(t,x) \tag{4.1}$$

in a domain $\Omega = \{(t,x) : 0 < t < T, \ 0 < x < 1\}$. Here $n-1 < \alpha, \beta < n$, and $f(t,x)$ is given a real function. We are interested in finding $u(t,x)$ that a regular solution of the Eq. (4.1) in Ω, satisfying

- regularity conditions: $u_{xx} \in C(\Omega)$, $\mathbb{M}_{0+}^{(\alpha,\beta)\mu} u(t,x) \in C(\Omega)$, $\left(\frac{t^p}{p}\right)^{n-\gamma} u(t,x) \in C(\bar{\Omega})$;
- initial conditions:

$$\lim_{t\to 0+} \left(t^{1-p}\frac{d}{dt}\right)^{n-k} {}_K I_{0t}^{(1-\mu)(n-\beta),p} u(t,x) = A_k(x), \quad 0 \leq x \leq 1, \quad k = \overline{1,n}; \tag{4.2}$$

- boundary conditions:

$$u(t,0) = u(t,1) = 0 \ , \ 0 < t \leq T. \tag{4.3}$$

Here $A_k(x)$ are given functions.
We search a solution of the problem (4.1),(4.2)-(4.3) as follows:

$$u(t,x) = \sum_{m=1}^{+\infty} v_m(t) \sin m\pi x \ . \tag{4.4}$$

Substituting (4.4) into (4.1) and (4.2) we will obtain the following Cauchy problem for $u_m(t)$:

$$\begin{cases} \mathbb{M}_{0+}^{(\alpha,\beta)\mu} v_m(t) + (m\pi)^2 v_m(t) = f_m(t), \\ \lim\limits_{t\to 0+} \left(t^{1-p}\dfrac{d}{dt}\right)^{n-k} {}_K I_{0+}^{(1-\mu)(n-\beta),p} v_m(t) = A_{k,m}, \end{cases} \tag{4.5}$$

where $A_{k,m} = 2\int\limits_0^1 A_k(x)\sin m\pi x dx$, $f_m(t) = 2\int\limits_0^1 f(t,x)\sin m\pi x dx$.
The solution of (4.5) can be represented as follows:

$$\begin{aligned} v_m(t) = &\sum_{j=1}^{n} A_{j,m} \left(\frac{t^p}{p}\right)^{\gamma-j} E_{\sigma,\gamma+1-j}\left[-(m\pi)^2\left(\frac{t^p}{p}\right)^{\sigma}\right] + \\ &+ \int_0^t s^{p-1}\left(\frac{t^p - s^p}{p}\right)^{\sigma-1} E_{\sigma,\sigma}\left[-(m\pi)^2\left(\frac{t^p - s^p}{p}\right)^{\sigma}\right] f_m(s)ds \ , \end{aligned} \tag{4.6}$$

where $\gamma = \beta + \mu\,(n-\beta)$, $\sigma = \beta + \mu\,(\alpha-\beta)$.

In the following subsection, we consider the inverse source problem for sub-diffusion $(0 < \alpha, \beta < 1)$ equations.

4.2. Inverse source problem.

Let us assume that $0 < \alpha, \beta < 1$ and $f(t,x) = h(x)$. We are interested to find a pair of functions $\{u(t,x),\ h(x)\}$, satisfying Eq. (4.1) in Ω together with conditions (4.2),(4.3) and the following over-determination condition

$$u(\xi,x) = \varphi(x)\,,\ 0 \le x \le 1 \tag{4.7}$$

for a fixed $\xi \in (0,T]$, where $\varphi(x)$ is a given function . Based on (4.4),(4.6), considering that $n = 1$ one can get

$$\begin{aligned} u(t,x) = &\sum_{m=1}^{+\infty} \sin m\pi x \left[A_{1,m}\left(\frac{t^p}{p}\right)^{\gamma-1} E_{\sigma,\gamma}\left[-(m\pi)^2\left(\frac{t^p}{p}\right)^{\sigma}\right] + \right. \\ &\left. + h_m \int_0^t s^{p-1}\left(\frac{t^p - s^p}{p}\right)^{\sigma-1} E_{\sigma,\sigma}\left[-(m\pi)^2\left(\frac{t^p - s^p}{p}\right)^{\sigma}\right] ds\right]. \end{aligned} \tag{4.8}$$

Considering that

$$\int_0^t s^{p-1}\left(\frac{t^p - s^p}{p}\right)^{\sigma-1} E_{\sigma,\sigma}\left[-(m\pi)^2\left(\frac{t^p - s^p}{p}\right)^{\sigma}\right] ds = \left(\frac{t^p}{p}\right)^{\sigma} E_{\sigma,\sigma+1}\left[-(m\pi)^2\left(\frac{t^p}{p}\right)^{\sigma}\right],$$

we deduce

$$\begin{aligned} u(t,x) = &\sum_{m=1}^{+\infty} \sin m\pi x \left[A_{1,m}\left(\frac{t^p}{p}\right)^{\gamma-1} E_{\sigma,\gamma}\left[-(m\pi)^2\left(\frac{t^p}{p}\right)^{\sigma}\right] + \right. \\ &\left. + h_m \left(\frac{t^p}{p}\right)^{\sigma} E_{\sigma,\sigma+1}\left[-(m\pi)^2\left(\frac{t^p}{p}\right)^{\sigma}\right]\right]. \end{aligned} \tag{4.9}$$

Using over-determination condition (4.7), we find h_m as follows:

$$h_m = \left(\frac{\xi^p}{p}\right)^{\mu(1-\alpha)-1} \frac{\varphi_m\left(\frac{\xi^p}{p}\right)^{1-\gamma} - A_{1,m}E_{\sigma,\sigma}\left[-(m\pi)^2\left(\frac{\xi^p}{p}\right)^{\sigma}\right]}{E_{\sigma,\sigma+1}\left[-(m\pi)^2\left(\frac{\xi^p}{p}\right)^{\sigma}\right]}. \tag{4.10}$$

It is known that if $0 < \sigma < 1$, then $E_{\sigma,\sigma+1}(...)$ has no zeros . If $0 < \alpha, \beta < 1$, then definitely $0 < \sigma < 1$.
If we substitute (4.10) into (4.9), we get

$$u(t,x) = \sum_{m=1}^{+\infty} \sin m\pi x \left[A_{1,m}\left(\frac{t^p}{p}\right)^{\gamma-1} E_{\sigma,\gamma}\left[-(m\pi)^2\left(\frac{t^p}{p}\right)^{\sigma}\right] + \left(\frac{t^p}{p}\right)^{\sigma}\left(\frac{\xi^p}{p}\right)^{\mu(1-\alpha)-1} \times \right.$$
$$\left. \frac{\varphi_m\left(\frac{\xi^p}{p}\right)^{1-\gamma} - A_{1,m}E_{\sigma,\sigma}\left[-(m\pi)^2\left(\frac{\xi^p}{p}\right)^{\sigma}\right]}{E_{\sigma,\sigma+1}\left[-(m\pi)^2\left(\frac{\xi^p}{p}\right)^{\sigma}\right]} E_{\sigma,\sigma+1}\left[-(m\pi)^2\left(\frac{t^p}{p}\right)^{\sigma}\right] \right]. \tag{4.11}$$

To prove the uniform convergence of infinite series corresponding to $h(x), u(t,x), u_{xx}(t,x)$ and $\mathbb{M}_{0t}^{(\alpha,\beta)\mu}u(t,x)$, we will use the following estimates:

$$\left|E_{\alpha,\beta}(-z)\right| \le \frac{C}{1+|z|} \le \frac{C}{|z|},$$
$$\sum_{m=1}^{+\infty}\left|A_{1,m}\right| < \sum_{m=1}^{+\infty}\frac{1}{m\pi}\left|A_{1,m}^{(1)}\right| < \sum_{m=1}^{+\infty}\frac{1}{(m\pi)^2} + \|A_1{}'(x)\|_{L_2}^2, \tag{4.12}$$
$$\sum_{m=1}^{+\infty}|\varphi_m| < \sum_{m=1}^{+\infty}\frac{1}{m\pi}\left|\varphi_m^{(1)}\right| < \sum_{m=1}^{+\infty}\frac{1}{(m\pi)^2} + \|\varphi'(x)\|_{L_2}^2.$$

Based on (4.10), considering (4.12) we get

$$|h(x)| \le \sum_{m=1}^{+\infty}|h_m| < \left(\frac{\xi^p}{p}\right)^{\mu(1-\alpha)-1}\left(C_1 + \|\varphi'(x)\|_{L_2}^2 + \|A_1{}'(x)\|_{L_2}^2\right).$$

Similarly, based on (4.11), considering (4.12) one can get

$$|u(t,x)| \le \sum_{m=1}^{+\infty}|v_m(t)| \le \left(\frac{t^p}{p}\right)^{\gamma-1}\|A_1(x)\|_C +$$
$$\left(\frac{t^p}{p}\right)^{\sigma}\left(\left(\frac{\xi^p}{p}\right)^{\mu(1-\alpha)-1}\|A_1(x)\|_C + \left(\frac{\xi^p}{p}\right)^{\mu(1-\alpha)-\gamma}\|\varphi(x)\|_C\right),$$

$$|u_{xx}(t,x)| \le \sum_{m=1}^{+\infty}(m\pi)^2|v_m(t)| < \left(\frac{t^p}{p}\right)^{\gamma-1}\left(C_2 + \|A_1{}'(x)\|_{L_2}^2\right) +$$
$$\left(\frac{t^p}{p}\right)^{\sigma}\left[\left(\frac{\xi^p}{p}\right)^{\mu(1-\alpha)-1}\left(C_3 + \|A_1{}'(x)\|_{L_2}^2\right) + \left(\frac{\xi^p}{p}\right)^{\mu(1-\alpha)-\gamma}\left(C_4 + \|\varphi'(x)\|_{L_2}^2\right)\right].$$

The convergence of an infinite series corresponding to $\mathbb{M}_{0+}^{(\alpha,\beta)\mu}u(t,x)$ can be obtained for Eq.(4.1).
Also, we must note that the uniqueness of the solution to the direct problem can be proved using the maximum principle that was proposed in [21]. The inverse source problem can be proved based on the completeness of the Fourier-sinus series in $L_2(0,1)$.

Example. Let $A_1(x) = \varphi(x) = \sin \pi x$. Then the solution of the inverse source problem will have the form

$$u(t,x) = \sin \pi x \left[\left(\frac{t^p}{p}\right) E_{\sigma,\gamma}\left[-\pi^2 \left(\frac{t^p}{p}\right)^{\sigma}\right] + \right.$$
$$\left. + \left(\frac{t^p}{p}\right)^{\sigma} \left(\frac{\xi^p}{p}\right)^{\mu(1-\alpha)-1} \frac{\left(\frac{\xi^p}{p}\right)^{1-\gamma} - E_{\sigma,\sigma}\left[-\pi^2 \left(\frac{\xi^p}{p}\right)^{\sigma}\right]}{E_{\sigma,\sigma+1}\left[-\pi^2 \left(\frac{\xi^p}{p}\right)^{\sigma}\right]} E_{\sigma,\sigma+1}\left[-\pi^2 \left(\frac{t^p}{p}\right)^{\sigma}\right] \right],$$

where $\gamma = \beta + (1-\beta)$, $\sigma = \beta + \mu(\alpha - \beta)$, $0 \le \mu \le 1$, $p \in \mathbb{R}^+$, $\xi \in (0, T]$.
1) $\alpha = 0,7$; $\beta = 0,7$; $p = 2$; $\mu = 0,5$; $\xi = 1$
2) $\alpha = 0,9$; $\beta = 0,8$; $p = 2$; $\mu = 0,5$; $\xi = 1$
With these values, we will present numerical results in the corresponding graphs. Figures 1 illustrate the numerical results calculated for $u(t,x)$ with the 2 settings.

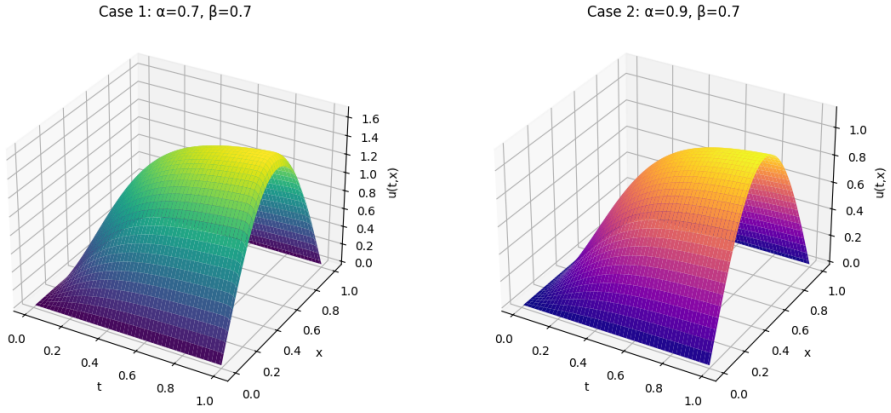


Figure 1. The numerical solutions of $u(x,t)$ when $T = 1$.

Acknowledgements

The authors would like to express their sincere gratitude to Professor E. T. Karimov for his valuable advice and constructive suggestions during the preparation of this article.

Funding

This work received no specific grant from any funding agency in the public, commercial, or not-for-profit sectors.

Availability of data and materials

Not applicable.

Competing interests

The authors declare that they have no competing interests.

Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

References

[1] Mainardi, F.: Fractional Calculus and Waves in Linear Viscoelasticity: An Introduction to Mathematical Models. World Scientific. Singapore (2010).

[2] Baleano, D., Guvens, Z. B., Machado, J. T.: New Trends in Nanotechnology and Fractional Calculus Applications. Springer. Singapore (2010).

[3] Tarasov, V. E.: Fractional Dynamics: Applications of Fractional Calculus to Dynamics of Particles, Fields and Media. Springer. New York(2010).

[4] Atanackovic, T., Pilipovic, S., Stankovic, B., Zorica, D.: Fractional Calculus with Applications in Mechanics: Vibrations and Diffusion Processes. Wiley-ISTE. London (2014).

[5] Korbel, J., Luchko, Y.: *Modeling financial processes with a space-time fractional diffusion equation of varying order.* Fractional Calculus and Applied Analysis. **19** (19), 1414-1433 (2016). https://DOI: 10.1515/fca-2016-0073

[6] Copot, D., Tenreiro, M. J., Bates, J.: *The role of fractional calculus in modeling biological phenomena: A review.* Communications in Nonlinear Science and Numerical Simulation. **51**, 141-159 (2017). https://doi.org/10.1016/j.cnsns.2017.04.001

[7] Samko, S. G., Kilbas, A.A., Marichev, O.I.: Fractional Integrals and Derivatives: Theory and Applications. Gordon and Breach Science Publishers. Philadelphia (1993).

[8] Kilbas, A. A., Srivastava, H. M., Trujillo, J. J.: Theory and Applications of Fractional Differential Equations. Gordon and Breach Science Publishers. Amsterdam (2006).

[9] Baleanu, D., Fernandez, A.: *On Fractional Operators and Their Classifications.* Mathematics. **7** (9), 830 (2019). https://doi.org/10.3390/math7090830

[10] Katugampola, U. N.: *New approach to a generalized fractional integral.*Bulletin of Mathematical Analysis and Applications **6** (4), 1-15. (2014). https://doi.org/10.48550/arXiv.1106.0965

[11] Lupinska, B.: *Properties of the Katugampola fractional operators.* Tatra Mountains Mathematical Publications. **79** (2), 135-148 (2021). https://doi.org/10.2478/tmmp-2021-0024

[12] Olivera, D. S., Olivera, E. D.: *Hilfer-Katugampola fractional derivatives.* Computational and Applied Mathematics. **37** (1), 3672-3690 (2018). https://doi.org/10.1007/s40314-017-0536-8

[13] Abbas, S., Mouffak, B., Hamidi, N., Yong, Z.: *Implicit coupled Hilfer-Hadamard fractional differential systems under weak topologies.* Advances in Difference Equations. 328 (2018). https://doi.org/10.1186/s13662-018-1787-4

[14] Bulavatsky, M. B.: *Closed form of the solutions of some boundary-value problems for anomalous diffusion equation with Hilfer's generalized derivative.* Cybernetics and Systems Analysis. **50** (4), 570-577 (2014). https://doi.org/10.1007/s10559-014-9645-1

[15] Karimov, E., Ruzhansky, M., Toshtemirov, B.: *Solvability of the boundary-value problem for a mixed equation involving hyper-Bessel fractional differential operator and bi-ordinal Hilfer fractional derivative.* Mathematical Methods in the Applied Sciences. **46** (1), 54-70 (2023). https://doi.org/10.1002/mma.8491

[16] Luchko, Y.: *General fractional integrals and derivatives and their applications.*Physica D Nonlinear Phenomena. **455** (9), 133906 (2023). https://doi.org/10.1016/j.physd.2023.133906

[17] Hilfer, R.: Applications of Fractional Calculus in Physics. World Scientific. Germany (2000).

[18] Salohiddinov, M.: Integral equations. Yangiyul polygraph service. Tashkent (2007). (in Uzbek).

[19] Gorenflo, R., Kilbas, A. A., Mainardi, F., Rogosin, S.: Mittag-Leffler functions, related topics and applications. Springer. London (2014).

[20] Magnus, R.: Fundamental Mathematical Analysis. Springer. Reykjavik (2020).

[21] Cao, L., Kong, H., Zeng, S. D.: *Maximum principles for time-fractional Caputo- -Katugampola diffusion equations.* Journal of Nonlinear Sciences and Applications. **10** (4), 2257–22679 (2017). https://doi.org/10.22436/jnsa.010.04.75

[22] Kerbal, S., Khasanov, Sh.: *On Kattugampola-Prabhakar fractional integral-differential operators.* Gulf Journal of Mathematics. **20** (1), 190–221 (2025). https://doi.org/10.56947/gjom.v20i.2856

Affiliations

Khaydarov Ibrokhim

Address: Fergana State University, Dept. of Applied Mathematics and Computer Science, 150100, Fergana-Uzbekistan.

e-mail: ibrohimbek@mail.ru

ORCID ID: https://orcid.org/0000-0002-3145-6911

Khasanov Shokhzodbek

Address: Fergana State University, Dept. of Mathematical Analysis and Differential Equations, 150100, Fergana-Uzbekistan.

e-mail: shohzodbekxasanov@gmail.com

ORCID ID: https://orcid.org/0009-0008-7900-5707

Uzbekistan Journal of Mathematics and Computer Science
Volume-2, Issue-3-special, Page 102–118 (2026)
DOI: https://doi.org/10.56143/ujmcs.v2i3s.9
Research article

Применение метода функции Грина к исследованию краевых задач в полубесконечных областях для уравнений четвертого порядка с кратными характеристиками

Б.И. Жамалов * **Д.М. Меликузиева**

Аннотация

Предметом исследования в данной статье являются граничные задачи в полуограниченной области для неоднородного дифференциального уравнения четвертого порядка с кратными характеристиками. Вопросы единственности регулярных решений разработанных задач решены на основе аппарата энергетических интегралов, в то время как теоремы существования обоснованы с применением классического метода разделения переменных. Искомые решения аналитически выражены через сконструированную функцию Грина в форме бесконечных функциональных рядов. В рамках работы строго доказана абсолютная, а также равномерная сходимость как самих рядов решений, так и рядов, составленных из соответствующих частных производных в рассматриваемых геометрических областях.

Ключевые слова: *Дифференциальные уравнения четвертого порядка; кратные характеристика; полубесконечная область; интеграл энергии; функция Грина; функциональный ряд; равномерная сходимость; частные производные.*

Предметная классификацие AMS (2020): *Основная: 35C10 ; Дополнительная: 35J15..*

1. Введение

Моделирование широкого круга задач газовой динамики, теории упругости, а также механики пластин и оболочек неразрывно связано с исследованием дифференциальных уравнений в частных производных высокого порядка. В контексте прикладных физических приложений дифференциальные уравнения четвертого порядка также представляют существенный научный интерес ([1-5]).

Вопросам систематизации и классификации дифференциальных уравнений в частных производных четвертого порядка посвящена известная монография Т. Д. Джураева и А. Сопуева [6].

Так, в публикации [7] предмет исследования составила краевая задача в ограниченной области для неоднородного уравнения четвертого порядка с кратными характеристиками, содержащего вторую временную производную.

Другой тип граничной задачи для уравнения четвертого порядка вида $u_{xxxx} - u_{tt} = f(x,t)$ изучен в работе [8].

Проблематика, связанная с уравнениями колебания балки, детально отражена в изысканиях [9-11].

Received : *28–april–2026,* ***Accepted :*** *31–july–2026*
*** Corresponding author***

Ряд авторов в работах [12-16] обратились к анализу краевых задач для уравнений третьего порядка с кратными характеристиками, включающих вторую производную по времени, в то время как в статьях [17-18] аналогичные уравнения исследовались в геометрии полуограниченных областей.

При этом граничные задачи для уравнений четвертого порядка, содержащих третью производную по временной переменной, до сих пор остаются малоизученными [19-22].

Тем не менее, в последнее время наблюдается рост интереса именно к уравнениям четвертого порядка с третьей производной по времени. Подобные математические структуры выступают фундаментальной основой в сфере нанотехнологий, в частности, при прецизионном расчете частотных характеристик колебаний микро- и наносенсоров. К примеру, С. Конг (Kong S.) совместно с соавторами в своих ключевых трудах провели подробный статический и динамический анализ микробалок, опираясь на градиентную теорию упругости (strain gradient elasticity theory) [23].

В отличие от вышеупомянутых исследований, в данной работе рассматривается краевая задача, поставленная для уравнения четвертого порядка с кратными характеристиками в полубесконечной области, путем построения функции Грина и её применения для получения решения

2. Постановка задач

В областях $D^+ = \{(x,y) : 0 < x < p,\ 0 < y < +\infty\}$ и $D^- = \{(x,y) : 0 < x < p,\ -\infty < y < 0\}$ рассмотрим уравнение

$$u_{xxxx}(x,y) - u_{yyy}(x,y) + au(x,y) = g(x,y), \tag{2.1}$$

где $a, p \in R$, $g(x,y)$ — заданная достаточно гладкая функция.

Задача A_1. В области D^+ требуется найти функцию $u(x,y)$, удовлетворяющую уравнению (2.1), принадлежащую классу $u(x,y) \in C^{4,3}_{x,y}(D^+) \cap C^{3,2}_{x,y}(D^+ \cup \Gamma_1)$, ограниченную при $y \to +\infty$ функции $u(x,y)$, $u_y(x,y)$ ограничена, равномерно по x, такую что $u_{xx} \in L_2(D^+)$ и удовлетворяющего следующим краевым условиям:

$$u(0,y) = u_x(0,y) = u_{xx}(p,y) = u_{xxx}(p,y) = 0, \quad 0 \le y < +\infty \tag{2.2}$$

$$u_y(x,0) = \psi_1(x), \quad u_{yy}(x,0) = \psi_2(x), \quad \lim_{y\to+\infty} u_{yy}(x,y) = 0, \quad 0 \le x \le p \tag{2.3}$$

где $\Gamma_1 = \partial D^+$- граница области D^+. $\psi_1(x)$ и $\psi_2(x)$- заданные достаточно гладкие функции.

Задача A_2. В области D^- требуется найти функцию $u(x,y)$, удовлетворяющую уравнению (2.1), принадлежащую классу $C^{4,3}_{x,y}(D^-) \cap C^{3,2}_{x,y}(D^- \cup \Gamma_2)$, ограниченную при $y \to -\infty$ функции $u(x,y)$, $u_y(x,y)$ ограничена, равномерно по x, такую что $u_{xx} \in L_2(D^-)$, удовлетворяющего условим (2.2) и:

$$u_{yy}(x,0) = \psi_3(x), \quad \lim_{y\to-\infty} u_y(x,y) = \lim_{y\to-\infty} u_{yy}(x,y) = 0, \quad 0 \le x \le p, \tag{2.4}$$

где $\Gamma_2 = \partial D^-$ — граница области D^-, $\psi_3(x)$ - заданная достаточно гладкая функция.

3. Единственность решения

Теорема 3.1. *Если задача A_1 (A_2) имеет решение, то при условии $a \ge 0$, оно единственно.*

Доказательство. Предположим обратное. Пусть задача $A_1(A_2)$ имеет два решения $u_1(x,y)$ и $u_2(x,y)$. Тогда функция $u(x,y) = u_1(x,y) - u_2(x,y)$ удовлетворяет уравнению (2.1) с однородными краевыми условиями. Докажем, что $u(x,y) \equiv 0$ в D^+ (D^-).

В области D^+ (D^-) справедливо тождество

$$\frac{\partial}{\partial x}\left(uu_{xxx}-u_xu_{xx}\right)+u_{xx}^2+\frac{\partial}{\partial y}\left(-uu_{yy}+\frac{1}{2}u_y^2\right)+au^2=0. \tag{3.1}$$

Интегрируя тождество (3.1) по области $D_d^+=\{(x,y):\ 0<x<p,\quad 0<y<d\}$, где $d>0$ имеем

$$\begin{aligned}&\int_0^d\left[u\left(p,y\right)u_{xxx}\left(p,y\right)-u_x\left(p,y\right)u_{xx}\left(p,y\right)\right]dy\\&-\int_0^d\left[u\left(0,y\right)u_{xxx}\left(0,y\right)-u_x\left(0,y\right)u_{xx}\left(0,y\right)\right]dy\\&+\int_0^p\int_0^d u_{xx}^2dxdy+\int_0^p\left(-u\left(x,d\right)u_{yy}\left(x,d\right)+u\left(x,0\right)u_{yy}\left(x,0\right)\right)dx\\&+\frac{1}{2}\int_0^p\left(u_y^2\left(x,d\right)-u_y^2\left(x,0\right)\right)dx+a\int_0^p\int_0^d u^2\left(x,y\right)dxdy=0.\end{aligned} \tag{3.2}$$

Если $d\to+\infty$, то $D_d^+\to D^+$. Учитывая однородные краевые условия задачи A_1, т.е. $\psi_1\left(x\right)=\psi_2\left(x\right)=0$ и свойства функции $u\left(x,y\right)$ при $y\to+\infty$, а также, что $u_{xx}\in L_2\left(D^+\right)$, из (3.2) получаем

$$\iint_{D^+}u_{xx}^2\left(x,y\right)dxdy+\frac{1}{2}\lim_{d\to+\infty}\int_0^p u_y^2\left(x,d\right)dx+a\iint_{D^+}u^2\left(x,y\right)dxdy=0.$$

Если $a>0$ из третьего слагаемого, получим $u\left(x,y\right)\equiv0,\quad \left(x,y\right)\in D^+\cup\Gamma_1$, а если $a=0$, из первого слагаемого имеем $u_{xx}\left(x,y\right)=0$. Следовательно, $u\left(x,y\right)=xf_1\left(y\right)+f_2\left(y\right)$, где $f_1\left(y\right),\quad f_2\left(y\right)-$ произвольные функции, удовлетворяющие условиям задачи A_1. С учётом условия (2.2) в области $D^+\cup\Gamma_1$ получаем $f_1\left(y\right)=f_2\left(y\right)=0$, тогда имеем $u\left(x,y\right)\equiv0$.

$D_c^-=\{(x,y):\ 0<x<p,\quad c<y<0\}$, где $c<0$ имеем если $c\to-\infty$, то $D_c^-\to D^-$. Учитывая однородные краевые условия задачи A_2, т.е. $\psi_3\left(x\right)=0$ и свойства функции $u\left(x,y\right)$ при $y\to-\infty$, а также, что $u_{xx}\in L_2\left(D^-\right)$, из (3.2) получаем

$$\iint_{D^-}u_{xx}^2\left(x,y\right)dxdy+\frac{1}{2}\int_0^p u_y^2\left(x,0\right)dx+a\iint_{D^-}u^2\left(x,y\right)dxdy=0.$$

Отсюда также легко следует, что $u\left(x,y\right)\equiv0$ в $D^-\cup\Gamma_2$.

Теоема 3.1 доказана.

4. Существование решения

Теорема 4.1. *При выполнения следующих условий:*

1) $\psi_i\left(x\right)\in C^5\left[0,p\right],\quad i=1,2,3;\quad \psi_i\left(0\right)=\psi'_i\left(p\right)=\psi_i{}'\left(0\right)=\psi''_i\left(p\right)=\psi_i{}'''\left(p\right)=\psi_i{}'''\left(0\right)=\psi_i{}^{(4)}=0;$

2) $\dfrac{\partial^2g\left(x,y\right)}{\partial x\partial y}\in C_b\left(D^+\cup\Gamma_1\right),\quad \dfrac{\partial^2g\left(x,y\right)}{\partial x\partial y}\in C_b\left(D^-\cup\Gamma_2\right);\quad g\left(0,y\right)=0,\ \lim\limits_{y\to\pm\infty}g\left(x,y\right)=0,\ \ g\left(x,0\right)=0,$

то решение задачи A_1 *(* A_2*) существует.*

Доказательство. Решение задачи A_1 (A_2) ищем в виде

$$u\left(x,y\right)=\sum_{n=1}^{\infty}X_n\left(x\right)Y_n\left(y\right). \tag{4.1}$$

Подставляя (4.1) в уравнение (2.1), получаем по $X(x)$ следующую задачу:

$$\begin{cases} X^{IV} - \lambda X = 0, \\ X(0) = X'(0) = X''(p) = X'''(p) = 0. \end{cases} \tag{4.2}$$

Собственные значение задачи (4.2) [11],

$$\lambda_n = \left(\frac{\pi}{p}\left(n - \frac{1}{2} + (-1)^n O\left(\frac{1}{n^2}\right)\right)\right)^4 \quad n \in N,$$

соответствующая собственная функция имеет вид [10]:

$$X_n(x) = \begin{cases} \dfrac{1}{\sqrt{p}\left|cth\left(\dfrac{\sqrt[4]{\lambda_n}p}{2}\right)\right|}\left(\dfrac{ch\left(\sqrt[4]{\lambda_n}\left(x-\dfrac{p}{2}\right)\right)}{sh\left(\dfrac{\sqrt[4]{\lambda_n}p}{2}\right)} + \dfrac{\sin\left(\sqrt[4]{\lambda_n}\left(x-\dfrac{p}{2}\right)\right)}{\cos\left(\dfrac{\sqrt[4]{\lambda_n}p}{2}\right)}\right), & n = 2m-1, \\ \dfrac{1}{\sqrt{p}\left|th\left(\dfrac{\sqrt[4]{\lambda_n}p}{2}\right)\right|}\left(-\dfrac{sh\left(\sqrt[4]{\lambda_n}\left(x-\frac{p}{2}\right)\right)}{ch\left(\dfrac{\sqrt[4]{\lambda_n}p}{2}\right)} + \dfrac{\cos\left(\sqrt[4]{\lambda_n}\left(x-\dfrac{p}{2}\right)\right)}{\sin\left(\dfrac{\sqrt[4]{\lambda_n}p}{2}\right)}\right), & n = 2m. \end{cases} \tag{4.3}$$

Известно [11], что система собственных функций (4.3) задачи (4.2) является полной и ортонормированной в $L_2(0,p)$.

Учитывая условия (2.3), приходим к следующей задаче по $Y(y)$ для A_1:

$$\begin{cases} Y''' - (a+\lambda_n)Y = g_n(y) \\ Y'(0) = \psi_{1n}, \quad Y''(0) = \psi_{2n}, \quad \lim\limits_{y\to+\infty} Y''(y) = 0 \end{cases} \tag{4.4}$$

здесь

$$\psi_{in} = \int_0^p \psi_i(x) X_n(x)\,dx, \quad i = 1,2,$$

$$g_n(y) = \begin{cases} \dfrac{1}{\sqrt{p}\left|\mathrm{cth}\left(\dfrac{\sqrt[4]{\lambda_n}p}{2}\right)\right|}\displaystyle\int_0^p g(x,y)\left(\dfrac{\mathrm{ch}\left(\sqrt[4]{\lambda_n}\left(x-\dfrac{p}{2}\right)\right)}{\mathrm{sh}\left(\dfrac{\sqrt[4]{\lambda_n}p}{2}\right)} + \dfrac{\sin\left(\sqrt[4]{\lambda_n}\left(x-\dfrac{p}{2}\right)\right)}{\cos\left(\dfrac{\sqrt[4]{\lambda_n}p}{2}\right)}\right)dx, & n = 2m-1; \\ \dfrac{1}{\sqrt{p}\left|\mathrm{th}\left(\dfrac{\sqrt[4]{\lambda_n}p}{2}\right)\right|}\displaystyle\int_0^p g(x,y)\left(-\dfrac{\mathrm{sh}\left(\sqrt[4]{\lambda_n}\left(x-\dfrac{p}{2}\right)\right)}{\mathrm{ch}\left(\dfrac{\sqrt[4]{\lambda_n}p}{2}\right)} + \dfrac{\cos\left(\sqrt[4]{\lambda_n}\left(x-\dfrac{p}{2}\right)\right)}{\sin\left(\dfrac{\sqrt[4]{\lambda_n}p}{2}\right)}\right)dx, & n = 2m. \end{cases}$$

Интегрируя по частям $g_n(y)$ и учитывая второе условие теоремы 4.1, имеем:

$$g_n(y) = \frac{1}{\sqrt[4]{\lambda_n}\sqrt{p}\left|\operatorname{cth}\left(\frac{\sqrt[4]{\lambda_n}p}{2}\right)\right|} g(x,y)\left(\frac{\operatorname{sh}\sqrt[4]{\lambda_n}\left(x-\frac{p}{2}\right)}{\operatorname{sh}\left(\frac{\sqrt[4]{\lambda_n}p}{2}\right)} - \frac{\cos\left(\sqrt[4]{\lambda_n}\left(x-\frac{p}{2}\right)\right)}{\cos\left(\frac{\sqrt[4]{\lambda_n}p}{2}\right)}\right)\Bigg|_{x=0}^{x=p} -$$

$$- \frac{1}{\sqrt[4]{\lambda_n}\sqrt{p}\left|\operatorname{cth}\left(\frac{\sqrt[4]{\lambda_n}p}{2}\right)\right|}\int_0^p g_x(x,y)\left(\frac{\operatorname{sh}\sqrt[4]{\lambda_n}\left(x-\frac{p}{2}\right)}{\operatorname{sh}\left(\frac{\sqrt[4]{\lambda_n}p}{2}\right)} - \frac{\cos\left(\sqrt[4]{\lambda_n}\left(x-\frac{p}{2}\right)\right)}{\cos\left(\frac{\sqrt[4]{\lambda_n}p}{2}\right)}\right)dx =$$

$$= \frac{2\sqrt{p}}{\pi(2n-1)}F_n(y), \quad n = 2m-1;$$

$$g_n(y) = \frac{1}{\sqrt[4]{\lambda_n}\sqrt{p}\left|\operatorname{th}\left(\frac{\sqrt[4]{\lambda_n}p}{2}\right)\right|} g(x,y)\left(-\frac{\operatorname{ch}\sqrt[4]{\lambda_n}\left(x-\frac{p}{2}\right)}{\operatorname{ch}\left(\frac{\sqrt[4]{\lambda_n}p}{2}\right)} + \frac{\sin\left(\sqrt[4]{\lambda_n}\left(x-\frac{p}{2}\right)\right)}{\sin\left(\frac{\sqrt[4]{\lambda_n}p}{2}\right)}\right)\Bigg|_{x=0}^{x=p} -$$

$$- \frac{1}{\sqrt[4]{\lambda_n}\sqrt{p}\left|\operatorname{th}\left(\frac{\sqrt[4]{\lambda_n}p}{2}\right)\right|}\int_0^p g_x(x,y)\left(-\frac{\operatorname{ch}\sqrt[4]{\lambda_n}\left(x-\frac{p}{2}\right)}{\operatorname{ch}\left(\frac{\sqrt[4]{\lambda_n}p}{2}\right)} + \frac{\sin\left(\sqrt[4]{\lambda_n}\left(x-\frac{p}{2}\right)\right)}{\sin\left(\frac{\sqrt[4]{\lambda_n}p}{2}\right)}\right)dx =$$

$$= \frac{2\sqrt{p}}{\pi(2n-1)}F_n(y), \quad n = 2m.$$

где

$$F_n(y) = -\frac{1}{\left|cth\left(\frac{\sqrt[4]{\lambda_n}p}{2}\right)\right|}\int_0^p g_x(x,y)\left(\frac{sh\sqrt[4]{\lambda_n}\left(x-\frac{p}{2}\right)}{sh\left(\frac{\sqrt[4]{\lambda_n}p}{2}\right)} - \frac{\cos\left(\sqrt[4]{\lambda_n}\left(x-\frac{p}{2}\right)\right)}{\cos\left(\frac{\sqrt[4]{\lambda}p}{2}\right)}\right)dx, \quad n = 2m-1;$$

$$F_n(y) = -\frac{1}{\left|th\left(\frac{\sqrt[4]{\lambda_n}p}{2}\right)\right|}\int_0^p g_x(x,y)\left(-\frac{ch\sqrt[4]{\lambda_n}\left(x-\frac{p}{2}\right)}{ch\left(\frac{\sqrt[4]{\lambda_n}p}{2}\right)} + \frac{\sin\left(\sqrt[4]{\lambda_n}\left(x-\frac{p}{2}\right)\right)}{\sin\left(\frac{\sqrt[4]{\lambda}p}{2}\right)}\right)dx, \quad n = 2m.$$

В последующих выкладках символ M будет использоваться для обозначения максимума из всех положительных известных величин, фигурирующих в оценках. Из $g_n(y)$ получим оценки

$$|g_n(y)| \le \frac{M}{n}|F_n(y).| \tag{4.5}$$

Интегрируя по частям $\psi_1(x)$ и $\psi_2(x)$ пять раза и учитывая первое условие теоремы 4.1, получим оценки:

$$|\psi_{in}| \le \frac{M}{n^5}|\Psi_{in}|, \ i = 1,2, \tag{4.6}$$

где

$$\Psi_{in} = \frac{1}{\sqrt{p}\left|cth\left(\frac{\sqrt[4]{\lambda_n}p}{2}\right)\right|}\int_0^p \psi_i^{(5)}(\eta)\left(\frac{sh\left(\sqrt[4]{\lambda_n}\left(x-\frac{p}{2}\right)\right)}{sh\left(\frac{\sqrt[4]{\lambda_n}p}{2}\right)} - \frac{\cos\left(\sqrt[4]{\lambda_n}\left(x-\frac{p}{2}\right)\right)}{\cos\left(\frac{\sqrt[4]{\lambda_n}p}{2}\right)}\right)dx, \quad i = 1,2; \ n = 2m-1;$$

$$\Psi_{in}=\frac{1}{\sqrt{p}\left|th\left(\frac{\sqrt[4]{\lambda_n}p}{2}\right)\right|}\int\limits_0^p{\psi_i}^{(5)}(\eta)\left(-\frac{ch\left(\sqrt[4]{\lambda_n}\left(x-\frac{p}{2}\right)\right)}{ch\left(\frac{\sqrt[4]{\lambda_n}p}{2}\right)}-\frac{\sin\left(\sqrt[4]{\lambda_n}\left(x-\frac{p}{2}\right)\right)}{\sin\left(\frac{\sqrt[4]{\lambda_n}p}{2}\right)}\right)dx,\quad i=1,2;\ \ n=2m.$$

Для нахождения решения задачи (4.4) применим функций Грина. С этой целью введем вспомогательную функцию

$$V(y)=Y(y)-\rho(y), \tag{4.7}$$

которая позволяет свести исходные граничные условия задачи (4.4) к однородному виду. При этом выражение $\rho(y)$ задается следующим образом:

$$\rho_n(y)=-\psi_{1n}e^{-y}-\frac{\psi_{1n}+\psi_{2n}}{2}e^{-y^2}. \tag{4.8}$$

Подставляя (4.7), (4.8) в (4.4), приходим к следующей задаче

$$\begin{cases}V'''-(a+\lambda_n)V=\lambda_n f_n(y),\\ V'(0)=V''(0)=\lim\limits_{y\to+\infty}V''(y)=0,\end{cases} \tag{4.9}$$

здесь

$$f_n(y)=\left(\frac{a+\lambda_n-4y^3+6y-1}{\lambda_n}e^{-y}+\frac{(a+\lambda_n)\left(4y^3-6y\right)}{2\lambda_n}e^{-y^2}\right)\psi_{1n}+\frac{(a+\lambda_n-1)\left(4y^3-6y\right)}{\lambda_n}e^{-y^2}\psi_{2n}+\frac{g_n(y)}{\lambda_n}.$$

С учётом (4.5) и (4.6) получаем следующие оценки

$$\begin{aligned}&|f_n(y)|\le\frac{M}{n^5}\left(\left(e^{-y}+e^{-y^2}\right)|\Psi_{1n}|+e^{-y^2}|\Psi_{2n}|+|F_n(y)|\right),\\ &|f_n(0)|\le\frac{M}{n^5}\left(|\Psi_{1n}|+|\Psi_{2n}|+|F_n(0)|\right),\\ &\left|{f_n}'(y)\right|\le\frac{M}{n^5}\left(e^{-y}|\Psi_{1n}|+\left(|\Psi_{1n}|+|\Psi_{2n}|\right)e^{-y^2}+{F'}_n(y)\right),\end{aligned} \tag{4.10}$$

Решение задачи (4.9) имеет следующий вид:

$$V_n(y)=\lambda_n\int\limits_0^{+\infty}G_n(y,\eta)\,f_n(\eta)\,d\eta \tag{4.11}$$

где функция $G_n(y,\eta)$ является функцией Грина задачи (4.9) и принадлежит следующему классу $C_b^1[0,+\infty)\cap C_b^3(0,+\infty)$ где $b=\lim\limits_{y\to+\infty}y$, [23].

Функция Грина задачи (4.9) удовлетворяет следующему уравнению:

$$G_{nyyy}(y,\eta)-(a+\lambda_n)G_n(y,\eta)=0, \tag{4.12}$$

а также следующим условиям:

$$G_{1ny}(0,\eta)=G_{1nyy}(0,\eta)=\lim_{y\to+\infty}G_{2nyy}(y,\eta)=0, \tag{4.13}$$

$$\begin{cases}G_{2n}(\eta,\eta)-G_{1n}(\eta,\eta)=0,\\ G_{2ny}(\eta,\eta)-G_{1ny}(\eta,\eta)=0,\\ G_{2nyy}(\eta,\eta)-G_{1nyy}(\eta,\eta)=1.\end{cases} \tag{4.14}$$

Искомую функцию Грина будем искать в следующем виде

$$G_n^+(y,\eta)=\begin{cases}G_{1n}^+(y,\eta),\ 0\le y<\eta,\\ G_{2n}^+(y,\eta),\ \eta\le y<\infty.\end{cases}\tag{4.15}$$

здесь $k_n=\sqrt[3]{\lambda_n+a}$,

$$G_n^+(y,\eta)=\begin{cases}-\dfrac{1}{3k_n^2}\left(e^{k_n(y-\eta)}+2e^{-k_n\left(\eta+\frac{y}{2}\right)}\cos\left(\dfrac{\sqrt{3}}{2}k_ny\right)\right),\quad 0\le y<\eta,\\ \\ -\dfrac{2e^{-k_n\left(\eta+\frac{y}{2}\right)}}{3k_n^2}\cos\left(\dfrac{\sqrt{3}}{2}k_ny\right)-\dfrac{2e^{\frac{k_n}{2}(\eta-y)}}{3k_n^2}\sin\left(\dfrac{\sqrt{3}}{2}k_n(y-\eta)+\dfrac{\pi}{6}\right),\ \eta\le y<\infty\end{cases}$$

Непосредственная проверка показывает, что заданная соотношением (4.15) функция в полной мере удовлетворяет всем критериям, заложенным в определении функции Грина. Далее перейдём к оценке функции $G_n^+(\xi,y)$,

$$\left|G_{1n}^+(y,\eta)\right|\le\frac{1}{3k_n^2}\left(e^{k_n(y-\eta)}+2e^{-k_n\left(\eta+\frac{y}{2}\right)}\right)\le\frac{M}{k_n^2},$$

$$\left|G_{2n}^+(y,\eta)\right|\le\frac{1}{3k_n^2}\left(2\,e^{-k_n\left(\eta+\frac{y}{2}\right)}+2e^{\frac{k_n}{2}(\eta-y)}\right)\le\frac{M}{k_n^2}.$$

Из этого следует следующая оценка,

$$\left|G_{n\eta}^+(y,\eta)\right|\le\frac{M}{k_n},\qquad\left|G_{n\eta\eta}^+(y,\eta)\right|\le M.\tag{4.16}$$

Следовательно, общее решение задачи A_1 имеет следующий вид:

$$u(x,y)=\sum_{n=1}^{\infty}\left(\rho_n(y)+\lambda_n\int_0^{+\infty}G_n(y,\eta)\,f_n(\eta)\,d\eta\right)X_n(x).\tag{4.17}$$

С учётом (4.6) оценим (4.8),

$$|\rho_n(y)|\le\frac{M}{n^5}\left(\left(e^{-y}+e^{-y^2}\right)|\Psi_{1n}|+e^{-y^2}|\Psi_{2n}|\right).\tag{4.18}$$

Учитывая (4.10), оценим (4.11), для этой цели воспользуемся следующим соотношением

$$G_n(y,\eta)=\frac{1}{a+\lambda_n}G_{nyyy}(y,\eta),$$

$$V_n(y)=\lambda_n\int_0^{+\infty}G_n(y,\eta)\,f_n(\eta)\,d\eta=\frac{\lambda_n}{a+\lambda_n}\left(\int_0^{\eta-\varepsilon}G_{n\eta\eta\eta}(y,\eta)\,f_n(\eta)\,d\eta+\right.$$
$$\left.+\int_{\eta+\varepsilon}^{\nu}G_{n\eta\eta\eta}(y,\eta)\,f_n(\eta)\,d\eta+\int_{\nu}^{+\infty}G_{n\eta\eta\eta}(y,\eta)\,f_n(\eta)\,d\eta\right).$$

Для функции Грина $G_n(y,\eta)$ точка $y=\eta$ является точкой разрыва первого рода. В то же время данная функция сохраняет непрерывность в точке $y=\nu$ для любого $\nu\in[0,+\infty)$.

$$V_n(y)=\frac{\lambda_n}{a+\lambda_n}\left(\lim_{\varepsilon\to0}\left(f_n(\eta-\varepsilon)\,G_{n\eta\eta}(y,\eta-\varepsilon)-f_n(0)\,G_{n\eta\eta}(y,0)-\int_0^{\eta-\varepsilon}G_{n\eta\eta}(y,\eta)\,f_n{}'(\eta)\,d\eta\right)+\right.$$
$$+\tfrac{\lambda_n}{a+\lambda_n}\lim_{\varepsilon\to0}\left(f_n(\nu)\,G_{n\eta\eta}(y,\nu)-f_n(\eta+\varepsilon)\,G_{n\eta\eta}(y,\eta+\varepsilon)-\int_{\eta+\varepsilon}^{\nu}G_{n\eta\eta}(y,\eta)\,f_n{}'(\eta)\,d\eta\right)\Bigg)+$$
$$\left.+\tfrac{\lambda_n}{a+\lambda_n}\left(\lim_{B\to+\infty}\left(f_n(B)\,G_{n\eta\eta}(y,B)-f_n(\nu)\,G_{n\eta\eta}(y,\nu)-\int_{\nu}^{B}G_{n\eta\eta}(y,\eta)\,f_n{}'(\eta)\,d\eta\right)\right.\right.$$

Учитывая (4.10) и (4.16), получим оценку для $V_n(y)$,

$$|V_n(y)| \le \frac{M}{n^5}\left(|\Psi_{1n}| + |\Psi_{2n}|\right). \tag{4.19}$$

Принимая во внимание (4.18) и (4.19) оценим (4.17) в виде,

$$|u(x,y)| \le \sum_{n=1}^{\infty}\left(|\rho_n(y)| + |V_n(y)|\right)|X_n(x)| \le \frac{M}{n^5}\left(|\Psi_{1n}| + |\Psi_{2n}|\right) < \infty.$$

Оценка собственной функции имеет вид: $|X_n(x)| \le M, \quad 0 < M = const.$

$$|X_n(x)| \le \frac{1}{\sqrt{p}\left|cth\left(\frac{\sqrt[4]{\lambda_n}p}{2}\right)\right|}\left(\frac{\left|ch\frac{\sqrt[4]{\lambda_n}p}{2}\right|}{\left|sh\frac{\sqrt[4]{\lambda_n}p}{2}\right|} + \frac{1}{\left|\cos\left(\frac{\sqrt[4]{\lambda_n}p}{2}\right)\right|}\right) \le \frac{1}{\sqrt{p}} + \frac{1}{\sqrt{p}\left|cth\left(\frac{\sqrt[4]{\lambda_n}p}{2}\right)\right|\left|\cos\left(\frac{\sqrt[4]{\lambda_n}p}{2}\right)\right|},$$

Следовательно, для $n = 2m-1$ имеем:

$$\lim_{n\to\infty}\left|\cos\frac{\sqrt[4]{\lambda_n}p}{2}\right| = \lim_{m\to\infty}\left|\cos\frac{\pi(4m-3)}{2p}\cdot\frac{p}{2}\right| = \lim_{m\to\infty}\left|\cos\left(\pi m - \frac{3\pi}{4}\right)\right| = \left|\cos\frac{3\pi}{4}\right| = \frac{\sqrt{2}}{2}.$$

Отсюда при $n = 2m$ получаем:

$$|X_n(x)| \le \frac{1}{\sqrt{p}\left|th\left(\frac{\sqrt[4]{\lambda_n}p}{2}\right)\right|}\left(\frac{\left|sh\frac{\sqrt[4]{\lambda_n}p}{2}\right|}{\left|ch\frac{\sqrt[4]{\lambda_n}p}{2}\right|} + \frac{1}{\left|\sin\left(\frac{\sqrt[4]{\lambda_n}p}{2}\right)\right|}\right) \le \frac{1}{\sqrt{p}} + \frac{1}{\sqrt{p}\left|th\left(\frac{\sqrt[4]{\lambda_n}p}{2}\right)\right|\left|\sin\left(\frac{\sqrt[4]{\lambda_n}p}{2}\right)\right|},$$

$$\lim_{n\to\infty}\left|\sin\frac{\sqrt[4]{\lambda_n}p}{2}\right| = \lim_{m\to\infty}\left|\sin\frac{\pi(4m-1)}{2p}\cdot\frac{p}{2}\right| = \lim_{m\to\infty}\left|\sin\left(\pi m - \frac{\pi}{4}\right)\right| = \left|\sin\frac{\pi}{4}\right| = \frac{\sqrt{2}}{2}.$$

Теперь оценим $u_{yyy}(x,y)$,

$$u_{yyy}(x,y) = \sum_{n=1}^{\infty}\left(\rho'''_n(y) + \lambda_n\int_0^{+\infty}G_{n\eta\eta\eta}(y,\eta)\,f_n(\eta)\,d\eta\right)X_n(x). \tag{4.20}$$

$\rho'''(y)$ имеет следующий вид,

$$\rho'''(y) = \psi_{1n}e^{-y} + 2\left(\psi_{1n} + \psi_{2n}\right)\left(2y^3 - 3y\right)e^{-y^2}. \tag{4.21}$$

Исходя из (4.6) оценим (4.21),

$$|\rho'''(y)| \le \frac{M}{n^5}\left(|\Psi_{1n}| + |\Psi_{2n}|\right). \tag{4.22}$$

Тогда $V'''(y)$ имеет следующий вид,

$$V_n'''(y)=\lambda_n\int\limits_0^{+\infty}G_{n\eta\eta\eta}(y,\eta)\,f_n(\eta)\,d\eta=\lambda_n\left(\lim_{\varepsilon\to0}f_n(\eta)\,G_{n\eta\eta}(y,\eta)\Big|_0^{\eta-\varepsilon}-\int\limits_0^{\eta-\varepsilon}G_{n\eta\eta}(y,\eta)\,f_n'(\eta)\,d\eta+\right.$$
$$\left.+\int\limits_{\eta+\varepsilon}^{+\infty}G_{n\eta\eta}(y,\eta)\,f_n'(\eta)\,d\eta\right)=\lambda_n\left(\lim_{\varepsilon\to0}\left(f_n(\eta-\varepsilon)\,G_{n\eta\eta}(y,\eta-\varepsilon)-f_n(0)\,G_{n\eta\eta}(y,0)\right)-\right.$$
$$\left.-\int\limits_0^{\eta-\varepsilon}G_{n\eta\eta}(y,\eta)\,f_n'(\eta)\,d\eta+\int\limits_{\eta+\varepsilon}^{\nu}G_{n\eta\eta}(y,\eta)\,f_n'(\eta)\,d\eta+\int\limits_{\nu}^{+\infty}G_{n\eta\eta}(y,\eta)\,f_n'(\eta)\,d\eta\right)=$$
$$=\lambda_n\lim_{\varepsilon\to0}\left(\left(f_n(\eta-\varepsilon)\,G_{n\eta\eta}(y,\eta-\varepsilon)-f_n(0)\,G_{n\eta\eta}(y,0)\right)-\int\limits_0^{\eta-\varepsilon}G_{n\eta\eta}(y,\eta)\,f_n'(\eta)\,d\eta\right)+$$
$$+\lambda_n\left(\lim_{\varepsilon\to0}\left(\left(f_n(\nu)\,G_{n\eta\eta}(y,\nu)-f_n(\eta+\varepsilon)\,G_{n\eta\eta}(y,\eta+\varepsilon)\right)-\int\limits_{\eta+\varepsilon}^{c}G_{n\eta\eta}(y,\eta)\,f_n'(\eta)\,d\eta\right)\right)+$$
$$+\lambda_n\left(\lim_{B\to+\infty}\left(f_n(B)\,G_{n\eta\eta}(y,B)-f_n(\nu)\,G_{n\eta\eta}(y,\nu)\right)-\int\limits_{\nu}^{B}G_{n\eta\eta}(y,\eta)\,f_n'(\eta)\,d\eta\right).$$

Учитывая (4.10) и (4.16) оценим $V'''(y)$,

$$\left|V_n'''(y)\right|\le\frac{1}{n}\left(|\Psi_{1n}|+|\Psi_{2n}|\right). \tag{4.23}$$

На основаним (4.22) и (4.23) оценим (4.20) в виде

$$|u_{yyy}(x,y)|\le M\sum_{n=1}^{\infty}\frac{1}{n}\left(|\Psi_{1n}|+|\Psi_{2n}|\right).$$

Используя неравенство Коши-Буняковского и Бесселя, получим

$$|u_{yyy}(x,y)|\le M\sqrt{\sum_{n=1}^{\infty}\frac{1}{n^2}}\left(\sqrt{\sum_{n=1}^{\infty}|\Psi_{1n}|^2}+\sqrt{\sum_{n=1}^{\infty}|\Psi_{2n}|^2}\right)\le M\sqrt{\frac{\pi^2}{6}}\left(\left\|\psi_1^{(5)}(x)\right\|+\left\|\psi_2^{(5)}(x)\right\|\right)<\infty.$$

так как

$$\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6},\quad\sum_{n=1}^{\infty}|\Psi_{in}|^2\le\left\|\psi_i^{(5)}(x)\right\|^2_{L_2(0<x<p)},\quad i=1,2. \tag{4.24}$$

Следовательно, ряд, соответствующий функции $u_{yyy}(x,y)$, сходится абсолютная и равномерно в области D^+. Из уравнения (4.17) найдём $u_{xxxx}(x,y)$ и также покажем его сходимость,

$$|u_{xxxx}(x,y)|\le\frac{1}{\sqrt{p}}\sum_{n=1}^{\infty}\frac{\left(|\rho_n(y)|+\int_0^{+\infty}|G_n(y,\eta)|\,|f_n(\eta)|\,d\eta\right)}{\left|\operatorname{cth}\left(\frac{\sqrt[4]{\lambda_n}p}{2}\right)\right|\left(\frac{\pi}{2p}(2n-1)\right)^4}\times$$
$$\times\left|\frac{\operatorname{ch}\left(\sqrt[4]{\lambda_n}\left(x-\frac{p}{2}\right)\right)}{\operatorname{sh}\left(\frac{\sqrt[4]{\lambda_n}p}{2}\right)}+\frac{\sin\left(\sqrt[4]{\lambda_n}\left(x-\frac{p}{2}\right)\right)}{\cos\left(\frac{\sqrt[4]{\lambda_n}p}{2}\right)}\right|\le M\sum_{n=1}^{\infty}\frac{1}{n}\left(|\Psi_{1n}|+|\Psi_{2n}|\right),\quad n=2m-1,$$

$$|u_{xxxx}(x,y)| \le \frac{1}{\sqrt{p}} \sum_{n=1}^{\infty} \frac{\left(|\rho_n(y)| + \int_0^{+\infty} |G_n(y,\eta)|\,|f_n(\eta)|\,d\eta\right)}{\left|\text{th}\left(\frac{\sqrt[4]{\lambda_n}p}{2}\right)\right| \left(\frac{\pi}{2p}(2n-1)\right)^4} \times$$

$$\times \left| -\frac{\text{sh}\left(\sqrt[4]{\lambda_n}\left(x-\frac{p}{2}\right)\right)}{\text{ch}\left(\frac{\sqrt[4]{\lambda_n}p}{2}\right)} + \frac{\cos\left(\sqrt[4]{\lambda_n}\left(x-\frac{p}{2}\right)\right)}{\sin\left(\frac{\sqrt[4]{\lambda_n}p}{2}\right)} \right| \le M \sum_{n=1}^{\infty} \frac{1}{n} \left(|\Psi_{1n}| + |\Psi_{2n}|\right), \quad n = 2m.$$

Используя неравенство Коши-Буняковского и Бесселя, получим

$$|u_{xxxx}(x,y)| \le M \sqrt{\sum_{n=1}^{\infty} \frac{1}{n^2}} \left(\sqrt{\sum_{n=1}^{\infty} |\Psi_{1n}|^2} + \sqrt{\sum_{n=1}^{\infty} |\Psi_{2n}|^2} \right) \le \sqrt{\frac{\pi^2}{6}} \left(\left\| \psi_1^{(5)}(x) \right\| + \left\| \psi_2^{(5)}(x) \right\| \right) < \infty.$$

Здесь (4.24).

Следовательно, $u_{xxxx}(x)$ также сходится абсолютно и равномерно в области D^+.

Теперь учитывая условия (2.4), приходим к следующей задаче по $Y(y)$ для A_2:

$$\begin{cases} Y''' - (a + \lambda_n) Y = g_n(y) \\ Y''(0) = \psi_{3n}, \quad \lim\limits_{y \to -\infty} Y'(y) = \lim\limits_{y \to -\infty} Y''(y) = 0 \end{cases} \tag{4.25}$$

здесь

$$\psi_{3n} = \int_0^p \psi_3(x) X_n(x)\,dx,$$

$$g_n(y) = \int_0^p g(x,y) X_n(x)\,dx$$

Интегрируя по частям $g_n(y)$ и учитывая второе условие теоремы 4.1, имеем:

$$g_n(y) = \frac{1}{\sqrt{p}\left|cth\left(\frac{\sqrt[4]{\lambda_n}p}{2}\right)\right|} \int_0^p g(x,y) \left(\frac{ch\left(\sqrt[4]{\lambda_n}\left(x-\frac{p}{2}\right)\right)}{sh\left(\frac{\sqrt[4]{\lambda_n}p}{2}\right)} + \frac{\sin\left(\sqrt[4]{\lambda_n}\left(x-\frac{p}{2}\right)\right)}{\cos\left(\frac{\sqrt[4]{\lambda_n}p}{2}\right)} \right) dx =$$

$$= \frac{1}{\sqrt[4]{\lambda_n}\sqrt{p}\left|cth\left(\frac{\sqrt[4]{\lambda_n}p}{2}\right)\right|} g(x,y) \left(\frac{sh\sqrt[4]{\lambda_n}\left(x-\frac{p}{2}\right)}{sh\left(\frac{\sqrt[4]{\lambda_n}p}{2}\right)} - \frac{\cos\left(\sqrt[4]{\lambda_n}\left(x-\frac{p}{2}\right)\right)}{\cos\left(\frac{\sqrt[4]{\lambda_n}p}{2}\right)} \right) \Bigg|_{x=0}^{x=p} -$$

$$- \frac{1}{\sqrt[4]{\lambda_n}\sqrt{p}\left|cth\left(\frac{\sqrt[4]{\lambda_n}p}{2}\right)\right|} \cdot \int_0^p g_x(x,y) \left(\frac{sh\sqrt[4]{\lambda_n}\left(x-\frac{p}{2}\right)}{sh\left(\frac{\sqrt[4]{\lambda_n}p}{2}\right)} - \frac{\cos\left(\sqrt[4]{\lambda_n}\left(x-\frac{p}{2}\right)\right)}{\cos\left(\frac{\sqrt[4]{\lambda_n}p}{2}\right)} \right) dx = \frac{2\sqrt{p}}{\pi(2n-1)} F_n(y); \quad n = 2m-1,$$

где

$$F_n(y) = -\frac{1}{\left|cth\left(\frac{\sqrt[4]{\lambda_n}p}{2}\right)\right|} \int_0^p g_x(x,y) \left(\frac{sh\sqrt[4]{\lambda_n}\left(x-\frac{p}{2}\right)}{sh\left(\frac{\sqrt[4]{\lambda_n}p}{2}\right)} - \frac{\cos\left(\sqrt[4]{\lambda_n}\left(x-\frac{p}{2}\right)\right)}{\cos\left(\frac{\sqrt[4]{\lambda}p}{2}\right)} \right) dx, \quad n = 2m-1.$$

$$g_n(y)=\frac{1}{\sqrt{p}\left|th\left(\frac{\sqrt[4]{\lambda_n}p}{2}\right)\right|}\int_0^p g(x,y)\left(-\frac{sh\left(\sqrt[4]{\lambda_n}\left(x-\frac{p}{2}\right)\right)}{ch\left(\frac{\sqrt[4]{\lambda_n}p}{2}\right)}+\frac{\cos\left(\sqrt[4]{\lambda_n}\left(x-\frac{p}{2}\right)\right)}{\sin\left(\frac{\sqrt[4]{\lambda_n}p}{2}\right)}\right)dx=$$

$$=\frac{1}{\sqrt[4]{\lambda_n}\sqrt{p}\left|th\left(\frac{\sqrt[4]{\lambda_n}p}{2}\right)\right|}g(x,y)\left(-\frac{ch\sqrt[4]{\lambda_n}\left(x-\frac{p}{2}\right)}{sh\left(\frac{\sqrt[4]{\lambda_n}p}{2}\right)}+\frac{\sin\left(\sqrt[4]{\lambda_n}\left(x-\frac{p}{2}\right)\right)}{\sin\left(\frac{\sqrt[4]{\lambda_n}p}{2}\right)}\right)\Bigg|_{x=0}^{x=p}-$$

$$-\frac{1}{\sqrt[4]{\lambda_n}\sqrt{p}\left|th\left(\frac{\sqrt[4]{\lambda_n}p}{2}\right)\right|}\cdot\int_0^p g_x(x,y)\left(-\frac{ch\sqrt[4]{\lambda_n}\left(x-\frac{p}{2}\right)}{ch\left(\frac{\sqrt[4]{\lambda_n}p}{2}\right)}+\frac{\sin\left(\sqrt[4]{\lambda_n}\left(x-\frac{p}{2}\right)\right)}{\sin\left(\frac{\sqrt[4]{\lambda_n}p}{2}\right)}\right)dx=\frac{2\sqrt{p}}{\pi(2n-1)}F_n(y),\quad n=2m.$$

где

$$F_n(y)=-\frac{1}{\left|th\left(\frac{\sqrt[4]{\lambda_n}p}{2}\right)\right|}\int_0^p g_x(x,y)\left(-\frac{ch\sqrt[4]{\lambda_n}\left(x-\frac{p}{2}\right)}{ch\left(\frac{\sqrt[4]{\lambda_n}p}{2}\right)}+\frac{\sin\left(\sqrt[4]{\lambda_n}\left(x-\frac{p}{2}\right)\right)}{\sin\left(\frac{\sqrt[4]{\lambda}p}{2}\right)}\right)dx,\quad n=2m.$$

Из $g_n(y)$ получим оценки

$$|g_n(y)|\le\frac{M}{n}|F_n(y)| \tag{4.26}$$

Интегрируя по частям $\psi_3(x)$ пять раза и учитывая первое условие теоремы 4.1, получим оценки:

$$|\psi_{3n}|\le\frac{M}{n^5}|\Psi_{in}|, \tag{4.27}$$

где

$$\Psi_{3n}=\int_0^p {\psi_3}^{(5)}(x)\,X_n(x)\,dx.$$

Для отыскания решения граничной задачи (4.25) воспользуемся методом функции Грина. С целью гомогенизации (приведения к однородному виду) заданных краевых условий введем новую переменную по правилу:

$$V_1(y)=Y(y)-\rho(y), \tag{4.28}$$

где аналитическая структура функции $\rho(y)$ записывается следующим образом:

$$\rho_{1n}(y)=\psi_{3n}e^y \tag{4.29}$$

Подставляя (4.28), (4.29) в (4.25), приходим к следующей задаче

$$\begin{cases}V'''-(a+\lambda_n)V=\lambda_n f_n(y),\\ V''(0)=\lim\limits_{y\to-\infty}V'(y)=\lim\limits_{y\to-\infty}V''(y)=0,\end{cases} \tag{4.30}$$

здесь

$$f_n(y)=\left(\frac{a+\lambda_n-1}{\lambda_n}\right)e^y\psi_{3n}+\frac{g_n(y)}{\lambda_n}.$$

Принимая во внимание (4.26) и (4.27) получаем следующие оценки

$$\begin{aligned} |f_n(y)| &\le \frac{M}{n^5}\left(e^y|\Psi_{3n}| + |F_n(y)|\right), \\ |f_n(0)| &\le \frac{M}{n^5}\left(|\Psi_{3n}| + |F_n(0)|\right), \\ \left|f_n'(y)\right| &\le \frac{M}{n^5}\left(e^y|\Psi_{3n}| + \left|F_n'(y)\right|\right). \end{aligned} \tag{4.31}$$

Решение задачи (4.30) имеет следующий вид:

$$V_{1n}(y) = \lambda_n \int_{-\infty}^{0} G_n(y,\eta) f_n(\eta)\, d\eta \tag{4.32}$$

где функция $G_n(y,\eta)$ является функцией Грина задачи (4.30) и принадлежит следующему классу $C_b^1(-\infty;0] \cap C_b^3(-\infty;0)$ где $b = \lim\limits_{y\to-\infty} y$.

Функция Грина задачи (4.30) удовлетворяет уравнению (4,12), а также следующим условиям

$$G_{1nyy}(0,\eta) = \lim_{y\to-\infty} G_{2ny}(y,\eta) = \lim_{y\to-\infty} G_{2nyy}(y,\eta) = 0. \tag{4.33}$$

$$\begin{cases} G_{2n}(\eta,\eta) - G_{1n}(\eta,\eta) = 0, \\ G_{2ny}(\eta,\eta) - G_{1ny}(\eta,\eta) = 0, \\ G_{2nyy}(\eta,\eta) - G_{1nyy}(\eta,\eta) = 1. \end{cases} \tag{4.34}$$

Искомую функцию Грина будем искать в следующем виде

$$G_n^-(y,\eta) = \begin{cases} G_{1n}^-(y,\eta), & -\infty < y \le \eta, \\ G_{2n}^-(y,\eta), & \eta < y \le 0. \end{cases} \tag{4.35}$$

$$G_n^-(y,\eta) = \begin{cases} -\dfrac{1}{3k_n^2}\left(e^{k_n(y-\eta)} + 2e^{-k_n\left(\eta+\frac{y}{2}\right)}\cos\left(\dfrac{\sqrt{3}}{2}k_n y\right)\right), \\ -\dfrac{2}{3k_n^2}\left(e^{-k_n\left(\eta+\frac{y}{2}\right)}\cos\left(\dfrac{\sqrt{3}}{2}k_n y\right) + e^{\frac{k_n}{2}(\eta-y)}\sin\left(\dfrac{\sqrt{3}}{2}k_n(y-\eta)+\dfrac{\pi}{6}\right)\right). \end{cases}$$

Можно проверить, что функция, определенная формулой (4.35), обладает всеми свойствами, указанными в определении функции Грина. Теперь оценим $G_n^-(y,\eta)$,

$$\left|G_{1n}^-(y,\eta)\right| \le \frac{1}{3k_n^2}\left(e^{k_n(y-\eta)} + 2e^{-k_n\left(\eta+\frac{y}{2}\right)}\right) \le \frac{M}{k_n^2}$$

$$\left|G_{2n}^-(y,\eta)\right| \le \frac{1}{3k_n^2}\left(2\,e^{-k_n\left(\eta+\frac{y}{2}\right)} + 2e^{\frac{k_n}{2}(\eta-y)}\right) \le \frac{M}{k_n^2}$$

Из этого следует следующая оценка,

$$\left|G_n^-(y,\eta)\right| \le \frac{M}{k_n^2},$$

также

$$\left|G_{n\eta}^-(y,\eta)\right| \le \frac{M}{k_n}, \quad \left|G_{n\eta\eta}^-(y,\eta)\right| \le M, \tag{4.36}$$

Следовательно, общее решение задачи A_2 имеет следующий вид:

$$u(x,y) = \sum_{n=1}^{\infty}\left(\rho_{1n}(y) + \lambda_n \int_{-\infty}^{0} G_n(y,\eta) f_n(\eta)\, d\eta\right) X_n(x). \tag{4.37}$$

С учётом (4.27) оценим (4.29),

$$|\rho_{1n}(y)| \leq \frac{M}{n^5}|\Psi_{3n}|. \tag{4.38}$$

Учитывая (4.31), оценим (4.32), для этой цели воспользуемся следующим соотношением

$$V_{1n}(y) = \lambda_n \int\limits_{-\infty}^{0} G_n(y,\eta) f_n(\eta)\, d\eta = \frac{\lambda_n}{a+\lambda_n} \int\limits_{-\infty}^{0} G_{n\eta\eta\eta}(y,\eta) f_n(\eta)\, d\eta =$$
$$= \frac{\lambda_n}{a+\lambda_n}\left(\int\limits_{-\infty}^{\nu} G_{n\eta\eta\eta}(y,\eta) f_n(\eta)\, d\eta + \lim_{\varepsilon\to 0}\left(\int\limits_{\nu}^{\eta-\varepsilon} G_{n\eta\eta\eta}(y,\eta) f_n(\eta)\, d\eta + \int\limits_{\eta+\varepsilon}^{0} G_{n\eta\eta\eta}(y,\eta) f_n(\eta)\, d\eta\right)\right)$$
$$= \frac{\lambda_n}{a+\lambda_n}\lim_{B\to-\infty}\left(f_n(\nu) G_{n\eta\eta}(y,\nu) - f_n(B) G_{n\eta\eta}(y,B) - \int\limits_{B}^{\nu} G_{n\eta\eta}(y,\eta) f_n{}'(\eta)\, d\eta\right) +$$
$$+\frac{\lambda_n}{a+\lambda_n}\lim_{\varepsilon\to 0}\left(f_n(\eta-\varepsilon) G_{n\eta\eta}(y,\eta-\varepsilon) - f_n(\nu) G_{n\eta\eta}(y,\nu) - \int\limits_{\nu}^{\eta-\varepsilon} G_{n\eta\eta}(y,\eta) f_n{}'(\eta)\, d\eta\right) +$$
$$+\frac{\lambda_n}{a+\lambda_n}\lim_{\varepsilon\to 0}\left(f_n(0) G_{n\eta\eta}(y,0) - f_n(\eta+\varepsilon) G_{n\eta\eta}(y,\eta+\varepsilon) - \int\limits_{\eta+\varepsilon}^{0} G_{n\eta\eta}(y,\eta) f_n{}'(\eta)\, d\eta\right).$$

Для функции Грина $G_n(y,\eta)$ точка $y=\eta$ является точкой разрыва первого рода. В то же время данная функция сохраняет непрерывность в точке $y=\nu$ для любого $\nu \in (-\infty, 0]$

Следовательно

$$|V_{1n}(y)| \leq \frac{M}{n^5}(|\Psi_{3n}|) \tag{4.39}$$

Принимая во внимание (4.38) и (4.39) оценим (4.37) в виде,

$$|u(x,y)| \leq \sum_{n=1}^{\infty}(|\rho_{1n}(y)| + |V_{1n}(y)|)|X_n(x)| \leq \frac{M}{n^5}(|\Psi_{3n}|) < \infty$$

Теперь оценим $u_{yyy}(x,y)$,

$$u_{yyy}(x,y) = \sum_{n=1}^{\infty}\left(\rho_{1n}{}'''(y) + \lambda_n \int\limits_{-\infty}^{0} G_{nyyy}(y,\eta) f_n(\eta)\, d\eta\right) X_n(x). \tag{4.40}$$

Оценим $\rho'''(y)$ с учетом соотношения (4.27),

$$|\rho_{1n}{}'''(y)| \leq \frac{M}{n^5}(|\Psi_{3n}|). \tag{4.41}$$

Тогда $V_{1n}{}'''(y)$ имеет следующий вид,

$$V_{1n}{}'''(y) = \lambda_n \int\limits_{-\infty}^{0} G_{n\eta\eta\eta}(y,\eta) f_n(\eta)\, d\eta =$$
$$= \lambda_n\left(\lim_{B\to-\infty}\left(f_n(\nu) G_{n\eta\eta}(y,\nu) - f_n(B) G_{n\eta\eta}(y,B) - \int\limits_{B}^{\nu} G_{n\eta\eta}(y,\eta) f_n{}'(\eta)\, d\eta\right)\right) +$$
$$+\lambda_n\left(\lim_{\varepsilon\to 0}\left(f_n(\eta-\varepsilon) G_{n\eta\eta}(y,\eta-\varepsilon) - f_n(\nu) G_{n\eta\eta}(y,\nu) - \int\limits_{\nu}^{\eta-\varepsilon} G_{n\eta\eta}(y,\eta) f_n{}'(\eta)\, d\eta\right)\right) +$$
$$++\lambda_n\left(\lim_{\varepsilon\to 0}\left(f_n(0) G_{n\eta\eta}(y,0) - f_n(\eta+\varepsilon) G_{n\eta\eta}(y,\eta+\varepsilon) - \int\limits_{\eta+\varepsilon}^{0} G_{n\eta\eta}(y,\eta) f_n{}'(\eta)\, d\eta\right)\right);$$

На основании (4.31) и (4.36) оценка $V_{1n}{}'''(y)$, имеет вид

$$\left|V_{1n}{}'''(y)\right| \leq \frac{1}{n}(|\Psi_{3n}|). \tag{4.42}$$

С учётом оценок (4.41) и (4.42) оценим (4.40),

$$|u_{yyy}(x,y)| \le M \sum_{n=1}^{\infty} \frac{1}{n} \left(|\Psi_{3n}|\right)$$

Используя неравенство Коши-Буняковского и Бесселя, получим

$$|u_{yyy}(x,y)| \le M \sqrt{\sum_{n=1}^{\infty} \frac{1}{n^2}} \left(\sqrt{\sum_{n=1}^{\infty} |\Psi_{3n}|^2} \right) \le M \sqrt{\frac{\pi^2}{6}} \left(\left\| \psi_3^{(5)}(x) \right\| \right) < \infty,$$

здесь

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}, \ \sum_{n=1}^{\infty} |\Psi_{3n}|^2 \le \left\| \psi_{3n}^{(5)}(x) \right\|^2_{L_2(0<x<p)}. \tag{4.43}$$

Следовательно, ряд, соответствующий функции $u_{yyy}(x,y)$, сходится абсолютная и равномерно в области D^-.

Вычислив частную производную четвертого порядка по x от уравнения (4.37), докажем её сходимость,

$$|u_{xxxx}(x,y)| \le \frac{1}{\sqrt{p}} \sum_{n=1}^{\infty} \frac{\left(|\rho_{1n}(y)| + \int_{-\infty}^{0} |G_n(y,\eta)| \, |f_n(\eta)| \, d\eta \right)}{\left| \operatorname{cth}\left(\frac{\sqrt[4]{\lambda_n} p}{2} \right) \right| \left(\frac{\pi}{2p}(2n-1) \right)^4} \times$$

$$\times \left| \frac{\operatorname{ch}\left(\sqrt[4]{\lambda_n} \left(x - \frac{p}{2} \right) \right)}{\operatorname{sh}\left(\frac{\sqrt[4]{\lambda_n} p}{2} \right)} + \frac{\sin\left(\sqrt[4]{\lambda_n} \left(x - \frac{p}{2} \right) \right)}{\cos\left(\frac{\sqrt[4]{\lambda_n} p}{2} \right)} \right| \le M \sum_{n=1}^{\infty} \frac{1}{n} \left| \Psi_{3n} \right|, \quad n = 2m-1;$$

$$|u_{xxxx}(x,y)| \le \frac{1}{\sqrt{p}} \sum_{n=1}^{\infty} \frac{\left(|\rho_{1n}(y)| + \int_{-\infty}^{0} |G_n(y,\eta)| \, |f_n(\eta)| \, d\eta \right)}{\left| \operatorname{th}\left(\frac{\sqrt[4]{\lambda_n} p}{2} \right) \right| \left(\frac{\pi}{2p}(2n-1) \right)^4} \times$$

$$\times \left| -\frac{\operatorname{sh}\left(\sqrt[4]{\lambda_n} \left(x - \frac{p}{2} \right) \right)}{\operatorname{ch}\left(\frac{\sqrt[4]{\lambda_n} p}{2} \right)} + \frac{\cos\left(\sqrt[4]{\lambda_n} \left(x - \frac{p}{2} \right) \right)}{\sin\left(\frac{\sqrt[4]{\lambda_n} p}{2} \right)} \right| \le M \sum_{n=1}^{\infty} \frac{1}{n} \left| \Psi_{3n} \right|, \quad n = 2m.$$

Используя неравенство Коши-Буняковского и Бесселя, получим

$$|u_{xxxx}(x,y)| \le M \sqrt{\sum_{n=1}^{\infty} \frac{1}{n^2}} \left(\sqrt{\sum_{n=1}^{\infty} |\Psi_{3n}|^2} \right) \le M \sqrt{\frac{\pi^2}{6}} \left(\left\| \psi_3^{(5)}(x) \right\| \right) < \infty.$$

Следовательно, $u_{xxxx}(x,y))$ также сходится абсолютно и равномерно в области D^-.

Здесь (4.43).

Теорема 4.1 доказана.

5. Заключение

В рамках данного исследования проведен всесторонний и детальный анализ краевых задач, поставленных в полуограниченной области для неоднородного дифференциального уравнения четвертого порядка с кратными характеристиками, содержащего третью частную производную по времени. В ходе изысканий были достигнуты следующие фундаментальные научные и практические результаты:

1. В результате специальной адаптации метода энергетических интегралов (функционалов) получены априорные оценки, гарантирующие единственность решения в полубесконечной области, и строго доказана соответствующая теорема.

2. Успешно сконструирована специализированная функция Грина для данной задачи. С помощью построенной функции искомое решение выражено в явном аналитическом виде через интегральные операторы и бесконечные функциональные ряды.

3. Полностью обосновано существование задачи в классе регулярных решений. При этом строго показана абсолютная и равномерная сходимость рядов решений, а также рядов, состоящих из входящих в уравнение высших частных производных в рассматриваемой полуограниченной геометрической области.

6. Благодарность

Авторы выражают искреннюю и глубокую признательность редактору журнала и анонимным рецензентам за детальное рассмотрение рукописи. Их ценные научные замечания, конструктивные комментарии и рекомендации способствовали существенному повышению качества изложения и академического уровня представленной работы.

7. Финансирование

Данное научное исследование выполнено исключительно в рамках личной творческой инициативы авторов. Работа не получала финансовой поддержки со стороны каких-либо внешних фондов, государственных грантов или сторонних коммерческих организаций.

Список литературы

[1] Турбин, М.В.: *Исследование начально краевой задачи для модели движения жидкости Герше - Балки* . Вест. Воронеж. Гос. Ун-та. Сер.Физ. Мат. № 2. - С. 246-257.(2013).
[2] Уизем, Дж.: *Линейные и нелинейные волны.*-М.: Мир.-622 с.(1977).
[3] Шабров, С.А.: *Об оценках функций влияния одной математической модели четвёртого порядка.* Вест. Воронеж. Гос. Ун-та. Сер. Физ. Мат. **20**(2), - С. 168-179. (2015)
[4] Benney, D.J., Luke, J.C.: *Interactions of permanent waves of finite amplitude*. J. Math. Phys. **43**, 309-313. doi.org/10.1002/sapm1964431309 (1964).
[5] Tursunov, D.A., Erkaboyev, U.Z.: *Asymptotic expansions of solutions to Dirichlet problem for elliptic equation with singularities.* Ufa Mathematical Journal. **8** (1), pp. 97 - 107. DOI: 10.13108/2016-8-1-97
[6] Джураев, Т.Д., Сопуев, А.: *К теории дифференциальных уравнений в частных производных четвёртого порядка.* Ташкент. «Фан». -144 с (2000).
[7] Аманов, Д, Мурзамбетова, М.Б.: *Краевая задача для уравнения четвёртого порядка с младшим членом.* Вестн. Удмуртск. ун-та. Матем. Мех. Компьют. науки. Выпуск **1** -С. 3–10. DOI: 10.20537/vm130101 (2013).
[8] Аманов, Д., Бекиев, А.Б., Отарова, Ж.А.: *Краевая задача для уравнения четвёртого порядка.*Узб.мат.журнал. -№4. -С.11-18 (2015).
[9] Сабитов, К. Б.: *Колебания балки с заделанными концами.* Вестн. Сам. гос. техн. ун-та. Сер. Физ. мат. Науки. **19**(2). -С.311–324 https://doi.org/10.14498/vsgtu1406 (2015).
[10] Сабитов, К. Б.: *К теории начально-граничных задач для уравнения стержней и балок.* Диффер. уравн. **53**(1). -С.89–100 (2017).
[11] Сабитов, К. Б., Фадеева, О. В: *Начально-граничная задача для уравнения вынужденных колебаний консольной балки.* Вестн. Сам. гос. техн. ун-та. Сер. Физ.-мат. науки. **25**(1). -С. 51–66. DOI: https://doi.org/10.14498/vsgtu1845 (2021).
[12] Dzhuraev, T.D., Apakov, Yu.P.: *On the theory of the third- order equation with multiple characteristics containing the second time derivative.* Ukrainian Mathematical Journal. **62**, -pp.43–55. https://doi.org/10.1007/s11253-010-0332-8 (2010).
[13] Apakov, Yu.P.,Rutkauskas, S.: *On a boundary problem to third order PDE with multiple characteristics.* Nonlinear Analysis: Modeling and Control.Vilnius. **16**(3). -pp.255–269. DOI:10.15388/NA.16.3.14092 (2011).
[14] Apakov, Yu. P.: *On the solution of a boundary-value problem for a third-order equation with multiple characteristics.* Ukrainian Mathematical Journal. **64**(1). -pp.1–11. https://doi.org/10.1007/s11253-012-0625-1 (2012).

[15] Apakov, Yu. P.: *On unique solvability of boundary-value problem for a viscous transonic equation.* Lobachevskii Journal of Mathematics. **41**. -pp.1754–1761. https://doi.org/10.1134/S1995080220090036 (2020).
[16] Apakov, Yu.P. , Umarov, R.A.: *Solution of the Boundary Value Problem for a Third Order Equation with Little Terms Construction of the Green's Function.* Lobachevskii Journal of Mathematics.**43**.-pp.738–748. DOI:10.1134/S199508022206004X (2022).
[17] Апаков, Ю.П.: *К решению краевых задач для уравнения в неограниченных областях* . Доклады Академии Наук Республики Узбекистан. Ташкент. –№3. С. 17-20 (2006).
[18] Apakov, Yu.P., Hamitov, A.A.: *On solution of the boundary value problems posed for an equation with the third-order multiple characteristics in semi-bounded domains in three dimensional space.* Boletin de la Sociedad Matematica Mexicana. **29**(58). pp.2-14. https://doi.org/10.1007/s40590-023-00523-1 (2023).
[19] Аманов, Д., Скоробогатова, Э.Р.: *Краевые задачи для уравнения четвёртого порядка.* Вестник КазНУ сер.мат.,мех., инф. **4**(63).-С.16-20 (2009).
[20] Apakov, Yu.P., Melikuzieva, D.M.: *On a problem for a fourth-order with multiple characteristics containing the third time derivate.* Lobachevskii Journal of Mathematics. **44**(8). pp.3218-3224. DOI:10.1134/S1995080223080061(2023).
[21] Apakov, Yu.P., Melikuzieva, D.M.: *On a boundary problem for the fourth order equation with the third derivative with respect to time.* Bulletin of the Karaganda University. Mathematics Series. **4**(112).pp.30-40. DOI:10.31489/2023M4/30-40 (2023).
[22] Apakov, Yu.P., Melikuzieva, D.M.: *On the solution of a boundary problem for a fourth order equation containing a third time derivative in semi-bounded domains*. Uzbek Mathematical Journal. **68**(3). pp 5-11. DOI: 10.29229 / uzmj. 2024-3-1 (2024)
[23] Kong, S., Zhou, S., Nie, Z., Wang. K.:*Static and dynamic analysis of micro beams based on strain gradient elasticity theory.* International Journal of Engineering Science, **46**(3), 487-498, DOI: 10.1016/j.ijengsci.2007.10.004 (2008).
[24] Краснов, М.М., Киселев, А.М., Макаренко, Г.И.:*Интегралное уравнения: задача и примеры с подробным решениями, в сб.* Едиториал УРСС, М., 2003.

Solution of boundary value problems for a fourth-order equation with multiple characteristics in a semi-bounded domain by constructing the Green's function

Jamalov B.I., Melikuzieva D.M.

Abstract

The subject of this study is boundary value problems in a semi-bounded domain for a fourth-order inhomogeneous differential equation with multiple characteristics. The uniqueness of regular solutions for the formulated problems is established using the energy integral method, while the existence theorems are justified by applying the classical method of separation of variables. The desired solutions are analytically expressed in the form of infinite functional series using the constructed Green's function. Furthermore, the absolute and uniform convergence of both the solution series themselves and the series composed of the corresponding partial derivatives is rigorously proved within the considered geometric domains.

Keywords

Fourth-order differential equations, multiple characteristics, semi-infinite domain, energy integral, Green's function, functional series, uniform convergence, partial derivatives.

Affiliations

Jamalov B.I.
Address: Namangan State Technical University, Dept. of Higher Mathematics, 160103, Namangan, Uzbekistan.
e-mail: b-i-jamalov@mail.ru
ORCID ID: https://orcid.org/0009-0009-0718-2496

Melikuzieva D.M.
Address: Namangan State Technical University, Dept. of Higher Mathematics, 160103, Namangan, Uzbekistan.
e-mail: meliquziyevadilshoda@gmail.com
ORCID ID: https://orcid.org/0009-0004-8759-7571

Uzbekistan Journal of Mathematics and Computer Science
Volume-2, Issue-3-special, Page 119–135 (2026)
DOI: https://doi.org/10.56143/ujmcs.v2i3s.10
Research article

Periodic Gibbs measures for the soft-core Widom–Rowlinson model and their extremality

B.Z. Tojiboev

Abstract

We consider both hard-core and soft-core Widom–Rowlinson models with spin values $\{-1, 0, 1\}$ on a Cayley tree of order $k \geq 2$, focusing on the Gibbs measures of these models. The models depend on three parameters: the order k of the tree, the interaction strength θ (describing $J < 0$ and $J > 0$ cases), and the particle intensity λ. The hard-core Widom–Rowlinson model corresponds to the case $\theta = 0$.

For $k = 2$ and $J > 0$, we prove the existence of two and four non-translation-invariant (non-TI) periodic Gibbs measures within the corresponding regions of the parameters θ and λ. Since the explicit forms of the four solutions to the system of functional equations are obtained, the conditions for the extremality and non-extremality of these periodic Gibbs measures are established. In the case $k = 3$ and $J > 0$, under the condition $\theta > 2$, explicit expressions for two critical values $\lambda_{\mathrm{cr},i}(\theta)$ $(i = 1, 2)$ are derived. Using these critical values, we show that there exist exactly two or exactly four non-translation-invariant periodic Gibbs measures in the corresponding regions of the parameter space.

Keywords: *Cayley tree, Widom–Rowlinson model, Gibbs-measure, translation-invariant Gibbs measure, periodic Gibbs measure, extremality, non-extremality, phase transition.*

AMS Subject Classification (2020): *Primary:82B20; Secondary: 60K35, 82B26, 37B10.*

1. Introduction

The Widom–Rowlinson model was originally introduced in Euclidean space as a model of point particles carrying positive or negative charges [20]. In its hard-core version, particles with opposite signs are prohibited from being located within a prescribed distance of each other. Under this hard-core constraint on a graph, particles of opposite signs cannot occupy adjacent vertices. By relaxing the hard-core constraint, one obtains the soft-core variant. The analysis of this model becomes more intricate due to the presence of an additional parameter that controls the ferromagnetic or antiferromagnetic interaction between particles of opposite signs.

Let us now focus on models defined on trees. For earlier studies of such models, we refer to [1, 2, 3]. This approach relies on a one-to-one correspondence between boundary laws and splitting Gibbs measures. In the Widom–Rowlinson model, boundary laws are represented by two-dimensional positive vectors that satisfy a nonlinear recurrence relation on the tree. Each boundary law determines a corresponding transition matrix for a Markov chain indexed by the tree [7, 17]. The problem of establishing the extremality of Gibbs measures is considerably more delicate and is usually studied separately [5, 11, 23, 24].

Received : *28–april–2026,* ***Accepted :*** *31–july–2026*
**** Corresponding author***

In [21], Gibbs measures for the hard- and soft-core Widom–Rowlinson models on a Cayley tree were investigated in detail. In particular, a system of functional equations associated with the model was derived, and necessary and sufficient conditions for the existence and uniqueness of translation-invariant Gibbs measures were established. Moreover, the existence of multiple Gibbs measures for certain parameter values was proved, and the occurrence of phase transitions was demonstrated.

This paper is a direct continuation of the studies initiated in [21], where periodic Gibbs measures for the soft-core Widom–Rowlinson model on a Cayley tree were investigated. Positive solutions of the system of functional equations associated with the model are analyzed. Based on these equations, the existence of splitting periodic Gibbs measures is studied. Furthermore, conditions for the extremality and non-extremality of the obtained periodic Gibbs measures are established.The structure of the paper is organized as follows. In Section 2, we introduce the basic definitions and formulate the problem. Section 3 is devoted to translation-invariant splitting Gibbs measures (TISGMs), where some known results from [21] are recalled. Section 4 deals with periodic Gibbs measures; in particular, for a fixed invariant set, we identify non-translation-invariant periodic Gibbs measures for $k = 2$ and $k = 3$. Finally, Sections 5 and 6 are devoted to establishing conditions for the extremality and non-extremality of the measures obtained in the case $k = 2$.

2. Preliminaries

A Cayley tree $\Im^k$ of order $k \geq 1$ is an infinite, connected, acyclic regular graph in which each vertex has degree $k + 1$. Let $\Im^k = (V, L, i)$, where V is the set of vertices $\Im^k$, L is the set of its edges, and i is the incidence function that assigns, to every edge $l \in L$, its endpoints $x, y \in V$. If $i(l) = \{x, y\}$, then x and y are called the *nearest neighbor vertices* and are denoted by $l = \langle x, y\rangle$. The set of nearest neighbor pairs $\langle x, x_1\rangle, \langle x_1, x_2\rangle, \ldots, \langle x_{d-1}, y\rangle$ is called a *path* from x to y. The distance $d(x, y)$ between vertices x and y on a Cayley tree is the number of edges of the shortest path connecting the vertices x and y, i.e., $d(x, y) = \min\{d \mid \exists x = x_0, x_1, \ldots, x_{d-1}, x_d = y \in V$ such that $\langle x_0, x_1\rangle, \ldots, \langle x_{d-1}, x_d\rangle\}$.

For fixed $x^0 \in V$, we set

$$W_n = \{x \in V \mid d(x, x^0) = n\}, \quad V_n = \{x \in V \mid d(x, x^0) \leq n\}, \quad L_n = \{l = \langle x, y\rangle \in L \mid x, y \in V_n\}.$$

A point y is called a direct successor of a point x if $x \prec y$ and $d(x, y) = 1$. Note that in $\Im^k$, every vertex $x \neq x^0$ has k direct successors, and the vertex x^0 has $k + 1$ successors. The set of direct successors of a vertex x is denoted by $S(x)$, i.e., if $x \in W_n$, then

$$S(x) = \{y_i \in W_{n+1} \mid d(x, y_i) = 1, i = 1, 2, \ldots, k\}.$$

Let $\Phi = \{-1, 0, 1\}$ and $\sigma \in \Omega = \Phi^V$ be a configuration, i.e., $\sigma = \{\sigma(x) \in \Phi : x \in V\}$. In other words, in this model, to every vertex x, there is assigned one of the values $\sigma(x) \in \Phi = \{-1, 0, 1\}$. The set of all configurations on V is denoted by Ω. Similarly, one can define configurations in V_n (W_n, resp.), and the set of all configurations in V_n (W_n, resp.) is denoted by Ω_{V_n} (Ω_{W_n}, resp.).

Consider the set Φ as the set of vertices of some graph G. The activity set for the graph G is the function $\lambda : G \to \mathbb{R}_+$ from the set G to the set of positive real numbers. The value of λ_i of the function λ at a vertex $i \in \{-1, 0, 1\}$ is called its *activity*.

It is known from [21] that the Hamiltonian for the Soft-Core Widom–Rowlinson (SCWR) model of step n, i.e. on the configuration set Ω_{V_n}, is given by

$$H_n^{sc}(\sigma) = J \sum_{\{x,y\} \in L_n} \mathbf{1}(\sigma(x)\sigma(y) = -1) - \frac{\ln \lambda}{\beta} \sum_{x \in V_n} \sigma^2(x),$$

where $\beta = \frac{1}{T}$ and $T > 0$ is the temperature. The parameter $J \in \mathbb{R}$ can be seen as repulsion or attraction between particles of different charges depending on the sign of J and $\lambda > 0$ as an activity.

The associated finite volume Gibbs measure on Ω_{V_n} with external fields on the boundary, $\{h^x = (h_{-1,x}, h_{0,x}, h_{1,x}) \in \mathbb{R}^3 : x \in \Im^k\}$ is defined by

$$\mu_{n,\beta}(\sigma) = \frac{1}{Z_n} \exp\left(-\beta H_n^{sc}(\sigma) + \sum_{x \in W_n} h_{\sigma(x),x}\right).$$

The sequence $\mu_{n,\beta}$, $n \geq 1$ is said to be compatible if for all $n \geq 2$ and all $\sigma^{n-1} \in \Omega_{V_{n-1}}$ as well as $\omega^n \in \Omega_{W_n}$

$$\sum_{\omega^n \in \Omega_{W_n}} \mu_{n,\beta}(\sigma^{n-1} \vee \omega^n) = \mu_{n-1,\beta}(\sigma^{n-1})$$

holds. Here

$$\sigma^{n-1} \vee \omega^n = \begin{cases} \sigma^{n-1}, & \text{if } x \in V_{n-1}, \\ \omega^n, & \text{if } x \in W_n. \end{cases}$$

For a sequence of compatible finite volume Gibbs measures $\{\mu_{n,\beta}\}_{n=1}^{\infty}$, by Kolmogorov's extension theorem (see [25]), there exists a unique measure μ defined on the whole tree with $\mu(\sigma\big|_{V_n} = \sigma^n) = \mu_{n,\beta}(\sigma^n)$. Following [15], we call μ a splitting Gibbs measure. On a Cayley tree, the absence of cycles allows for a recursive decomposition of the system via a system of functional equations (boundary laws). Consequently, splitting Gibbs measures (SGMs) are closely related to limiting Gibbs measures (LGMs), often coinciding with physical LGMs under appropriate boundary conditions. In contrast, on the $\mathbb{Z}^d$ lattice, the presence of loops prevents any splitting property, meaning the concept of an SGM is absent. In this setting, DLR states or generic LGMs serve as the fundamental objects obtained via the thermodynamic limit. Thus, while the concept of SGMs closely coincides with that of LGMs on trees, these two notions are fundamentally distinct on lattices.

The finite volume Gibbs measure for the Hard-Core Widom–Rowlinson model we get for $J > 0$ by $\lim_{\beta \to \infty} \mu_{n,\beta}^{sc} = \mu_n^{hc}$.

The following theorem gives conditions to make the finite volume Gibbs measures compatible.

Theorem 2.1. *[21] Let $0 < \beta < \infty$, $J \in \mathbb{R}$ and $\lambda > 0$. The sequence of probability measures $\{\mu_{n,\beta}^{sc}\}_{n=1}^{\infty}$ is compatible if and only if for any $x \in V \setminus \{x^0\}$ the following system of equations holds:*

$$\begin{cases} \tilde{h}_{+,x} = \ln \lambda + \sum_{y \in S(x)} f(\tilde{h}_{+,y}, \tilde{h}_{-,y}, \theta), \\ \tilde{h}_{-,x} = \ln \lambda + \sum_{y \in S(x)} f(\tilde{h}_{-,y}, \tilde{h}_{+,y}, \theta), \end{cases} \tag{2.1}$$

where $\theta = \exp(-J\beta)$,

$$f(x, y, \theta) = \ln \frac{1 + e^x + \theta e^y}{1 + e^x + e^y}$$

and $\tilde{h}_{\pm,y} = \ln \lambda + h_{\pm 1,y} - h_{0,y}$.

3. Translation-invariant Gibbs measures

We seek translation-invariant solutions to the system (2.1). In the case of a TISGM, the external field vectors h^x are independent of x; that is, $h^x = h = (h_1, h_2, h_3) \in \mathbb{R}^3$ for all $x \in \Im^k$. The measure corresponding to such a solution is a TISGM. By introducing two new variables, $z_1 = e^{\tilde{h}_+}$ and $z_2 = e^{\tilde{h}_-}$, system (2.1) can be reduced to the following system of equations:

$$\begin{cases} z_1 = \lambda \left(\dfrac{1 + z_1 + \theta z_2}{1 + z_1 + z_2} \right)^k, \\ z_2 = \lambda \left(\dfrac{1 + \theta z_1 + z_2}{1 + z_1 + z_2} \right)^k. \end{cases} \tag{3.1}$$

The following results are known from [21].

- For the SCWR model with $J < 0$, there exist a critical value $\theta_{\rm cr}(k)$ and two critical values $\lambda_{{\rm cr},i}(k,\theta)$ $(i = 1, 2)$ such that for the number N_1 of TISGMs, the following assertions hold:

$$N_1 = \begin{cases} 1, & if \quad \theta \le \theta_{\rm cr} \ or \ \theta > \theta_{\rm cr} \ and \ \lambda \in (0, \lambda_{{\rm cr},2}) \cup (\lambda_{{\rm cr},1}, +\infty), \\ 2, & if \quad \theta > \theta_{\rm cr} \ and \ \lambda \in \{\lambda_{{\rm cr},2}, \lambda_{{\rm cr},1}\}, \\ 3, & if \quad \theta > \theta_{\rm cr} \ and \ \lambda \in (\lambda_{{\rm cr},2}, \lambda_{{\rm cr},1}), \end{cases}$$

where $\theta_{cr} = 2\left(\frac{k+1}{k-1}\right)^2 - 1$ and $\lambda_{cr,i} := \lambda_{cr,i}(k,\theta) = \frac{2^k z_i}{(1+\theta)^{k+1}} \left(\frac{1+\theta+2z_i}{2(1+z_i)}\right)^k$, where z_1 and z_2 are the solutions of the equation $2z^2 + [4 - (\theta - 1)(k - 1)]z + \theta + 1 = 0$.

- For the SCWR model with $J > 0$ and $k = 2$, there exists a critical value $\lambda_{\rm cr}(2) := \frac{1}{1-3\theta}\left(\frac{3}{2}\right)^2$ such that for the number N_2 of TISGMs, the following assertions hold:

$$N_2 = \begin{cases} 1, & if \quad \theta \ge \frac{1}{3} \ or \ \theta < \frac{1}{3} \ and \ \lambda \le \lambda_{\rm cr}(2), \\ 3, & if \quad \theta < \frac{1}{3} \ and \ \lambda > \lambda_{\rm cr}(2). \end{cases}$$

- For the SCWR model with $J > 0$ and $k = 3$, there exists a critical value $\lambda_{\rm cr}(3) := \frac{1}{2-4\theta}\left(\frac{4}{3}\right)^3$ such that for the number N_3 of TISGMs, the following assertions hold:

$$N_3 = \begin{cases} 1, & if \quad \theta \ge \frac{1}{2} \ or \ \theta < \frac{1}{2} \ and \ \lambda \le \lambda_{\rm cr}(3), \\ 3, & if \quad \theta < \frac{1}{2} \ and \ \lambda > \lambda_{\rm cr}(3). \end{cases}$$

- For the SCWR model with $J > 0$ and $k \ge 4$, the following assertions hold:
  1) If $\theta \ge \theta'_{\rm cr}(k)$, or if $\theta < \theta'_{\rm cr}(k)$ and $\lambda < \lambda'_{\rm cr}(k)$, then there exists a unique TISGM;
  2) If $\theta'_{\rm cr}(k) > \theta \ge \theta_{\rm cr}(k)$, or if $\theta < \theta_{\rm cr}(k)$ and $\lambda'_{\rm cr}(k) < \lambda \le \lambda_{\rm cr}(k)$, then there exists at least one TISGM;
  3) For $\theta < \theta_{\rm cr}(k)$ and $\lambda > \lambda_{\rm cr}(k)$, there exist at least three TISGMs,

  where

$$\theta'_{\rm cr}(k) := \frac{k-1}{k}, \quad \lambda'_{\rm cr}(k) = \frac{1}{k - 1 - k\theta}, \quad \text{and} \quad \lambda_{\rm cr}(k) = \frac{1}{k - 1 - \theta(k+1)} \left(\frac{k+1}{k}\right)^k.$$

Remark 3.1. For $\theta = 0$, the system of equations (3.1) reduces to the three-state hard-core model studied in [16, 18], which corresponds to the case where the activity graph G is a hinge graph.

4. Periodic Gibbs Measures in the Case $J > 0$

In this section, we study periodic Gibbs measures for the SCWR model.

It is well known that there exists a one-to-one correspondence between the vertex set V of a Cayley tree of order $k \geq 1$ and the group G_k, which is the free product of $k+1$ cyclic groups of order two with generators $a_1, a_2, \ldots, a_{k+1}$ (see [10]). Each element $x \in G_k$ (except for the identity element e) can be uniquely represented as a reduced word $x = a_{i_1} a_{i_2} \ldots a_{i_n}$, where $i_m \in \{1, \ldots, k+1\}$ for all $1 \leq m \leq n$ and $i_m \neq i_{m+1}$ for all $1 \leq m \leq n-1$. The number n of generators in this representation is called the length of the word x and is denoted by $|x|$.

Let $\widehat{G}_k$ be a normal subgroup of G_k of finite index $r \geq 1$, and let $G_k/\widehat{G}_k = \{H_1, \ldots, H_r\}$ denote the corresponding quotient group.

Definition 4.1. A collection of quantities $z = \{z_x, x \in G_k\}$ is said to be $\widehat{G}_k$-periodic if $z_{yx} = z_x$ for $\forall x \in G_k, y \in \widehat{G}_k$.

Definition 4.2. A measure μ is called $\widehat{G}_k$-periodic if it corresponds to a $\widehat{G}_k$-periodic collection of quantities z.

We rewrite the system of equations (2.1) as follows:

$$\begin{cases} h_{+,x} = \ln\lambda + \sum\limits_{y \in S(x)} \ln \dfrac{1 + e^{h_{+,y}} + \theta e^{h_{-,y}}}{1 + e^{h_{+,y}} + e^{h_{-,y}}}, \\ h_{-,x} = \ln\lambda + \sum\limits_{y \in S(x)} \ln \dfrac{1 + \theta e^{h_{+,y}} + e^{h_{-,y}}}{1 + e^{h_{+,y}} + e^{h_{-,y}}}, \end{cases} \tag{4.1}$$

where $h_{+,x} = \ln z_{1,x}$ and $h_{-,x} = \ln z_{2,x}$. We consider periodic solutions of the system (4.1).

Let the function $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ with $h = (h_+, h_-) \mapsto F(h) = (F_1(h), F_2(h))$ be defined by

$$F_1(h) = \ln \frac{1 + e^{h_+} + \theta e^{h_-}}{1 + e^{h_+} + e^{h_-}}, \quad F_2(h) = \ln \frac{1 + \theta e^{h_+} + e^{h_-}}{1 + e^{h_+} + e^{h_-}}.$$

Then, we have the following assertion.

Proposition 4.1. *The function $F(h)$ is injective.*

Proof. We show that $F(h) = F(l)$ if and only if $h_+ = l_+$ and $h_- = l_-$.

Necessity. Let $F(h) = F(l)$. Then, $F_1(h) = F_1(l)$ and $F_2(h) = F_2(l)$, where $h = (h_+, h_-)$ and $l = (l_+, l_-)$. From these relations, we obtain the following system of equations:

$$\begin{cases} (z_1 - s_1)s_2 - (z_2 - s_2)(s_1 + 1) = 0, \\ (z_1 - s_1)(s_2 + 1) - (z_2 - s_2)s_1 = 0, \end{cases} \tag{4.2}$$

where $z_1 = e^{h_+}$, $z_2 = e^{h_-}$, $s_1 = e^{l_+}$ and $s_2 = e^{l_-}$. It is easy to see that the determinant of system (4.2) is nonvanishing:

$$\Delta = \begin{vmatrix} s_2 & -(s_1+1) \\ s_2 + 1 & -s_1 \end{vmatrix} = s_1 + s_2 + 1 > 0.$$

Therefore, system (4.2) has a unique solution $z_i = s_i$ for $i = 1, 2$.

Sufficiency is obvious. This completes the proof.

Let $G_k^{(2)}$ be the subgroup of G_k consisting of words of even length.

Theorem 4.1. *Let H_0 be a normal subgroup of a finite index in G_k. Then, for the SCWR model, any H_0-periodic splitting Gibbs measure is either translation-invariant or $G_k^{(2)}$-periodic.*

Proof. This theorem can be proved by using Proposition 4.1., similarly to the proof of Theorem 2 in [12]. This completes the proof.

It follows from Theorem 4.1. that if periodic Gibbs measures exist, then they are $G_k^{(2)}$-periodic.

In [21], the following system of functional equations for periodic Gibbs measures was studied:

$$\begin{cases} z_1 = \lambda\left(\dfrac{1+t_1+\theta t_2}{1+t_1+t_2}\right)^k, \qquad z_2 = \lambda\left(\dfrac{1+t_2+\theta t_1}{1+t_1+t_2}\right)^k, \\ t_1 = \lambda\left(\dfrac{1+z_1+\theta z_2}{1+z_1+z_2}\right)^k, \qquad t_2 = \lambda\left(\dfrac{1+z_2+\theta z_1}{1+z_1+z_2}\right)^k. \end{cases} \tag{4.3}$$

Consider the mapping $W : \mathbb{R}^4 \to \mathbb{R}^4$ defined by the following system of equations:

$$\begin{cases} z_1' = \lambda\left(\dfrac{1+t_1+\theta t_2}{1+t_1+t_2}\right)^k, \\ z_2' = \lambda\left(\dfrac{1+t_2+\theta t_1}{1+t_1+t_2}\right)^k, \\ t_1' = \lambda\left(\dfrac{1+z_1+\theta z_2}{1+z_1+z_2}\right)^k, \\ t_2' = \lambda\left(\dfrac{1+z_2+\theta z_1}{1+z_1+z_2}\right)^k. \end{cases} \tag{4.4}$$

Note that the system of equations (4.3) can be written as $z = W(z)$. Thus, to solve the system of equations (4.3), it is necessary to find the fixed points of the mapping W.

The following lemma holds.

Lemma 4.1. *The following sets are invariant with respect to the mapping W:*

$$I_1 = \{(z_1, z_2, t_1, t_2) \in \mathbb{R}^4 : z_1 = z_2 = t_1 = t_2\},$$
$$I_2 = \{(z_1, z_2, t_1, t_2) \in \mathbb{R}^4 : z_1 = z_2,\ t_1 = t_2\},$$
$$I_3 = \{(z_1, z_2, t_1, t_2) \in \mathbb{R}^4 : z_1 = t_1,\ z_2 = t_2\},$$
$$I_4 = \{(z_1, z_2, t_1, t_2) \in \mathbb{R}^4 : z_1 = t_2,\ z_2 = t_1\}.$$

The following theorem is known.

Theorem 4.2. *[21] For $k \geq 6$, $\theta < \frac{k^2-6k+1}{(k+1)^2}$ and $\lambda \in (\lambda^-(k,\theta), \lambda^+(k,\theta))$ on I_2 there are at least three $G_k^{(2)}$-periodic SGMs μ_0, μ_1, μ_2, where*

$$\lambda^{\pm}(k,\theta) = s^{\pm} \cdot \left(\frac{1+2s^{\pm}}{1+(1+\theta)s^{\pm}}\right)^k, \quad s^{\pm} = \frac{k-3-(k+1)\theta \pm \sqrt{(1-\theta)[k^2-6k+1-(k+1)^2\theta]}}{4(1+\theta)}.$$

For the invariant sets I_1, I_2, and I_3, the following theorem holds.

Theorem 4.3. *For the SCWR model, the following assertions hold:*

1. *Let $k \geq 2$. On the invariant sets I_1 and I_3, any $G_k^{(2)}$-periodic SGM is a TISGM.*
2. *Let $k \in \{2,3,4,5\}$ and $J < 0$. On the invariant set I_2, any $G_k^{(2)}$-periodic SGM coincides with a TISGM.*

Proof.

1. The assertions for the cases I_1 and I_3 are obvious.

2. Let us consider the case I_2. In this case, the system of equations (4.3) takes the form:

$$\begin{cases} z_1 = \lambda \left(\dfrac{1 + (1+\theta) z_2}{1 + 2z_2} \right)^k, \\ z_2 = \lambda \left(\dfrac{1 + (1+\theta) z_1}{1 + 2z_1} \right)^k. \end{cases} \tag{4.5}$$

Assume that $z_1 > 0$, $z_2 > 0$, $\lambda > 0$, and $\theta \in (0,1) \cup (1, \infty)$. We show that the system of equations (4.5) admits a solution of the form $z_1 = z_2$.

Introducing the notation $x = \sqrt[k]{z_1}$ and $y = \sqrt[k]{z_2}$, we rewrite the system of equations (4.5) as

$$\begin{cases} x = \sqrt[k]{\lambda} \left(\dfrac{1 + (1+\theta) y^k}{1 + 2y^k} \right), \\ y = \sqrt[k]{\lambda} \left(\dfrac{1 + (1+\theta) x^k}{1 + 2x^k} \right). \end{cases}$$

In this system of equations, dividing the first equation by the second and performing some algebraic transformations, we obtain

$$(x-y) \left(1 - 2xy \sum_{i=0}^{k-2} x^{k-2-i} y^i + (1+\theta) \sum_{i=0}^{k} x^{k-i} y^i + 2(1+\theta) x^k y^k \right) = 0. \tag{4.6}$$

It follows that $x = y$ or

$$1 - 2xy \sum_{i=0}^{k-2} x^{k-2-i} y^i + (1+\theta) \sum_{i=0}^{k} x^{k-i} y^i + 2(1+\theta) x^k y^k = 0. \tag{4.7}$$

In the first case, the system of equations (4.5) has a unique solution, and the measure corresponding to this solution coincides with the TISGM.

In the second case, we rewrite equation (4.7) with respect to θ as follows:

$$\theta(k) = \frac{2xy \sum_{i=0}^{k-2} x^{k-2-i} y^i - 1}{2x^k y^k + \sum_{i=0}^{k} x^{k-i} y^i} - 1$$

and we show that this expression for θ is always negative for $k \in \{2,3,4,5\}$.

Indeed,

$$\theta(2) = -\frac{2x^2y^2 + xy + 1 + (x-y)^2}{2x^2y^2 + x^2 + xy + y^2} < 0, \qquad \theta(3) = -\frac{2x^3y^3 + (x+y)(x-y)^2 + 1}{2x^3y^3 + x^3 + x^2y + xy^2 + y^3} < 0,$$

$$\theta(4) = -\frac{x^4y^4 + (x-y)^2(x^2 + xy + y^2) + (x^2y^2 - \frac{1}{2})^2 + \frac{3}{4}}{2x^4y^4 + x^4 + x^3y + x^2y^2 + xy^3 + y^4} < 0,$$

$$\theta(5) = -\frac{x^5y^5 + (x-y)^2(x^3 + xy^2 + y^3)}{2x^5y^5 + x^5 + x^4y + x^3y^2 + x^2y^3 + xy^4 + y^5} < 0.$$

Hence, equation (4.6) has solutions only of the form $x = y$ for $k = 2,3,4,5$. Therefore, on I_2 and for $k \in \{2,3,4,5\}$, any $G_k^{(2)}$-periodic SGM is a TISGM. This completes the proof of Theorem 4.3.

The case I_4. In this case, from the system of equations (4.3), we obtain the following system:

$$\begin{cases} z_1 = \lambda \left(\frac{1+z_2+\theta z_1}{1+z_1+z_2}\right)^k, \\ z_2 = \lambda \left(\frac{1+z_1+\theta z_2}{1+z_1+z_2}\right)^k. \end{cases} \tag{4.8}$$

Subtracting the second equation of (4.8) from the first and performing some algebraic transformations, we obtain

$$(z_1 - z_2)\left(1 - \frac{\lambda}{(1+z_1+z_2)^k}(\theta-1)\left[\sum_{i=0}^{k-1}(1+z_2+\theta z_1)^{k-1-i}(1+z_1+\theta z_2)^i\right]\right) = 0. \tag{4.9}$$

If $\theta \in (0,1)$, then equation (4.9) has a unique solution $z_1 = z_2$, which corresponds to a TISGM.
Let $\theta > 1$. Then, from equation (4.9), it follows that $z_1 = z_2$ or

$$(1+z_1+z_2)^k = \lambda(\theta-1)\left[\sum_{i=0}^{k-1}(1+z_2+\theta z_1)^{k-1-i}(1+z_1+\theta z_2)^i\right]. \tag{4.10}$$

Now we solve equation (4.10) in the case $k = 2$. Then, we have

$$(1+z_1+z_2)^2 = \lambda(\theta-1)\left[1-\theta+(1+\theta)(1+z_1+z_2)\right]. \tag{4.11}$$

Solving this quadratic equation with respect to $1+z_1+z_2$, we obtain

$$1+z_1+z_2 = \frac{(\theta-1)}{2}\cdot g_{\pm}(\theta,\lambda), \tag{4.12}$$

where $g_{\pm}(\theta,\lambda) = \lambda(\theta+1) \pm \sqrt{\lambda^2(\theta+1)^2 - 4\lambda} :> 0$.
Substituting this formula into the first equation of (4.8), we get

$$z_1 = \lambda\left(\frac{2z_1 + g_{\pm}(\theta,\lambda)}{g_{\pm}(\theta,\lambda)}\right)^2 \quad \text{or} \quad z_1^2 + \left(g_{\pm}(\theta,\lambda) - \frac{g_{\pm}^2(\theta,\lambda)}{4\lambda}\right)z_1 + \frac{g_{\pm}^2(\theta,\lambda)}{4} = 0. \tag{4.13}$$

Equation (4.13) has the following solutions:

$$z_1^{(i)} = \frac{1}{16\lambda}\cdot\left(g_+(\theta,\lambda) \pm \sqrt{g_+^2(\theta,\lambda) - 8\lambda g_+(\theta,\lambda)}\right)^2, \quad i = 1,2,$$

$$z_1^{(j)} = \frac{1}{16\lambda}\cdot\left(g_-(\theta,\lambda) \pm \sqrt{g_-^2(\theta,\lambda) - 8\lambda g_-(\theta,\lambda)}\right)^2, \quad j = 3,4.$$

To determine the existence domain of the above solutions, we solve the inequality:

$$[g_{\pm}(\theta,\lambda) - 8\lambda]\cdot g_{\pm}(\theta,\lambda) \geq 0 \quad \text{or} \quad g_{\pm}(\theta,\lambda) - 8\lambda \geq 0.$$

The solutions $z_1^{(i)}$ $(i = 1,2)$ determined above are positive on M_1, M_2, and M_3, while the solutions $z_1^{(j)}$ $(j = 3,4)$ are positive on M_2, where

$$M_1 = \left\{(\theta,\lambda): \quad \theta \in (3,7), \quad \lambda \in \left(\frac{1}{4(\theta-3)}, \infty\right)\right\},$$

$$M_2 = \left\{(\theta,\lambda): \quad \theta \in [7,\infty), \quad \lambda \in \left[\frac{4}{(\theta+1)^2}, \frac{1}{4(\theta-3)}\right]\right\},$$

$$M_3 = \left\{(\theta,\lambda): \quad \theta \in [7,\infty), \quad \lambda \in \left(\frac{1}{4(\theta-3)}, \infty\right)\right\}.$$

Substituting the obtained solutions $z_1^{(i)}$ and $z_1^{(j)}$ into (4.12), we find the corresponding values of z_2. Thus, equation (4.11) has the following solutions:

a) On the set M_1, there exist two solutions: $(z_1^{(1)}, z_1^{(2)})$ and $(z_1^{(2)}, z_1^{(1)})$.
b) On the set M_2, there exist four solutions: $(z_1^{(1)}, z_1^{(2)})$, $(z_1^{(2)}, z_1^{(1)})$, $(z_1^{(3)}, z_1^{(4)})$, and $(z_1^{(4)}, z_1^{(3)})$.
c) On the set M_3, there exist two solutions: $(z_1^{(1)}, z_1^{(2)})$ and $(z_1^{(2)}, z_1^{(1)})$.

Let μ_1, μ_2, μ_3, and μ_4 be the measures corresponding to the solutions $(z_1^{(1)}, z_1^{(2)})$, $(z_1^{(2)}, z_1^{(1)})$, $(z_1^{(3)}, z_1^{(4)})$, and $(z_1^{(4)}, z_1^{(3)})$, respectively.

The following theorem holds.

Theorem 4.4. *Let $k = 2$. For the SCWR model on I_4, the following assertions hold:*

1. *Let $0 < \theta < 1$ and $\lambda > 0$. Then any $G_k^{(2)}$-periodic Gibbs measure coincides with the TISGM.*
2. *If $\theta > 1$, then for the number $N_{per}(2)$ of $G_k^{(2)}$-periodic non-TI Gibbs measures, the following assertions hold*

$$N_{per}(2) = \begin{cases} \emptyset & \text{if} \quad \theta \in (1,3] \text{ and } \lambda \in (0,\ \infty), \\ 2\ (\mu_1, \mu_2), & \text{if} \quad \theta \in (3,\infty) \text{ and } \lambda \in \left(\frac{1}{4(\theta-3)},\ \infty\right), \\ 4\ (\mu_1, \mu_2, \mu_3, \mu_4), & \text{if} \quad \theta \in [7,\infty) \text{ and } \lambda \in \left[\frac{4}{(\theta+1)^2},\ \frac{1}{4(\theta-3)}\right]. \end{cases}$$

Case $k = 3$. In this case, setting $u = \sqrt[3]{z_1} > 0$, $v = \sqrt[3]{z_2} > 0$, and $a = \sqrt[3]{\lambda}$, we rewrite the system of equations (4.8) in the following form:

$$\begin{cases} u = a\frac{1+v^3+\theta u^3}{1+u^3+v^3}, \\ v = a\frac{1+u^3+\theta v^3}{1+u^3+v^3}. \end{cases} \tag{4.14}$$

Let us denote the sum and the product of u and v by

$$s = u + v, \quad p = uv. \tag{4.15}$$

Assuming $u \neq v$, we divide the first equation of (4.14) by the second to obtain

$$1 + (u+v)(u^2+v^2) - \theta uv(u+v) = 0. \tag{4.16}$$

By using the algebraic identities $u^2 + v^2 = s^2 - 2p$ and $u^3 + v^3 = s(s^2 - 3p)$, from (4.16) we obtain the following expression for p:

$$p = \frac{1+s^3}{(\theta+2)s}. \tag{4.17}$$

Adding the first equation of system (4.14) to the second, we have

$$u + v = a \cdot \frac{2 + (1+\theta)(u^3+v^3)}{1+u^3+v^3}.$$

Using (4.15), this equality can be written as

$$s = a \cdot \frac{2 + (1+\theta)s(s^2-3p)}{1+s(s^2-3p)}. \tag{4.18}$$

Substituting the expression for p from (4.17) into (4.18), we obtain

$$s = a \cdot \frac{(1+\theta)s^3 - 1}{s^3+1}. \tag{4.19}$$

Let s be a solution to equation (4.19). Then, to find the corresponding values of u and v according to (4.15), we solve the system

$$u + v = s, \quad uv = \frac{1 + s^3}{(\theta + 2)s}.$$

This system can be reduced to the quadratic equation

$$u^2 - su + \frac{1 + s^3}{(\theta + 2)s} = 0.$$

The discriminant of this equation is non-negative if and only if $(\theta - 2)s^3 - 4 \geq 0$. From this inequality, it follows that θ and s must satisfy $\theta > \theta_c(3) := 2, \quad s^3 \geq \frac{4}{\theta - 2}$.

Solving equation (4.19) for a, we obtain

$$a = s \cdot \frac{s^3 + 1}{(1 + \theta)s^3 - 1} =: \eta(s). \tag{4.20}$$

The derivative of the function $\eta(s)$ is given by

$$\eta'(s) = \frac{(\theta + 1)s^6 - 2(\theta + 3)s^3 - 1}{((1 + \theta)s^3 - 1)^2}.$$

The equation $\eta'(s) = 0$ has a unique positive solution

$$s_1 = \sqrt[3]{\frac{\theta + 3 + \sqrt{\theta^2 + 7\theta + 10}}{\theta + 1}},$$

and the function $\eta(s)$ attains its minimum at the point s_1.

Hence, we define the critical value as

$$a_{cr}^{(1)} = \eta(s_1) = \sqrt[3]{\frac{\theta + 3 + \sqrt{\theta^2 + 7\theta + 10}}{\theta + 1}} \cdot \frac{\sqrt{\theta^2 + 7\theta + 10} - \theta + 1}{3(\theta + 1)},$$

$$a_{cr}^{(2)} = \max_{\sqrt[3]{\frac{4}{\theta-2}} \leq s \leq s_1} \eta(s) = \eta\left(\sqrt[3]{\frac{4}{\theta - 2}}\right) = \frac{2}{3}\sqrt[3]{\frac{1}{2\theta - 4}}.$$

If $a < a_{cr}^{(1)}$, equation (4.20) has no solutions; if $a = a_{cr}^{(1)}$ or $a \geq a_{cr}^{(2)}$, equation (4.20) has a unique solution; and if $a_{cr}^{(1)} < a < a_{cr}^{(2)}$, equation (4.20) has two solutions.

Let us denote

$$\lambda_{cr}^{(1)} = \left[a_{cr}^{(1)}\right]^3 = \frac{\theta + 3 + \sqrt{\theta^2 + 7\theta + 10}}{\theta + 1} \cdot \left(\frac{\sqrt{\theta^2 + 7\theta + 10} - \theta + 1}{3(\theta + 1)}\right)^3,$$

$$\lambda_{cr}^{(2)} = \left[a_{cr}^{(2)}\right]^3 = \frac{1}{2\theta - 4} \cdot \left(\frac{2}{3}\right)^3.$$

The following theorem holds for the system of equations (4.14):

Theorem 4.5. *Let $k = 3$ and $\lambda > 0$. For the SCWR model on I_4, the following assertions hold:*

1. *Let $0 < \theta < 1$. Then any $G_k^{(2)}$-periodic Gibbs measure coincides with the TISGM.*
2. *If $\theta > 1$, then for the number $N_{per}(3)$ of $G_k^{(2)}$-periodic non-TI Gibbs measures, the following assertions hold*

$$N_{per}(3) = \begin{cases} \emptyset, & \text{if} \quad 1 < \theta \leq 2 \text{ or } \theta > 2 \text{ and } \lambda < \lambda_{cr}^{(1)}, \\ 2, & \text{if} \quad \theta > 2 \text{ and } \lambda = \lambda_{cr}^{(1)} \text{ or } \theta > 2 \text{ and } \lambda > \lambda_{cr}^{(2)}, \\ 4, & \text{if} \quad \theta > 2 \text{ and } \lambda_{cr}^{(1)} < \lambda \leq \lambda_{cr}^{(2)}. \end{cases}$$

5. Extremality and Non-Extremality of Periodic Splitting Gibbs Measures

To investigate the non-extremality of the periodic Gibbs measures, we employ the method proposed in [24]. For this purpose, we consider a Markov chain with the state space $\{-1, 0, +1\}$ and the transition probability matrix $P_\mu = (P_{\sigma(x)\sigma(y)})$, which is determined by the found Gibbs measure μ. Here, $P_{\sigma(x)\sigma(y)}$ denotes the probability of transitioning from state $\sigma(x)$ to state $\sigma(y)$:

$$P_{\sigma(x),\sigma(y)} = \frac{\exp\left\{-J\beta\left(\mathbf{1}(\sigma(x)\cdot\sigma(y) = -1) + \frac{\ln\lambda}{\beta}\cdot\sigma^2(y)\right) + h_{\sigma(y),y}\right\}}{\sum\limits_{\sigma(y)\in\{-1,0,1\}} \exp\left\{-J\beta\left(\mathbf{1}(\sigma(x)\cdot\sigma(y) = -1) + \frac{\ln\lambda}{\beta}\cdot\sigma^2(y)\right) + h_{\sigma(y),y}\right\}}.$$

From this, it follows that:

$$P_{-1,-1} = \frac{z_1}{z_1+\theta z_2+1}, \qquad P_{-1,0} = \frac{1}{z_1+\theta z_2+1}, \qquad P_{-1,1} = \frac{\theta z_2}{z_1+\theta z_2+1},$$

$$P_{0,-1} = \frac{z_1}{z_1+z_2+1}, \qquad P_{0,0} = \frac{1}{z_1+z_2+1}, \qquad P_{0,1} = \frac{z_2}{z_1+z_2+1},$$

$$P_{1,-1} = \frac{\theta z_1}{\theta z_1+z_2+1}, \qquad P_{1,0} = \frac{1}{\theta z_1+z_2+1}, \qquad P_{1,1} = \frac{z_2}{\theta z_1+z_2+1},$$

where $z_1 = \exp(h_{-1,y} - J\ln\lambda - h_{0,y})$ and $z_2 = \exp(h_{1,y} - J\ln\lambda - h_{0,y})$.

Thus, the transition probability matrix P_μ is expressed in the following form:

$$\mathbb{P}_\mu = \begin{pmatrix} \frac{z_1}{z_1+\theta z_2+1} & \frac{1}{z_1+\theta z_2+1} & \frac{\theta z_2}{z_1+\theta z_2+1} \\ \frac{z_1}{z_1+z_2+1} & \frac{1}{z_1+z_2+1} & \frac{z_2}{z_1+z_2+1} \\ \frac{\theta z_1}{\theta z_1+z_2+1} & \frac{1}{\theta z_1+z_2+1} & \frac{z_2}{\theta z_1+z_2+1} \end{pmatrix}. \tag{5.1}$$

However, in the case of periodic measures, the matrix $\mathbb{P}$ depends on $z = (z_1, z_2)$ and $t = (t_1, t_2)$ where $z \neq t$, and $\left(z_1^{(1)}, z_1^{(2)}\right)$, $\left(z_1^{(2)}, z_1^{(1)}\right)$, $\left(z_1^{(3)}, z_1^{(4)}\right)$, $\left(z_1^{(4)}, z_1^{(3)}\right)$ are the $G_k^{(2)}$ - periodic solutions (non-TI) of the system (4.8) for $k = 2$. More precisely, we have the relation $\mathbb{P} \equiv \mathbb{P}_{z,t} = \mathbf{P}_{\mu_i}$, where $\mathbf{P}_{\mu_i}$ is the transition probability matrix defined by the given $G_k^{(2)}$ - periodic Gibbs measure μ_i (see [23]). It should be noted that the matrix $\mathbf{P}_{\mu_i}$ is equal to the product of two transition probability matrices:

$$\mathbf{P}_{\mu_i} = \mathbb{P}_z\mathbb{P}_t = \begin{pmatrix} \frac{z_1}{z_1+\theta z_2+1} & \frac{1}{z_1+\theta z_2+1} & \frac{\theta z_2}{z_1+\theta z_2+1} \\ \frac{z_1}{z_1+z_2+1} & \frac{1}{z_1+z_2+1} & \frac{z_2}{z_1+z_2+1} \\ \frac{\theta z_1}{\theta z_1+z_2+1} & \frac{1}{\theta z_1+z_2+1} & \frac{z_2}{\theta z_1+z_2+1} \end{pmatrix} \cdot \begin{pmatrix} \frac{t_1}{t_1+\theta t_2+1} & \frac{1}{t_1+\theta t_2+1} & \frac{\theta t_2}{t_1+\theta t_2+1} \\ \frac{t_1}{t_1+t_2+1} & \frac{1}{t_1+t_2+1} & \frac{t_2}{t_1+t_2+1} \\ \frac{\theta t_1}{\theta t_1+t_2+1} & \frac{1}{\theta t_1+t_2+1} & \frac{t_2}{\theta t_1+t_2+1} \end{pmatrix}.$$

Using the solution (z_1, z_2, t_1, t_2) and the I_4 invariant, the transition matrix $\mathbf{P}_{\mu_i}$ can be expressed as follows:

$$\mathbf{P}_{\mu_i} = \begin{pmatrix} \frac{z_2}{C}\left(\frac{z_1}{A}+\frac{1}{B}+\frac{\theta^2 z_2}{C}\right) & \frac{1}{C}\left(\frac{z_1}{A}+\frac{1}{B}+\frac{\theta z_2}{C}\right) & \frac{z_1}{C}\left(\frac{\theta z_1}{A}+\frac{1}{B}+\frac{\theta z_2}{C}\right) \\ \frac{z_2}{B}\left(\frac{z_1}{A}+\frac{1}{B}+\frac{\theta z_2}{C}\right) & \frac{1}{B}\left(\frac{z_1}{A}+\frac{1}{B}+\frac{z_2}{C}\right) & \frac{z_1}{B}\left(\frac{\theta z_1}{A}+\frac{1}{B}+\frac{z_2}{C}\right) \\ \frac{z_2}{A}\left(\frac{\theta z_1}{A}+\frac{1}{B}+\frac{\theta z_2}{C}\right) & \frac{1}{A}\left(\frac{\theta z_1}{A}+\frac{1}{B}+\frac{z_2}{C}\right) & \frac{z_1}{A}\left(\frac{\theta^2 z_1}{A}+\frac{1}{B}+\frac{z_2}{C}\right) \end{pmatrix}, \tag{5.2}$$

where $A = \theta z_1 + z_2 + 1$, $B = z_1 + z_2 + 1$, $C = z_1 + \theta z_2 + 1$.

Consequently, the matrix $\mathbf{P}_{\mu_i}$ defines a Markov chain corresponding to a two-step transition on the Cayley tree of order k^2, which can be viewed as a chain on the tree's even-level vertices.

5.1. Non-Extremality Conditions for the Splitting Gibbs Measures

The Kesten-Stigum sufficient condition for the non-extremality of the Gibbs measure μ_i $(i = 1, 2, 3, 4)$, corresponding to the matrix $\mathbf{P}_{\mu_i}$ is given by $k^2 s_1^2 > 1$, where s_1 is the second largest eigenvalue of $\mathbf{P}_{\mu_i}$ in absolute value.

Since $\mathbf{P}_{\mu_i}$ is a stochastic matrix, its largest eigenvalue is $s_0 = 1$. The remaining two eigenvalues are given by:

$$s_1(z_1, z_2, \theta) = \frac{(\theta-1)^2}{2(z_1+z_2+1)^2(z_1+\theta z_2+1)^2(\theta z_1+z_2+1)^2}\left(E + H\sqrt{D}\right),$$

$$s_2(z_1, z_2, \theta) = \frac{(\theta-1)^2}{2(z_1+z_2+1)^2(z_1+\theta z_2+1)^2(\theta z_1+z_2+1)^2}\left(E - H\sqrt{D}\right),$$

where

$$E = ((\theta+1)z_1z_2(z_1+z_2))^2 + 2z_1^2z_2^2(z_1+z_2+1)(\theta^2+4\theta+2) + (z_1^2+z_2^2)(2z_1+2z_2+4z_1z_2+1)$$
$$+ 2z_1z_2(z_1^3+z_2^3+\theta z_1^2+\theta z_2^2+\theta z_1+\theta z_2) + z_1^4+z_2^4,$$

$$D = ((\theta+1)z_1z_2(z_1+z_2))^2 + (z_1-z_2)^2(z_1+z_2+1)^2 + 2z_1z_2(z_1^2+z_2^2)(z_1+z_2+1)$$
$$- 2\theta z_1z_2\left((z_1-z_2)^2 - 3z_1z_2(z_1+z_2) + z_1^3+z_2^3\right),$$

$$H = (\theta+1)z_1z_2(z_1+z_2+2) + z_1^2+z_2^2+z_1+z_2.$$

Lemma 5.1. *Let $k = 2$. Then, for any solution (z_1, z_2) to the system of equations (4.8), the following equality holds:*

$$s = \max\{|s_1(z_1, z_2, \theta)|, |s_2(z_1, z_2, \theta)|\} = s_1(z_1, z_2, \theta),$$

where the pair (z_1, z_2) takes the values $\left(z_1^{(1)}, z_1^{(2)}\right)$, $\left(z_1^{(2)}, z_1^{(1)}\right)$, $\left(z_1^{(3)}, z_1^{(4)}\right)$ and $\left(z_1^{(4)}, z_1^{(3)}\right)$.

The Kesten–Stigum condition for determining the non-extremality region of the Gibbs measures is given as follows:

$$s_1(z_1, z_2, \theta) > \frac{1}{2}. \tag{5.3}$$

By means of this inequality, the non-extremality regions of the measures μ_i are determined for the given parameter values. Since the function $s_1(z_1, z_2, \theta)$ is symmetric with respect to the variables z_1 and z_2, it is sufficient to verify the condition (5.3) only for the pairs of solutions $\left(z_1^{(1)}, z_1^{(2)}\right)$ and $\left(z_1^{(3)}, z_1^{(4)}\right)$. Due to the complexity of the inequality (5.3), solving it analytically is significantly difficult. Therefore, we employ numerical computations and graphical analysis.

1. We examine the inequality (5.3) for the solution $\left(z_1^{(1)}, z_1^{(2)}\right)$ defined in the domains M_1, M_2, and M_3. To this end, we consider the following function:

$$l_1(z_1^{(1)}, z_1^{(2)}, \theta, \lambda) = s_1(z_1^{(1)}, z_1^{(2)}, \theta) - \frac{1}{2}.$$

By comparing the function l_1 with $l = 0$ in the domains M_1, M_2, and M_3 using Python software, it is determined that there is no domain where these functions take positive values (see Fig. 1., a)–c)).

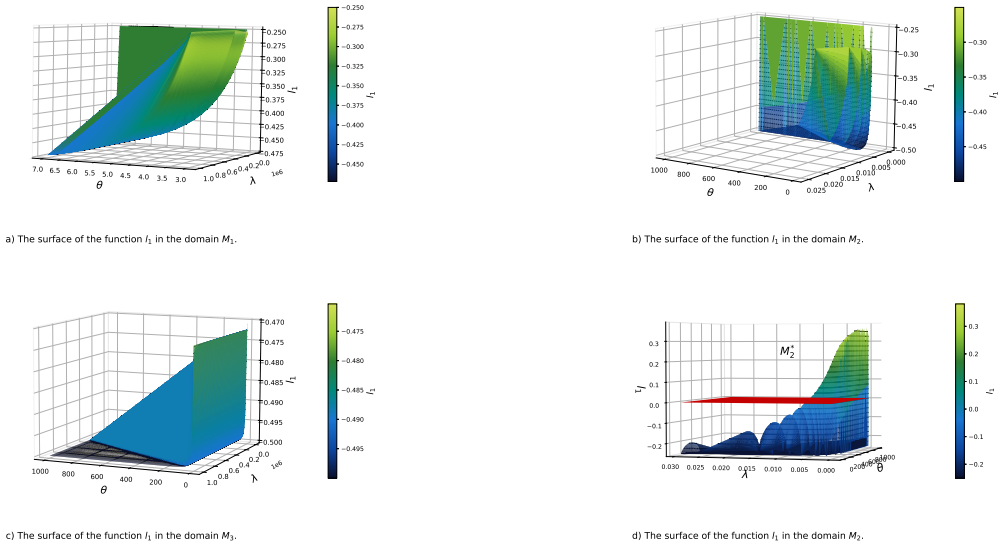
Remark 5.1. The measures μ_1 and μ_2 do not possess a non-extremality domain. Consequently, these measures may be extreme.

2. Next, we examine the inequality (5.3) for the solution $\left(z_1^{(3)}, z_1^{(4)}\right)$ in the domain M_2.

For the solution $\left(z_1^{(3)}, z_1^{(4)}\right)$, the domain where the function

$$l_1(z_1^{(3)}, z_1^{(4)}, \theta, \lambda) = s_1(z_1^{(3)}, z_1^{(4)}, \theta) - \frac{1}{2}$$

takes positive values is determined, and this domain satisfies inequality (5.3) (see Fig.1., d)).



a) The surface of the function l_1 in the domain M_1.

b) The surface of the function l_1 in the domain M_2.

c) The surface of the function l_1 in the domain M_3.

d) The surface of the function l_1 in the domain M_2.

Fig. 1.

Theorem 5.1. *For the measures μ_3 and μ_4 defined in the domain M_2, there exist subdomain M_2^* ($M_2^* \subsetneq M_2$), respectively, such that in these domain the measures μ_3 and μ_4 are not extreme.*

5.2. Extremality of the Gibbs measures

To study the extremality, we employ the methods and necessary definitions from [22]. The deletion of any edge $\langle x^0, x^1\rangle = l \in L$ from the Cayley tree $\Im^k$ breaks the tree into two components $\Im^{k_0}_{x^0}$ and $\Im^{k_1}_{x^1}$, each of which is called a *semi-infinite tree* or a *Cayley semitree*.

Consider the finite complete subtree S containing all initial points of the semitree $\Im^{k_0}_{x^0}$. The boundary ∂S of the subtree S consists of the nearest neighbors of its vertices, which lie in $\Im^{k_0}_{x^0} \setminus S$. We identify the subtree S with its vertex set. We denote the set of all edges in S by A and ∂A by $E(A)$.

In [22], two key quantities κ and γ were introduced, which play an important role in the study of the extremality of translation-invariant Gibbs measures. These quantities are properties of the set $\{\mu_x\}$ of Gibbs measures, where x is a fixed boundary condition and S is an arbitrary initial complete finite subtree of $\Im^{k_0}_{x^0}$. Given an initial subtree $S \subset \Im^{k_0}_{x^0}$ and any vertex $x \in S$, we write S_x for the (maximal) subtree of S with initial point x. If x is not an initial point of S, then μ_x^S denotes the Gibbs measure in which the "ancestor" x has spin s and the configuration on the lower boundary of S_x (i.e., on $\partial S_x \setminus \{x\}$, the ancestor x) is determined by l.

We define the norm distance between two measures μ_1 and μ_2 on Ω by

$$\|\mu_1-\mu_2\|_x=\frac{1}{2}\sum_{i\in\{-1,0,1\}}|\mu_1(\sigma(x)=i)-\mu_2(\sigma(x)=i)|\,.$$

Let $\xi^{x,s}$ be the configuration obtained from ξ by setting the spin at x to s. Following [22], we define

$$\kappa=\kappa(\mu)=\sup_{x\in\Gamma^k}\max_{\xi,\xi':\xi\equiv\xi' \text{ on } S\setminus\{x\}}\left\|\mu_A^{\xi}-\mu_A^{\xi'}\right\|_x,$$

$$\gamma=\gamma(\mu)=\sup_{A\subset\Gamma^k}\max_{\substack{y\in\partial A\\ x\in A}}\left\|\mu_A^{\xi,y,s}-\mu_A^{\xi}\right\|_x,$$

where the maximum is over all boundary conditions ξ, all $y\in\partial A$, all neighbors $x\in A$ of each vertex y, and all spins $s\in\{-1,0,1\}$. A sufficient condition for the extremality of a Gibbs measure μ is $k\kappa(\mu)\gamma(\mu)<1$; for the $G_k^{(2)}$ -periodic measures under consideration, this condition reads $k^2\kappa(\mu)\gamma(\mu)<1$ (see [23]).

The value of κ is defined by the following formula:

$$\kappa=\frac{1}{2}\max_{i\in\{-1,0,1\}}\sum_j|P_{ij}-P_{ji}|.$$

It follows that $|P_{il}-P_{jl}|=0$ for $i=j$. For $i\neq j$, (5.2) gives $\sum_{l\in\{-1,0,1\}}|P_{il}-P_{jl}|$.

Let $i=-1$ and $j=0$, then

$$\sum_{l\in\{-1,0,1\}}|P_{-1l}-P_{0l}|=\begin{cases}\dfrac{2z_1z_2(\theta-1)^2\Big((\theta+1)z_1z_2(z_1+z_2+2)+z_1^2+z_2^2+z_1+z_2\Big)}{(\theta z_2+z_1+1)^2(\theta z_1+z_2+1)(z_1+z_2+1)^2}, & if\quad z_1>z_2,\\ \dfrac{2z_2^2(\theta-1)^2\Big((\theta+1)z_1^2(z_1+z_2+2)+(2z_1+1)(z_2+1)+\theta z_1\Big)}{(\theta z_2+z_1+1)^2(\theta z_1+z_2+1)(z_1+z_2+1)^2}, & if\quad z_1<z_2.\end{cases}$$

Let $i=-1$ and $j=1$, then

$$\sum_{l\in\{-1,0,1\}}|P_{-1l}-P_{1l}|=\begin{cases}\dfrac{2z_1(\theta-1)^2\Big((\theta+1)^2(z_1+z_2+1)z_1z_2^2+\theta(z_1+3z_2+2)z_1z_2+(z_1+z_2)(z_1+z_2+z_2^2)+(2z_1z_2+1)z_1\Big)}{(\theta z_2+z_1+1)^2(\theta z_1+z_2+1)^2(z_1+z_2+1)}, & if\quad z_1>z_2,\\ \dfrac{2z_2(\theta-1)^2\Big((\theta+1)^2(z_1+z_2+1)z_1^2z_2+\theta(3z_1+z_2+2)z_1z_2+(z_1+z_2)(z_1+z_2+z_1^2)+(2z_1z_2+1)z_2\Big)}{(\theta z_2+z_1+1)^2(\theta z_1+z_2+1)^2(z_1+z_2+1)}, & if\quad z_1<z_2.\end{cases}$$

Let $i=0$ and $j=1$, then

$$\sum_{l\in\{-1,0,1\}}|P_{0l}-P_{1l}|=\begin{cases}\dfrac{2z_1^2(\theta-1)^2\Big((\theta+1)z_2^2(z_1+z_2+2)+(2z_2+1)(z_1+1)+\theta z_2\Big)}{(\theta z_1+z_2+1)^2(z_1+z_2+1)^2(\theta z_2+z_1+1)}, & if\quad z_1>z_2,\\ \dfrac{2z_1z_2(\theta-1)^2\Big(\theta+1)z_1z_2(z_1+z_2+2)+z_1^2+z_2^2+z_1+z_2\Big)}{(\theta z_1+z_2+1)^2(z_1+z_2+1)^2(\theta z_2+z_1+1)}, & if\quad z_1<z_2.\end{cases}$$

As a result, we obtain:

$$\kappa=\begin{cases}\dfrac{z_1(\theta-1)^2\Big((\theta+1)^2(z_1+z_2+1)z_1z_2^2+\theta(z_1+3z_2+2)z_1z_2+(z_1+z_2)(z_1+z_2+z_2^2)+(2z_1z_2+1)z_1\Big)}{(\theta z_2+z_1+1)^2(\theta z_1+z_2+1)^2(z_1+z_2+1)}, & if\quad z_1>z_2,\\ \dfrac{z_2(\theta-1)^2\Big((\theta+1)^2(z_1+z_2+1)z_1^2z_2+\theta(3z_1+z_2+2)z_1z_2+(z_1+z_2)(z_1+z_2+z_1^2)+(2z_1z_2+1)z_2\Big)}{(\theta z_2+z_1+1)^2(\theta z_1+z_2+1)^2(z_1+z_2+1)} & if\quad z_1<z_2.\end{cases}$$

Next, we calculate the parameter γ for these solutions:

$$\gamma = \max\left\{\|\mu_A^{\eta^{y,-1}} - \mu_A^{\eta^{y,0}}\|_x, \|\mu_A^{\eta^{y,0}} - \mu_A^{\eta^{y,1}}\|_x, \|\mu_A^{\eta^{y,-1}} - \mu_y^{\eta^{y,1}}\|_x\right\},$$

where the total variation distances between the conditional measures are computed as follows:

$$\|\mu_A^{\eta^{y,-1}} - \mu_A^{\eta^{y,0}}\| = \frac{1}{2}\sum_{s\in\{-1,0,1\}} |\mu_A^{\eta^{y,-1}}(\sigma(x)=s) - \mu_A^{\eta^{y,0}}(\sigma(x)=s)| =$$

$$= \frac{1}{2}(|P_{-1,-1} - P_{0,-1}| + |P_{-1,0} - P_{0,0}| + |P_{-1,1} - P_{0,1}|) =$$

$$= \begin{cases} \frac{2z_1z_2(\theta-1)^2\left((\theta+1)z_1z_2(z_1+z_2+2)+z_1^2+z_2^2+z_1+z_2\right)}{(\theta z_2+z_1+1)^2(\theta z_1+z_2+1)(z_1+z_2+1)^2}, & if \quad z_1 > z_2, \\ \frac{2z_2^2(\theta-1)^2\left((\theta+1)z_1^2(z_1+z_2+2)+(2z_1+1)(z_2+1)+\theta z_1\right)}{(\theta z_2+z_1+1)^2(\theta z_1+z_2+1)(z_1+z_2+1)^2}, & if \quad z_1 < z_2, \end{cases}$$

$$\|\mu_A^{\eta^{y,-1}} - \mu_A^{\eta^{y,1}}\|_x = \frac{1}{2}\sum_{s\in\{-1,0,1\}} |P_{-1,s} - P_{1,s}| =$$

$$= \begin{cases} \frac{2z_1(\theta-1)^2\left((\theta+1)^2(z_1+z_2+1)z_1z_2^2+\theta(z_1+3z_2+2)z_1z_2+(z_1+z_2)(z_1+z_2+z_2^2)+(2z_1z_2+1)z_1\right)}{(\theta z_2+z_1+1)^2(\theta z_1+z_2+1)^2(z_1+z_2+1)}, & if \quad z_1 > z_2, \\ \frac{2z_2(\theta-1)^2\left((\theta+1)^2(z_1+z_2+1)z_1^2z_2+\theta(3z_1+z_2+2)z_1z_2+(z_1+z_2)(z_1+z_2+z_1^2)+(2z_1z_2+1)z_2\right)}{(\theta z_2+z_1+1)^2(\theta z_1+z_2+1)^2(z_1+z_2+1)}, & if \quad z_1 < z_2, \end{cases}$$

$$\|\mu_A^{\eta^{y,0}} - \mu_A^{\eta^{y,1}}\|_x = \frac{1}{2}\sum_{s\in\{-1,0,1\}} |P_{0,s} - P_{1,s}| = \begin{cases} \frac{2z_1^2(\theta-1)^2\left((\theta+1)z_2^2(z_1+z_2+2)+(2z_2+1)(z_1+1)+\theta z_2\right)}{(\theta z_1+z_2+1)^2(z_1+z_2+1)^2(\theta z_2+z_1+1)}, & if \quad z_1 > z_2, \\ \frac{2z_1z_2(\theta-1)^2\left(\theta+1)z_1z_2(z_1+z_2+2)+z_1^2+z_2^2+z_1+z_2\right)}{(\theta z_1+z_2+1)^2(z_1+z_2+1)^2(\theta z_2+z_1+1)}, & if \quad z_1 < z_2. \end{cases}$$

From the above, it follows that:

$$\gamma = \begin{cases} \frac{z_1(\theta-1)^2\left((\theta+1)^2(z_1+z_2+1)z_1z_2^2+\theta(z_1+3z_2+2)z_1z_2+(z_1+z_2)(z_1+z_2+z_2^2)+(2z_1z_2+1)z_1\right)}{(\theta z_2+z_1+1)^2(\theta z_1+z_2+1)^2(z_1+z_2+1)}, & if \quad z_1 > z_2, \\ \frac{z_2(\theta-1)^2\left((\theta+1)^2(z_1+z_2+1)z_1^2z_2+\theta(3z_1+z_2+2)z_1z_2+(z_1+z_2)(z_1+z_2+z_1^2)+(2z_1z_2+1)z_2\right)}{(\theta z_2+z_1+1)^2(\theta z_1+z_2+1)^2(z_1+z_2+1)} & if \quad z_1 < z_2. \end{cases}$$

It is easy to verify that the function γ is symmetric with respect to the variables z_1 and z_2. Therefore, without loss of generality, in what follows we present all analyses and graphical illustrations only for the case $z_1 > z_2$.

It is observed that for all measures corresponding to the identified solutions in the case $k = 2$, the relation $\kappa = \gamma$ holds. Thus, the condition $4\kappa\gamma < 1$ is equivalent to:

$$\gamma < 1/2. \tag{5.4}$$

This inequality determines the regions of extremality for the measures μ_i. Since the expression for the parameter γ and the solutions of system (4.8) are rather complicated, it is difficult to analyze this inequality analytically. Therefore, we use numerical calculations and graphical analysis to identify the regions of extremality.

1. We examine the inequality (5.4) for the solutions $\left(z_1^{(1)}, z_1^{(2)}\right)$ defined in the domains M_1, M_2, and M_3. To this end, we introduce the following function: $l_2 = \gamma(z_1^{(1)}, z_1^{(2)}, \theta) - \frac{1}{2}$. By numerically analyzing the values of l_2 in these domains via Python software, it is established that the function takes only negative values across all considered regions, i.e., $l_2 < 0$ (see Fig. 2., a)–c)).

2. Next, we investigate the inequality (5.4) for the solution $(z_1^{(3)}, z_1^{(4)})$ defined in the domain M_2. To this end, we introduce the following function: $l_2 = \gamma(z_1^{(3)}, z_1^{(4)}, \theta) - \frac{1}{2}$. By numerically analyzing this function via Python software, we establish the existence of a subregion $M_2^{**} \subset M_2$ such that $l_2 < 0$ (see Fig. 2., d)).

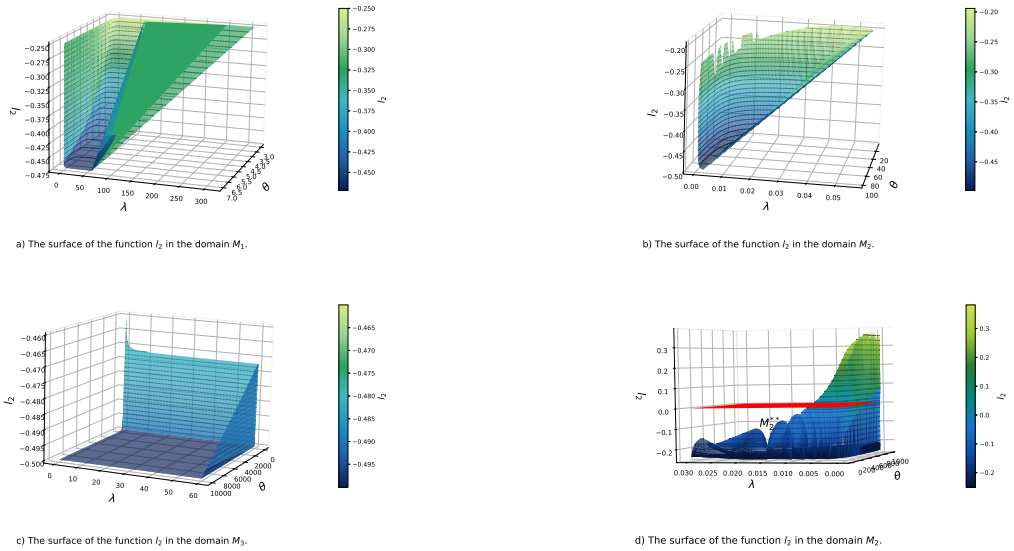


Fig. 2.

Theorem 5.2. *For the periodic Gibbs measures μ_i ($i = 1, 2, 3, 4$), the following assertions hold:*

1. *The measures μ_1 and μ_2 are extremal throughout the domains M_1, M_2, and M_3;*
2. *For the measures μ_3 and μ_4, there exists a proper subregion M_2^{**} ($M_2^{**} \subsetneq M_2$) in which these measures are extremal.*

References

[1] Bleher, P. M., Ganikhodjaev, N. N.: *On pure phases of the Ising model on the Bethe lattice*. Theor. Probab. Appl. **35**, 216–227 (1990).

[2] Bleher, P. M., Ruiz, J., Zagrebnov, V. A.: *On the purity of the limiting Gibbs state for the Ising model on the Bethe lattice*. J. Statist. Phys. **79**, 473–482 (1995).

[3] Higuchi, Y.: *Remarks on the limiting Gibbs states on a $(d+1)$-tree*. Publ. Res. Inst. Math. Sci. **13**, 335–348 (1977).

[4] Chayes, J. T., Chayes, L., Kotecký, R.: *The analysis of the Widom-Rowlinson model by stochastic geometric methods*. Comm. Math. Phys. **172**, 551–569 (1995).

[5] Formentin, M., Külske, C.: *Symmetric Entropy Bound on the Non-Reconstruction Regime of Markov Chains on Galton-Watson Trees*. Electron. Comm. Probab. **14**, 587–596 (2009).

[6] Gallavotti, G., Lebowitz, J.: *Phase Transitions in Binary Lattice Gases*. J. Math. Phys. **12**, 1129–1133 (1971).

[7] Georgii, H. O.: Gibbs Measures and Phase Transitions. Second edition, de Gruyter Studies in Mathematics, 9. Walter de Gruyter, Berlin (2011).

[8] Higuchi, Y., Takei, M.: *Some results on the phase structure of the two-dimensional Widom-Rowlinson model*. Osaka J. Math. **41**, 237–255 (2004).

[9] Jahnel, B., Külske, C.: *The Widom-Rowlinson model under spin flip: Immediate loss and sharp recovery of quasilocality*. Ann. Appl. Probab. **27**, 3845–3892 (2017).

[10] Ganikhodjaev, N. N., Rozikov, U. A.: *Description of periodic extreme Gibbs measures of some lattice models on a Cayley tree*. Theor. Math. Phys. **111** (1), 480–486 (1997).

[11] Külske, C., Rozikov, U. A.: *Fuzzy transformations and extremality of Gibbs measures for the potts model on a Cayley tree*. Random Struct. Alg. **50** (4), 636–678 (2017).
[12] Martin, J. B., Rozikov, U. A., Suhov, Y. M.: *A three state hard-core model on a Cayley tree*. Jour. Nonlinear Math. Phys. **12**, 432-448 (2005).
[13] Mazel, A., Suhov, Y., Stuhl, I., Zohren, S.: *Dominance of most tolerant species in multi-type lattice Widom-Rowlinson models*. J. Stat. Mech. **2014**, (2014).
[14] Mazel, A., Stuhl, I., Suhov, Y.: *Hard-core configurations on a triangular lattice and Eisenstein primes*. Preprint arXiv:1803.04041.
[15] Rozikov, U. A., Suhov, Y. M.: *Gibbs measures for SOS model on a Cayley tree*. Infin. Dimens. Anal. Quantum Probab. Relat. Top. **9**, 471–488 (2006).
[16] Rozikov, U. A., Shoyusupov, Sh. A.: *Fertile three state HC models on Cayley tree*. Theor. Math. Phys. **156**, 1319-1330 (2008).
[17] Rozikov, U. A.: Gibbs measures on Cayley trees. World Sci. Publ. Singapore. (2013).
[18] Rozikov, U. A., Khakimov, R. M.: *Gibbs measures for the fertile three-state hard core models on a Cayley tree*. Queueing Systems. **81**, 49-69 (2015).
[19] Ruelle, D.: *Existence of a Phase Transition in a Continuous Classical System*. Phys. Rev. Lett. **27**, 1040–1041 (1971).
[20] Widom, B., Rowlinson, J. S.: *New model for the study of liquid-vapor phase transition*. J. Chem. Phys. **52**, 1670–1684 (1970).
[21] Kissel, S., Külske, C., Rozikov, U.: *Hard-core and soft-core Widom–Rowlinson models on Cayley trees*. JSTAT/043204. (2019).
[22] Martinelli, F., Sinclair, A., Weitz, D.: *Fast mixing for independent sets, coloring and other models on trees*. Random Structures and Algorithms. **31**, 134–172 (2007).
[23] Rozikov, U. A., Khakimov, R. M., Makhammadaliev, M. T.: *Gibbs periodic measures for a two-state HC-model on a Cayley tree*. Journal of Mathematical Science, **278**, 647–660 (2024).
[24] Khatamov, N. M.: *Extremality of Gibbs measures for the HC-Blume–Capel model on the Cayley tree*. Math. Notes. **111** (5), 768–781 (2022).
[25] Haydarov, F.H.: *Kolmogorov extension theorem for non-probability measures on Cayley trees*. Reviews in Mathematical Physics. **36**(6), 1–12 (2024).

Affiliations

Bakhtiyor Z. Tojiboev
Address: Namangan State University, Department of Mathematics, 161100, Namangan, Uzbekistan.
e-mail: tbahtiyor2212@mail.ru
ORCID ID: https://orcid.org/0009-0002-7811-0624

Uzbekistan Journal of Mathematics and Computer Science
Volume-2, Issue-3-special, Page 136–144 (2026)
DOI: https://doi.org/10.56143/ujmcs.v2i3s.11
Research article

Построение решение краевой задачи для уравнения третьего порядка с кратными характеристиками в трехмерном пространстве

А.А. Хамитов* Ж.С. Эрматов

Аннотация

В работе исследуется краевая задача для дифференциального уравнения третьего порядка, обладающего кратными характеристиками, в трёхмерной области. Для доказательства единственности решения применяется метод энергетических интегралов. Существование решения устанавливается с использованием метода Фурье (разделения переменных). Полученное решение представляется в бесконечного функционального ряда, для которого обоснована возможность почленного дифференцирования по каждой из независимых переменных. При проверке условий равномерной сходимости ряда показано, что соответствующий «малый знаменатель» не обращается в нуль.

Ключевые слова: *уравнения с кратными характеристиками; краевая задача; единственность; существование; собственное значение; собственная функция, функциональный ряд, абсолютная и равномерная сходимость.*

Предметная классификацие AMS (2020): *Основная: 35C10 ; Дополнительная: 35G15.*

1. Введение

Дифференциальные уравнения третьего порядка с частными производными находят применение при моделировании процессов в нелинейной акустике, гидродинамике космической плазмы, а также в задачах фильтрации жидкости через пористые среды [1-2].

Первые результаты, посвящённые уравнениям третьего порядка с кратными характеристиками, были представлены в работах H. Block [3], E. Del Vecchio [4]. В дальнейшем L. Catabriga [5] для уравнения вида $D_x^{2n+1}u - D_y^2u = 0$ построил фундаментальное решение с использованием двойного несобственного интеграла, исследовал свойства соответствующего потенциала и решил ряд краевых задач.

В статьях [6–7] были получены фундаментальные решения для уравнений третьего порядка, содержащие вторые производные по времени. Эти решения выражаются через конфлюэнтные гипергеометрические функции, изучены их аналитические свойства и установлены асимптотические оценки при $|t| \to \infty$.

В работах [8–10] анализируются проблемы, связанные с возникновением малых знаменателей в процессе решения дифференциальных уравнений.

Received : *28–april–2026,* ***Accepted :*** *31–july–2026*
**** Corresponding author***

2. Постановка задачи

В области $D = \{(x, y, z) : 0 < x < p,\ 0 < y < q,\ 0 < z < r\}$ рассмотрим уравнения вида

$$L[u] \equiv u_{xxx} - u_{yy} - u_{zz} = 0, \tag{2.1}$$

где p, q, r - положительные вещественные числа ($p, q, r \in R^+$), и для данного уравнения рассматривается следующая краевая задача.

Задача A**.** Требуется найти регулярное решение уравнения (2.1) в области D, принадлежащее классу

$$u(x, y, z) \in C^{3,2,2}_{x,y,z}(D) \cap C^{2,1,1}_{x,y,z}\left(\overline{D}\right),$$

и удовлетворяющее краевым условиям:

$$u_y(x, 0, z) = u(x, q, z) = 0,\ u_z(x, y, 0) = u(x, y, r) = 0, \tag{2.2}$$

$$u(0, y, z) = \psi_1(y, z),\ u(p, y, z) = \psi_2(y, z),\ u_x(p, y, z) = \psi_3(y, z), \tag{2.3}$$

где $\psi_i(y, z)$, $\left(i = \overline{1, 3}\right)$ - заданные достаточно гладкие функции.

Следует отметить, что для двумерного аналога уравнения (2.1) корректные краевые задачи рассматривались в работах [11–12], а в трёхмерном случае — в исследованиях [13–16].

3. Единственность решения

Справедлива следующая теорема:

Теорема 3.1. *Если задача A имеет решение, то это решение единственно.*

Доказательство. Предположим противное: пусть существуют два различных решения $u_1(x, y, z)$ и $u_2(x, y, z)$. Тогда их разность

$$u(x, y, z) = u_1(x, y, z) - u_2(x, y, z)$$

удовлетворяет уравнению (2.1) и однородным краевым условиям. Докажем, что $u(x, y, z) \equiv 0$ в области $\overline{D}$.

В области D выполняется следующее тождество:

$$u\,L[u] \equiv \frac{\partial}{\partial x}\left(u\,u_{xx} - \frac{1}{2}\,u_x^2\right) - \frac{\partial}{\partial y}(u\,u_y) + u_y^2 - \frac{\partial}{\partial z}(u\,u_z) + u_z^2 = 0. \tag{3.1}$$

Проинтегрируем тождество (3.1) по области D и применим однородные граничные условия. В результате получим:

$$\frac{1}{2}\int_0^q\int_0^r u_x^2(0, y, z)dydz + \iiint_D u_y^2(x, y, z)\,dxdydz + \iiint_D u_z^2(x, y, z)\,dxdydz = 0. \tag{3.2}$$

Из (3.2) вытекает, что $u_y(x, y, z) = 0$, $u_z(x, y, z) = 0$, оттуда $u(x, y, z) = f(x)$, где $f(x)$ - некоторая функция, зависящая только от x, которая удовлетворяет уравнению (2.1) и однородным условиям задачи A. Подставляя $u(x, y, z) = f(x)$ в уравнение (2.1), получаем

$$f'''(x) = 0.$$

Общее решение этого уравнения имеет вид

$$f(x) = C_1x^2 + C_2x + C_3.$$

Из однородных краевых условий (2.3), т.е. $f(0) = f(p) = f'(p) = 0$, находим, что $f(x) \equiv 0$. Следовательно, $u(x, y, z) \equiv 0$ в области $\overline{D}$. Значит, $u_1(x, y, z) = u_2(x, y, z)$, что противоречит предположению о существовании двух различных решений.

Теорема 3.1 доказана.

4. Существование решения

Теорема 4.1. *Если для функции* $\psi_i(y,z)$, $i=\overline{1,3}$ *справедливы следующие условия:*

1) $\psi_i(y,z)\in C^{3,3}_{y,z}(0\le y\le q,\ 0\le z\le r)$, $i=\overline{1,3}$;

2)
$$\begin{cases} \psi_i(0,z)=\psi_i(q,z)=0, & \dfrac{\partial^2\psi_i(0,z)}{\partial y^2}=\dfrac{\partial^2\psi_i(q,z)}{\partial y^2}=0,\\ \psi_i(y,0)=\psi_i(y,r)=0, & \dfrac{\partial^2\psi_i(y,0)}{\partial z^2}=\dfrac{\partial^2\psi_i(y,r)}{\partial z^2}=0, \end{cases} \quad i=\overline{1,3},$$

то решение задачи A *существует в указанном классе функций.*

Доказательство. Решение задачи A будем искать в виде

$$u(x,y,z)=\mathrm{X}(x)\,V(y,z). \tag{4.1}$$

Подставляя (4.1) в уравнение (2.1) и применяя метод разделения переменных, приходим к следующему обыкновенному дифференциальному уравнению для функции $X(x)$:

$$X'''+\lambda X=0, \tag{4.2}$$

а для функции $V(y,z)$ получается краевая задача вида

$$\begin{cases} V_{yy}+V_{zz}+\lambda V=0,\\ V_y(0,z)=V(q,z)=0,\\ V_z(y,0)=V(y,r)=0, \end{cases} \tag{4.3}$$

где λ - параметр разделения.

Решение спектральной задачи (4.3) имеет вид:

$$V_{n,m}(y,z)=\frac{2}{\sqrt{qr}}\cos\left(\frac{\pi(2n-1)}{2q}y\right)\cos\left(\frac{\pi(2m-1)}{2r}z\right), \tag{4.4}$$

которые соответствуют собственным значениям

$$\lambda_{n,m}=\left(\frac{(2n-1)^2}{q^2}+\frac{(2m-1)^2}{r^2}\right)\frac{\pi^2}{4}, n,m\in N.$$

Известно, что система собственных функций (4.4) образует полный ортонормированный базис в пространстве $L_2(0<y<q,\ 0<z<r)$ [17].

Теперь переходим к решению уравнения (4.2). Нам известно, что характеристическое уравнение (4.2) имеет один вещественный $\tilde{k}_1=-\sqrt[3]{\lambda_{n,m}}$ и два комплексных $\tilde{k}_{2,3}=\left(\frac{1}{2}\pm\frac{\sqrt{3}}{2}i\right)\sqrt[3]{\lambda_{n,m}}$ корней. Тогда общее решение (4.2) имеет вид

$$X_{n,m}(x)=C_{1n,m}e^{-k_{n,m}x}+e^{\frac{1}{2}k_{n,m}x}\left(C_{2n,m}\cos\frac{\sqrt{3}}{2}k_{n,m}x+C_{3n,m}\sin\frac{\sqrt{3}}{2}k_{n,m}x\right), \tag{4.5}$$

здесь $C_{in,m}$, $i=\overline{1,3}$ - произвольные постоянные, $k_{n,m}=\sqrt[3]{\lambda_{n,m}}=\sqrt[3]{\left(\frac{(2n-1)^2}{q^2}+\frac{(2m-1)^2}{r^2}\right)\frac{\pi^2}{4}}$. Согласно (4.4) и (4.5), из равенства (4.1) следуют, что функции

$$u_{n,m}(x,y,z)=\frac{2}{\sqrt{qr}}\left[C_{1n,m}e^{-k_{n,m}x}+e^{\frac{1}{2}k_{n,m}x}\left(C_{2n,m}\cos\frac{\sqrt{3}}{2}k_{n,m}x+\right.\right.$$
$$\left.\left.+C_{3n,m}\sin\frac{\sqrt{3}}{2}k_{n,m}x\right)\right]\cos\left(\frac{\pi(2n-1)}{2q}y\right)\cos\left(\frac{\pi(2m-1)}{2r}z\right),$$

представляют собой частные решения уравнения (2.1), удовлетворяющим краевым условиям (2.2).

В силу линейности и однородности уравнения (2.1) суперпозиция построенных частных решений также является его решением. Поэтому решение задачи A будем искать в виде следующего ряда:

$$u(x,y,z)=\frac{2}{\sqrt{qr}}\sum_{n,m=1}^{+\infty}\left[C_{1n,m}e^{-k_{n,m}x}+e^{\frac{1}{2}k_{n,m}x}\left(C_{2n,m}\cos\frac{\sqrt{3}}{2}k_{n,m}x+\right.\right.$$
$$\left.\left.+C_{3n,m}\sin\frac{\sqrt{3}}{2}k_{n,m}x\right)\right]\cos\left(\frac{\pi(2n-1)}{2q}y\right)\cos\left(\frac{\pi(2m-1)}{2r}z\right). \tag{4.6}$$

Предполагая временно, что ряд (4.6) вместе со своими производными, входящими в уравнение (2.1), сходится равномерно, и налагая на функцию $u(x,y,z)$, определяемую этим рядом, краевые условия (2.3), приходим к следующим соотношениям:

$$\begin{cases} u(0,y,z)=\psi_1(y,z)=\dfrac{2}{\sqrt{qr}}\displaystyle\sum_{m,n=1}^{+\infty}\psi_{1n,m}\cos\left(\frac{\pi(2n-1)}{2q}y\right)\cos\left(\frac{\pi(2m-1)}{2r}z\right),\\ u(p,y,z)=\psi_2(y,z)=\dfrac{2}{\sqrt{qr}}\displaystyle\sum_{m,n=1}^{+\infty}\psi_{2n,m}\cos\left(\frac{\pi(2n-1)}{2q}y\right)\cos\left(\frac{\pi(2m-1)}{2r}z\right),\\ u_x(p,y,z)=\psi_3(y,z)=\dfrac{2}{\sqrt{qr}}\displaystyle\sum_{m,n=1}^{+\infty}\psi_{3n,m}\cos\left(\frac{\pi(2n-1)}{2q}y\right)\cos\left(\frac{\pi(2m-1)}{2r}z\right), \end{cases} \tag{4.7}$$

где

$$\begin{cases} C_{1n,m}+C_{2n,m}=\psi_{1n,m},\\ C_{1n,m}e^{-k_{n,m}p}+e^{\frac{1}{2}k_{n,m}p}\left(C_{2n,m}\cos\dfrac{\sqrt{3}}{2}k_{n,m}p+C_{3n,m}\sin\dfrac{\sqrt{3}}{2}k_{n,m}p\right)=\psi_{2n,m},\\ -k_{n,m}C_{1n,m}e^{-k_{n,m}p}+k_{n,m}e^{\frac{1}{2}k_{n,m}p}\left(C_{2n,m}\cos\left(\dfrac{\sqrt{3}}{2}k_{n,m}p+\dfrac{\pi}{3}\right)+\right.\\ \left.+C_{3n,m}\sin\left(\dfrac{\sqrt{3}}{2}k_{n,m}p+\dfrac{\pi}{3}\right)\right)=\psi_{3n,m}. \end{cases} \tag{4.8}$$

Из (4.7) видно, что числа $\psi_{in,m},\ (i=\overline{1,3})$ являются коэффициентами Фурье функции $\psi_i(y,z),\ (i=\overline{1,3})$ при разложении их в ряд Фурье по косинусам на $\{0<y<q,\ 0<z<r\}$, т.е.

$$\psi_{in,m}=\frac{2}{\sqrt{qr}}\int_0^q\int_0^r\psi_i(y,z)\cos\left(\frac{\pi(2n-1)}{2q}y\right)\cos\left(\frac{\pi(2m-1)}{2r}z\right)dydz,\quad i=\overline{1,3}. \tag{4.9}$$

Итак, для вычисления коэффициентов $C_{in,m}, i=\overline{1,3}$ получена система уравнений (4.8). Основной определитель данной системы равен

$$\Delta=\begin{vmatrix} 1 & 1 & 0\\ e^{-k_{n,m}p} & \alpha e^{\frac{1}{2}k_{n,m}p} & \beta e^{\frac{1}{2}k_{n,m}p}\\ -k_{n,m}e^{-k_{n,m}p} & \gamma k_{n,m}e^{\frac{1}{2}k_{n,m}p} & \delta k_{n,m}e^{\frac{1}{2}k_{n,m}p} \end{vmatrix}=\sqrt{3}k_{n,m}e^{k_{n,m}p}\bar{\Delta},$$

где

$$\bar{\Delta}=\frac{1}{2}-e^{-\frac{3}{2}k_{n,m}p}\sin\left(\frac{\sqrt{3}}{2}k_{n,m}p+\frac{\pi}{6}\right),\ \alpha=\cos\frac{\sqrt{3}}{2}k_{n,m}p,\ \beta=\sin\frac{\sqrt{3}}{2}k_{n,m}p,$$
$$\gamma=\cos\left(\frac{\sqrt{3}}{2}k_{n,m}p+\frac{\pi}{3}\right),\ \delta=\sin\left(\frac{\sqrt{3}}{2}k_{n,m}p+\frac{\pi}{3}\right),\ n,m\in N.$$

Теперь покажем, что $\bar{\Delta}\neq 0$ и $\min|\bar{\Delta}|>0$, для этого введем обозначение $f(\xi)=\frac{1}{2}-e^{-\sqrt{3}\xi}\sin\left(\xi+\frac{\pi}{6}\right)$, где $\xi=\frac{\sqrt{3}}{2}k_{n,m}p, k_{n,m}=\sqrt[3]{\left(\frac{(2n-1)^2}{q^2}+\frac{(2m-1)^2}{r^2}\right)\frac{\pi^2}{4}}>0, p>0, \xi=\frac{\sqrt{3}}{2}k_{n,m}p$ - принимает только дискретные значения:

а) $f'(\xi) > 0$, т.е. $f'(\xi) = 2e^{-\sqrt{3}\xi}\sin\xi > 0 \Rightarrow \sin\xi > 0$, отсюда получим, что при $0 < \xi < \pi$ - функция возрастает и минимальное значение принимает при $n = 1, m = 1 \Rightarrow \bar{\Delta} \geq \dfrac{1}{2} - e^{-\frac{3}{2}k_{1,1}p}\sin\left(\dfrac{\sqrt{3}}{2}k_{1,1}p + \dfrac{\pi}{6}\right) = \delta_1 > 0$;

б) при $\xi \geq \pi$ выполняется неравенство $\dfrac{1}{2} - e^{-\sqrt{3}\xi}\sin\left(\xi + \dfrac{\pi}{6}\right) > \delta_2 \Rightarrow e^{\sqrt{3}\xi} > \left(\dfrac{1}{2} - \delta_2\right) > 1 \Rightarrow 0 < \delta_2 < \min\limits_{\xi\geq\pi}\left(\dfrac{1}{2} - e^{-\sqrt{3}\xi}\right) = \dfrac{1}{2} - e^{-\sqrt{3}\pi}$.

Так как выполняются эти условия и минимальное значения $\bar{\Delta} = \delta = \min\{\delta_1; \delta_2\} > 0$, получим $\Delta \neq 0$. Следует отметить, что минимальное значение $\bar{\Delta} = \delta = \min\{\delta_1; \delta_2\} > 0$ даёт отсутствие "малого знаменателя".

В силу $\Delta \neq 0$, система уравнений (4.8) имеет единственное решение:

$$C_{1n,m} = \frac{1}{\Delta}\left[\frac{\sqrt{3}}{2}\psi_{1n,m}k_{n,m}e^{k_{n,m}p} - \psi_{2n,m}\delta k_{n,m}e^{\frac{1}{2}k_{n,m}p} + \psi_{3n,m}\beta e^{\frac{1}{2}k_{n,m}p}\right],$$

$$C_{2n,m} = \frac{1}{\Delta}\left[-k_{n,m}(\beta+\delta)\psi_{1n,m}e^{-\frac{1}{2}k_{n,m}p} + \psi_{2n,m}\delta k_{n,m}e^{\frac{1}{2}k_{n,m}p} - \psi_{3n,m}\beta e^{\frac{1}{2}k_{n,m}p}\right],$$

$$C_{3n,m} = \frac{1}{\Delta}\left[k_{n,m}(\alpha+\gamma)\psi_{1n,m}e^{-\frac{1}{2}k_{n,m}p} - \psi_{2n,m}k_{n,m}\left(e^{-k_{n,m}p} + \gamma e^{\frac{1}{2}k_{n,m}p}\right) + \right.$$
$$\left. + \psi_{3n,m}\left(\alpha e^{\frac{1}{2}k_{n,m}p} - e^{-k_{n,m}p}\right)\right].$$

Подставив значения $C_{in,m},\ i = \overline{1,3}$ в (4.6), получим

$$u(x,y,z) = \frac{2}{\sqrt{qr}}\sum_{n,m=1}^{+\infty}\left[\psi_{1n,m}B_{1n,m}(x) + \psi_{2n,m}B_{2n,m}(x) + \psi_{3n,m}B_{3n,m}(x)\right]\times$$
$$\times\cos\left(\frac{\pi(2n-1)}{2q}y\right)\cos\left(\frac{\pi(2m-1)}{2r}z\right), \tag{4.10}$$

где

$$B_{1n,m}(x) = \frac{\sqrt{3}k_{n,m}}{\Delta}\left(\frac{1}{2}e^{k_{n,m}(p-x)} - e^{-\frac{1}{2}k_{n,m}(p-x)}\sin\left(\frac{\sqrt{3}}{2}k_{n,m}(p-x) + \frac{\pi}{6}\right)\right),$$

$$B_{2n,m}(x) = \frac{k_{n,m}}{\Delta}\left(-e^{k_{n,m}(\frac{1}{2}p-x)}\sin\left(\frac{\sqrt{3}}{2}k_{n,m}p + \frac{\pi}{3}\right) - e^{k_{n,m}(\frac{1}{2}x-p)}\sin\frac{\sqrt{3}}{2}k_{n,m}x + \right.$$
$$\left. + e^{\frac{1}{2}k_{n,m}(p+x)}\sin\left(\frac{\sqrt{3}}{2}k_{n,m}(p-x) + \frac{\pi}{3}\right)\right),$$

$$B_{3n,m}(x) = \frac{1}{\Delta}\left(e^{k_{n,m}(\frac{1}{2}p-x)}\sin\frac{\sqrt{3}}{2}k_{n,m}p - e^{k_{n,m}(\frac{1}{2}x-p)}\sin\frac{\sqrt{3}}{2}k_{n,m}x - \right.$$
$$\left. - e^{\frac{1}{2}k_{n,m}(p+x)}\sin\left(\frac{\sqrt{3}}{2}k_{n,m}(p-x)\right)\right).$$

Теперь покажем, что ряд (4.10) и его частные производные u_{xxx}, u_{yy} и u_{zz} сходятся абсолютно и равномерно в области $\overline{D}$. Это означает, что функция $u(x,y,z)$, определяемая рядом (4,10), является решением задачи A.

Сначала покажем абсолютную и равномерную сходимость ряда (4.10). На основании (4.10) получим оценку

$$|u(x,y,z)| \leq M\left[\sum_{n,m=1}^{+\infty}|\psi_{1n,m}|\,|B_{1n,m}(x)| + \sum_{n,m=1}^{+\infty}|\psi_{2n,m}|\,|B_{2n,m}(x)| + \right.$$
$$\left. + \sum_{n,m=1}^{+\infty}|\psi_{3n,m}|\,|B_{3n,m}(x)|\right]. \tag{4.11}$$

В данной работе все оценки выполнены в замкнутой области $\overline{D}$, а максимальное значение всех известных положительных чисел, входящих в оценках, будем обозначать с одной буквой M.

Теперь применяя интегрирование по частям к (4.9) с учетом условия 2 теоремы (4.1), получаем

$$|\psi_{in,m}| = M\frac{|\Psi_{in,m}|}{n^3m^3}, i = \overline{1,3}, \tag{4.12}$$

где

$$\Psi_{in,m} = \frac{2}{\sqrt{qr}}\int\limits_0^q\int\limits_0^r \frac{\partial^6\psi_i(y,z)}{\partial y^3\partial z^3}\sin\left(\frac{\pi\,(2n-1)}{2q}y\right)\sin\left(\frac{\pi\,(2m-1)}{2r}z\right)dydz, i = \overline{1,3}.$$

Учитывая (4.12), оценка (4.11) принимает следующий вид

$$|u(x,y,z)| \le M\left[\sum_{n,m=1}^{+\infty}\frac{|\Psi_{1n,m}|}{n^3m^3}\,|B_{1n,m}\,(x)| + \sum_{n,m=1}^{+\infty}\frac{|\Psi_{2n,m}|}{n^3m^3}\,|B_{2n,m}\,(x)| + \sum_{n,m=1}^{+\infty}\frac{|\Psi_{3n,m}|}{n^3m^3}\,|B_{3n,m}\,(x)|\right]. \tag{4.13}$$

С учетом $x \in [0,p]$ оценим функции $B_{in,m}(x)$, $i = \overline{1,3}$:

$$|B_{1n,m}(x)| \le \frac{1}{|\bar{\Delta}|}\left[\frac{1}{2}e^{-k_{n,m}x} + e^{-\frac{1}{2}k_{n,m}(3p-x)}\right] \le M,$$

$$|B_{2n,m}(x)| \le \frac{1}{\sqrt{3}\,|\bar{\Delta}|}\left[e^{-k_{n,m}(\frac{1}{2}p+x)} + e^{-k_{n,m}(2p-\frac{1}{2}x)} + e^{-\frac{1}{2}k_{n,m}(p-x)}\right] \le M,$$

$$|B_{3n,m}(x)| \le \frac{1}{\sqrt{3}k_{n,m}\,|\bar{\Delta}|}\left[e^{-k_{n,m}(\frac{1}{2}p+x)} + e^{-k_{n,m}(2p-\frac{1}{2}x)} + e^{-\frac{1}{2}k_{n,m}(p-x)}\right] \le \frac{1}{k_{n,m}}M,$$

$$|B_{3n,m}(x)| \le \frac{1}{\sqrt{3}k_{n,m}\,|\bar{\Delta}|}\left[e^{-k_{n,m}(\frac{1}{2}p+x)} + e^{-k_{n,m}(2p-\frac{1}{2}x)} + e^{-\frac{1}{2}k_{n,m}(p-x)}\right] \le \frac{1}{k_{n,m}}M,$$

где

$$\bar{\Delta} = \frac{1}{2} - e^{-\frac{3}{2}k_{n,m}p}\sin\left(\frac{\sqrt{3}}{2}k_{n,m}p + \frac{\pi}{6}\right).$$

Учитывая оценки от функции $B_{in,m}(x)$, $i = \overline{1,3}$, тогда из (4.13), получим

$$|u(x,y,z)| \le M\left[\sum_{n,m=1}^{+\infty}\frac{|\Psi_{1n,m}|}{n^3m^3} + \sum_{n,m=1}^{+\infty}\frac{|\Psi_{2n,m}|}{n^3m^3} + \sum_{n,m=1}^{+\infty}\frac{|\Psi_{3n,m}|}{n^{\frac{10}{3}}m^{\frac{10}{3}}}\right] < \infty.$$

Отсюда следует, что ряд (4.10) сходится абсолютно и равномерно в области $\overline{D}$.

Далее покажем, что частные производные этого ряда, участвующие в уравнении (2.1), также обладают свойством абсолютной и равномерной сходимости в $\overline{D}$. Для этого продифференцируем ряд (4.10) почленно дважды по переменными y и z. В результате получаем следующие выражения для вторых производных:

$$\frac{\partial^2 u}{\partial y^2} = -\frac{\pi^2}{2\sqrt{q^5r}}(2n-1)^2 u\,(x,y,z)\,,$$

$$\frac{\partial^2 u}{\partial z^2} = -\frac{\pi^2}{2\sqrt{qr^5}}(2m-1)^2 u\,(x,y,z)\,.$$

Учитывая оценку $u(x,y,z)$, получим

$$\left|\frac{\partial^2 u}{\partial y^2}\right| \le M\left[\sum_{n,m=1}^{+\infty}\frac{|\Psi_{1n,m}|}{nm^3} + \sum_{n,m=1}^{+\infty}\frac{|\Psi_{2n,m}|}{nm^3} + \sum_{n,m=1}^{+\infty}\frac{|\Psi_{3n,m}|}{n^{\frac{4}{3}}m^{\frac{10}{3}}}\right],$$

$$\left|\frac{\partial^2 u}{\partial z^2}\right| \le M\left[\sum_{n,m=1}^{+\infty}\frac{|\Psi_{1n,m}|}{n^3m} + \sum_{n,m=1}^{+\infty}\frac{|\Psi_{2n,m}|}{n^3m} + \sum_{n,m=1}^{+\infty}\frac{|\Psi_{3n,m}|}{n^{\frac{10}{3}}m^{\frac{4}{3}}}\right].$$

Используя неравенства Коши- Буняковского и Бесселя, получим

$$\left|\frac{\partial^2 u}{\partial y^2}\right| \le M\left[\sqrt{\sum_{n,m=1}^{+\infty}\left(\frac{1}{nm^3}\right)^2}\left(\sqrt{\sum_{n,m=1}^{+\infty}|\Psi_{1n,m}|^2}+\sqrt{\sum_{n,m=1}^{+\infty}|\Psi_{2n,m}|^2}\right)+\sum_{n,m=1}^{+\infty}\frac{|\Psi_{3n,m}|}{n^{\frac{4}{3}}m^{\frac{10}{3}}}\right]\le$$
$$\le M\left[\left\|\frac{\partial^6\psi_1(y,z)}{\partial y^3\partial z^3}\right\|_{L_2(0<y<q,0<z<r)}+\left\|\frac{\partial^6\psi_2(y,z)}{\partial y^3\partial z^3}\right\|_{L_2(0<y<q,0<z<r)}+\sum_{n,m=1}^{+\infty}\frac{|\Psi_{3n,m}|}{n^{\frac{4}{3}}m^{\frac{10}{3}}}\right]<\infty,$$

$$\left|\frac{\partial^2 u}{\partial z^2}\right| \le M\left[\sqrt{\sum_{n,m=1}^{+\infty}\left(\frac{1}{n^3m}\right)^2}\left(\sqrt{\sum_{n,m=1}^{+\infty}|\Psi_{1n,m}|^2}+\sqrt{\sum_{n,m=1}^{+\infty}|\Psi_{2n,m}|^2}\right)+\sum_{n,m=1}^{+\infty}\frac{|\Psi_{3n,m}|}{n^{\frac{10}{3}}m^{\frac{4}{3}}}\right]\le$$
$$\le M\left[\left\|\frac{\partial^6\psi_1(y,z)}{\partial y^3\partial z^3}\right\|_{L_2(0<y<q,0<z<r)}+\left\|\frac{\partial^6\psi_2(y,z)}{\partial y^3\partial z^3}\right\|_{L_2((0<y<q,0<z<r))}+\sum_{n,m=1}^{+\infty}\frac{|\Psi_{3n,m}|}{n^{\frac{10}{3}}m^{\frac{4}{3}}}\right]<\infty,$$

где

$$\sum_{n,m=1}^{+\infty}|\Psi_{in,m}|^2 \le \left\|\frac{\partial^6\psi_i(y,z)}{\partial y^3\partial z^3}\right\|^2_{L_2(0<y<q,0<z<r)}, i=1,2,$$

$$\sum_{n=1}^{+\infty}\frac{1}{n^2}=\frac{\pi^2}{6},\ \sum_{m=1}^{+\infty}\frac{1}{m^2}=\frac{\pi^2}{6}.$$

Следовательно, ряд, соответствующий функциям $\frac{\partial^2 u}{\partial y^2}$, $\frac{\partial^2 u}{\partial z^2}$ сходится абсолютно и равномерно в области $\overline{D}$. Аналогично можно доказать абсолютную и равномерную сходимость ряда $\frac{\partial^3 u}{\partial x^3}$, или же это можно получить из $\left|\frac{\partial^3 u}{\partial x^3}\right| \le \left|\frac{\partial^2 u}{\partial y^2}\right| + \left|\frac{\partial^2 u}{\partial z^2}\right|$ и приведенного выше доказательства.

Теорема 4.1 доказана.

5. Заключение

В данной работе исследована краевая задача для дифференциального уравнения третьего порядка с кратными характеристиками в трёхмерной области. На основе проведённого анализа можно сформулировать следующие основные результаты:

1. Единственность решения. С помощью метода энергетических интегралов доказано, что если решение поставленной задачи существует, то оно единственно. При этом установлено, что разность двух возможных решений тождественно равна нулю в замкнутой области.

2. Существование решения. Методом разделения переменных построено явное решение в виде бесконечного двойного ряда по системе собственных функций соответствующей спектральной задачи. Получены формулы для коэффициентов ряда через коэффициенты Фурье заданных краевых функций.

3. Обоснование корректности. Доказана абсолютная и равномерная сходимость построенного ряда, а также рядов его частных производных, входящих в исходное уравнение. Это обосновывает возможность почленного дифференцирования и подтверждает, что построенная функция действительно является регулярным решением задачи в указанном классе гладкости.

4. Отсутствие малых знаменателей. Показано, что определитель системы для определения неизвестных коэффициентов не обращается в нуль, причём его модуль ограничен снизу положительной постоянной. Это гарантирует устойчивость решения и корректность используемого метода.

Таким образом, в работе получено полное аналитическое решение краевой задачи для дифференциального уравнения третьего порядка с кратными характеристиками, доказаны его существование и единственность в соответствующем функциональном классе. Полученные результаты могут быть использованы при исследовании

более общих краевых задач для уравнений подобного типа, а также в прикладных задачах нелинейной акустики, гидродинамики и теории фильтрации.

Благодарность

Авторы хотели бы выразить искреннюю благодарность редактору и рецензентам за их полезные комментарии и предложения.

Финансирование

Для этой работы нет финансирования.

Список литературы

[1] Юлдашев Т. К.: *Обратная задача для одного интегро-дифференциального уравнения Фредгольма в частных производных третьего порядка.* Вестн. Сам.гос. техн. ун-та. Сер. Физ.-мат. науки, - Самара. **34** (1), 56-65 (2014).
[2] Tursunov D. A., Omaralieva G. A.: *An Intermediate Boundary Layer in Singularly Perturbed First-Order Equations.* Trudy Instituta Matematiki i Mekhaniki UrO RAN. **28** (2), 193-200 (2022).
[3] Block H.: *Sur les equations lineaires aux derives parielles a carateristiques multiples.* Ark. Mat. Astron. Fus. Note 1. **13** (7), 1-51 (1912).
[4] Del Vecchio E.: *Sulleequazioni* $z_{xxx} - z_y + \varphi_1(x, y) = 0$, $z_{xxx} - z_{yy} + \varphi_2(x, y) = 0$. Memorie R. Accad. Sci. Ser.2. - Torino. **66**, 1-41 (1915).
[5] Cattabriga L.: *Potenziali di linea e di dominio per equazioni non paraboliche in due variabilia caratteristiche multiple.* Rendiconti del seminario matimatico della univ. di Padava. **31**, 1-45 (1961).
[6] Джураев Т. Д., Апаков Ю. П.: *Об автомодельном решении одного уравнения третьего порядка с кратными характеристиками.* Вестник Сам. гос. техн. ун-та. Сер. Физ.-мат. науки, - Самара. **15** (2), 18-26 (2007).
[7] Джураев Т. Д., Апаков Ю. П.: *К теории уравнения третьего порядка с кратными характеристиками, содержащего вторую производную по времени.* Украинский математический журнал. **62** (1), 40-51 (2010).
[8] Арнольд В. И.: *Малые знаменатели и проблемы устойчивости движения в классической и небесной механике.* УМН. **18** (6), 91-192 (1967).
[9] Козлов В. В.: *Условие вмороженности поля направлений, малые знаменатели и хаотизация стационарных течений вязкой жидкости.* ПММ. **63** (2), 237-244 (1999).
[10] Сабитов К. Б.: *Задача Дирихле для уравнений с частными производными.* Матем. заметки. **97** (2), 262-276 (2015).
[11] Apakov Yu. P., Umarov R. A.: *On the third boundary problem for a nonhomogeneous third order equation with multiple characteristics.* Uzbek Mathematical Journal. **67** (1), 4-14 (2023).
[12] Умаров Р. А.: *Об одной краевой задаче для неоднородного уравнения третьего порядка с переменными коэффициентами.* Научный вестник НамГУ. **2**, 16-27 (2024).
[13] Сабитов К. Б.: *Начально-граничные задачи для уравнения колебаний прямоугольной пластины.* Изв. вузов. Матем. **10**, 60-70 (2021).
[14] Уткина Е. А.: *Задача Дирихле для одного трехмерного уравнения.* Вестник Самарского государственного университета. Естественнонаучн. сер. **76** (1), 84-95 (2010).
[15] Apakov Yu. P., Hamitov A. A.: *Third Boundary Value Problem for an Equation with the Third Order Multiple Characteristics in Three Dimensional Space.* Lobachevskii Journal of Mathematics. **44** (2), 523-532 (2023).
[16] Apakov Yu. P., Hamitov A. A.: *Third boundary value problem for a third-order inhomogeneous equation with multiple characteristics in three-dimensional space.* Uzbek Mathematical Journal. **69** (1), 27-36 (2025).
[17] Ректорис К.: Вариационные методы в математической физике и технике. Москва, Мир (1985).

Solving the Boundary value Problem for a Third-Order Equation with Multiple Characteristics in Three-Dimensional Space

Khamitov Azizbek Akhmadjon ugli, Ermatov Jamoldin Salokhiddin ugli

Abstract

In this work, a boundary value problem in three-dimensional space is considered for a third-order equation with multiple characteristics. The uniqueness of the solution to the problem is proved using the method of energy integrals, and the existence of the solution is proved using the method of separation of variables. The solution is explicitly constructed as an infinite series, and the possibility of differentiating the series with respect to all variables is justified. The uniform convergence is established by proving that the "small denominator"is non-zero.

Keywords

Equations with multiple characteristics; boundary value problem; uniqueness; existence; eigenvalue; eigenfunction; functional series and uniform convergence.

Affiliations

Khamitov Azizbek Akhmadjon ugli
Address: Namangan State Technical University, Dept. of Higher Mathematics, 160103, Namangan, Uzbekistan.
e-mail: azizbek.khamitov.93@mail.ru
ORCID ID: https://orcid.org/0009-0000-2184-450X

Ermatov Jamoldin Salokhiddin ugli
Address: Namangan State Technical University, Dept. of Higher Mathematics, 160103, Namangan, Uzbekistan.
e-mail: ermatovjamoldin578@gmail.com
ORCID ID: https://orcid.org/0009-0003-7127-1828

Uzbekistan Journal of Mathematics and Computer Science
Volume-2, Issue-3-special, Page 145–159 (2026)
DOI: https://doi.org/10.56143/ujmcs.v2i3s.12
Research article

The Inverse Spectral Problem Method for the Periodic Differential–Difference Sine–Gordon Equation

B.A. Babajanov * A.K. Babadjanova and B. Fayzullaev

Abstract

This paper is devoted to the investigation of the periodic discrete sine–Gordon equation by means of the inverse spectral problem method. The associated discrete Lax pair is studied, and the spectral theory of the corresponding periodic linear difference problem is established. Explicit representations of fundamental solutions are obtained in terms of polynomial structures, and the spectral data are characterized via a hyperelliptic curve.
Using these results, trace formulas are derived that provide an explicit reconstruction of the solution in terms of the spectral parameters. An algorithm for solving the inverse spectral problem is formulated based on the evolution of the spectral data. Finally, we derive a system of evolution equations for the spectral parameters, which may be regarded as a discrete analogue of the Dubrovin equations.

Keywords: *Discrete sine–Gordon equation; inverse spectral problem; Lax pair; spectral data; trace formula; hyperelliptic curve; finite-gap integration; Dubrovin equations.*

AMS Subject Classification (2020): *Primary: 34A55 ; Secondary: 37K15; 39A14; 35Q55.*

1. Introduction

The Sine–Gordon equation appears in various physical and dynamical processes, such as a chain of coupled pendulums [1], Josephson tunnel junctions between superconductors [4], dislocations in crystals [2, 3], and biological cellular structures [5]. The SG equation, as well as its discrete version (dSG), has been extensively studied in the context of the inverse scattering transform (IST) [6, 8, 7] of which application to dSG equations with self-consistent sources can be found in [9, 10]. Using another different approaches, interesting results have also been obtained for the discrete integrable SG equation, for instance, solutions of the discrete SG equation have been derived using a method based on Riccati equation expansion [11], while the variable-coefficient Jacobi elliptic function method has been used to construct two families of exact travelling-wave solutions [12]. The dSG equation also arises naturally in various physical models; for example, its formulation on metric graphs provides an effective framework for describing the propagation of deformations in branched solid structures [13].

In this paper, we focus on the investigation of the dSG equation with periodic boundary conditions and develop the spectral theory for the associated linear spectral difference equation. Our approach follows the methods developed in [14, 15, 16, 17] to integrate the dSG equation within the class of periodic functions.

Received : *28–april–2026,* ***Accepted :*** *31–july–2026*
**** Corresponding author***

We consider the differential-difference sine-Gordon equation

$$\dot{\theta}_{n+1}(t) - \dot{\theta}_n(t) = 2sin\,\theta_{n+1}(t) + 2sin\theta_n(t), \quad n \in \mathbb{Z},\ t > 0 \tag{1.1}$$

in the class of N-periodic functions $\theta_{n+N}(t) = \theta_n(t), \quad n \in Z,\ t \geq 0$, with the initial data

$$\theta_n(0) = \theta_n^0,\ n \in Z. \tag{1.2}$$

where the sequence $\{\theta_n^0\}_{n\in\mathbb{Z}}$ is assumed to be N-periodic and N is natural number.

The Lax pair representation associated with Eq. (1.1) is given as

$$y_{n+1}(z,t) = L_n(z,t)y_n(z,t), \tag{1.3}$$
$$\dot{y}_n(z,t) = H_n(z,t)y_n(z,t)$$

where $L_n(z,t)$ and $H_n(z,t)$ are the following matrix

$$L_n(z,t) = zP_n(t) + \frac{1}{z}Q_n(t),$$
$$H_n(z,t) = \frac{z^2+1}{z^2-1}\sigma_3 + i\omega_n\sigma_2,$$
$$P_n(t) = \frac{1}{2}\begin{pmatrix} 1+\cos\theta_n(t) & \sin\theta_n(t) \\ \sin\theta_n(t) & 1-\cos\theta_n(t) \end{pmatrix},$$
$$Q_n(t) = \frac{1}{2}\begin{pmatrix} 1-\cos\theta_n(t) & -\sin\theta_n(t) \\ -\sin\theta_n(t) & 1+\cos\theta_n(t) \end{pmatrix}, \tag{1.4}$$
$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Here z is the non-zero spectral parameter and $\omega_n(t)$ satisfy the compatibility equations [9]

$$\dot{\theta}_n(t) = -(\omega_{n+1}(t) + \omega_n(t)),$$
$$\omega_{n+1}(t) - \omega_n(t) = -2\sin\theta_n(t). \tag{1.5}$$

Using (1.5) together with the periodicity of $\omega_n(t)$, we obtain

$$\sum_{n=0}^{N-1} \sin\theta_n(t) = 0. \tag{1.6}$$

The identity (1.6) is the closure condition for the periodic chain: the total increment over one period must vanish, and hence the forcing terms $\sin\theta_n(t)$ must balance each other. Since (1.5) determines $\omega_n(t)$ only up to an additive function of t, we impose the normalization condition

$$\omega_0(t) = f(t).$$

This condition fixes the additive freedom in $\omega_n(t)$ and ensures uniqueness.

The solution of the inverse spectral problem is based on the trace formulae and the evolution equations for the spectral data, which together provide an algorithm for reconstructing the periodic solution $\{\theta_n(t)\}_{n=-\infty}^{\infty}$ of (1.1)–(1.2).

In Section 2, we study the discrete spectral theory for the linear difference equation associated with the dSG equation. Section 3 is devoted to obtaining the trace formula for the shifted discrete linear system and presenting an algorithm for solving the inverse spectral problem. In the final section, we derive a discrete analogue of the Dubrovin system that describes the time evolution of the spectral data associated with the underlying linear problem.

2. Discrete spectral problem

Let us study the following discrete system, temporally omitting its dependence on time t

$$y_{n+1}(z) = L_n(z)y_n(z), \tag{2.1}$$

$$L_n = zP_n + \frac{1}{z}Q_n$$

where z is the is the non-zero spectral parameter and

$$P_n = \frac{1}{2}\begin{pmatrix} 1+\cos\theta_n & \sin\theta_n \\ \sin\theta_n & 1-\cos\theta_n \end{pmatrix},$$

$$Q_n = \frac{1}{2}\begin{pmatrix} 1-\cos\theta_n & -\sin\theta_n \\ -\sin\theta_n & 1+\cos\theta_n \end{pmatrix}.$$

Here $\theta_{n+N} = \theta_n$ and $y_n = (y_{n,1},\ y_{n,2})^T$, $n \in Z$. Let $c_n(z)$, $n \in Z$ and $s_n(z)$, $n \in Z$, be the solutions of system (2.1) determined by the initial conditions $c_0(z) = (1,\ 0)^T$ and $s_0(z) = (0,\ 1)^T$. Then these solutions admit the representation

$$c_n(z) = \frac{\gamma_n}{z^n}\begin{pmatrix} T_n(z^2) \\ S_n(z^2) \end{pmatrix}, \quad s_n(z) = \frac{\gamma_n}{z^n}\begin{pmatrix} B_n(z^2) \\ U_n(z^2) \end{pmatrix}. \tag{2.2}$$

Here γ_n is given by

$$\gamma_n = \cos\frac{\theta_0}{2}\prod_{j=1}^{n-1}\cos\frac{\theta_j-\theta_{j-1}}{2}\cdot\cos\frac{\theta_{n-1}}{2}, \qquad n \geq 1,$$

with the convention

$$\gamma_0 = 1.$$

The polynomial sequences $\{T_n(z^2)\}$, $\{S_n(z^2)\}$, $\{B_n(z^2)\}$, and $\{U_n(z^2)\}$, where each polynomial is of degree n, satisfy the following recurrence relations:

$$T_{n+1}(\lambda) = \frac{\cos\frac{\theta_{n-1}}{2}\left[\lambda(1+\cos\theta_n)+(1-\cos\theta_n)\right]}{2\cos\frac{\theta_n-\theta_{n-1}}{2}\cos\frac{\theta_n}{2}}T_n(\lambda) + \frac{(\lambda-1)\sin\frac{\theta_n}{2}\cos\frac{\theta_{n-1}}{2}}{\cos\frac{\theta_n-\theta_{n-1}}{2}}S_n(\lambda), \tag{2.3}$$

$$S_{n+1}(\lambda) = \frac{\cos\frac{\theta_{n-1}}{2}\left(\lambda(1-\cos\theta_n)+(1+\cos\theta_n)\right)}{2\cos\frac{\theta_n-\theta_{n-1}}{2}\cos\frac{\theta_n}{2}}S_n(\lambda) + \frac{(\lambda-1)\sin\frac{\theta_n}{2}\cos\frac{\theta_{n-1}}{2}}{\cos\frac{\theta_n-\theta_{n-1}}{2}}T_n(\lambda), \tag{2.4}$$

$$B_{n+1}(\lambda) = \frac{\cos\frac{\theta_{n-1}}{2}\left(\lambda(1+\cos\theta_n)+(1-\cos\theta_n)\right)}{2\cos\frac{\theta_n-\theta_{n-1}}{2}\cos\frac{\theta_n}{2}}B_n(\lambda) + \frac{(\lambda-1)\sin\frac{\theta_n}{2}\cos\frac{\theta_{n-1}}{2}}{\cos\frac{\theta_n-\theta_{n-1}}{2}}U_n(\lambda), \tag{2.5}$$

$$U_{n+1}(\lambda) = \frac{\cos\frac{\theta_{n-1}}{2}\left(\lambda(1-\cos\theta_n)+(1+\cos\theta_n)\right)}{2\cos\frac{\theta_n-\theta_{n-1}}{2}\cos\frac{\theta_n}{2}}U_n(\lambda) + \frac{(\lambda-1)\sin\frac{\theta_n}{2}\cos\frac{\theta_{n-1}}{2}}{\cos\frac{\theta_n-\theta_{n-1}}{2}}B_n(\lambda), \tag{2.6}$$

where $\lambda = z^2$, and the initial values are

$$T_1(\lambda) = \lambda + tg^2\frac{\theta_0}{2}, \tag{2.7}$$

$$S_1(\lambda) = (\lambda-1)tg\frac{\theta_0}{2}, \tag{2.8}$$

$$B_1(\lambda) = (\lambda-1)tg\frac{\theta_0}{2}, \tag{2.9}$$

$$U_1(\lambda) = \lambda tg^2\frac{\theta_0}{2} + 1. \tag{2.10}$$

We denote the fundamental matrix of the system (2.1) by

$$M_n(z) = \begin{pmatrix} c_{n,1}(z) & s_{n,1}(z) \\ c_{n,2}(z) & s_{n,2}(z) \end{pmatrix}. \tag{2.11}$$

The corresponding Wronskian is defined as

$$W_n(z) = \det M_n(z) = c_{n,1}(z)s_{n,2}(z) - c_{n,2}(z)s_{n,1}(z). \tag{2.12}$$

It follows that the Wronskian (2.12) is independent of both the discrete variable n and the spectral parameter z. Moreover, the matrix (2.11) can be rewritten in the form

$$M_n = L_{n-1} \cdot L_{n-2} \cdot .....L_1 \cdot L_0 \tag{2.13}$$

since

$$W_n(z) = det(M_n(z)) = \Pi_{j=0}^{n-1} det(L_j).$$

Since

$$\det(L_j) = 1,$$

it follows that

$$W_n(z) = W_0(z) = 1.$$

According to the recurrence relations (2.3)–(2.10) and the following multiplication properties of the matrices P_j and Q_j, for $j = 1, \ldots, n-1$, we obtain

$$P_0c_0 = \frac{1}{2}\begin{pmatrix} 1+\cos\theta_0 \\ \sin\theta_0 \end{pmatrix} = \cos\frac{\theta_0}{2}\begin{pmatrix} \cos\frac{\theta_0}{2} \\ \sin\frac{\theta_0}{2} \end{pmatrix},$$

....

$$P_{n-1}P_{n-2}...P_1P_0c_0 = \cos\frac{\theta_0}{2}\cos\frac{\theta_1-\theta_0}{2}...\cos\frac{\theta_{n-1}-\theta_{n-2}}{2}\begin{pmatrix} \cos\frac{\theta_{n-1}}{2} \\ \sin\frac{\theta_{n-1}}{2} \end{pmatrix},$$

$$P_{n-1}\cdot P_{n-2}\cdot ...P_1\cdot Q_0c_0 = -\sin\frac{\theta_0}{2}\sin\frac{\theta_1-\theta_0}{2}\cos\frac{\theta_2-\theta_1}{2}...\cos\frac{\theta_{n-1}-\theta_{n-2}}{2}\begin{pmatrix} \cos\frac{\theta_{n-1}}{2} \\ \sin\frac{\theta_{n-1}}{2} \end{pmatrix},$$

$$P_{n-1}\cdot P_{n-2}\cdot ...Q_1\cdot P_0c_0 = -\cos\frac{\theta_0}{2}\sin\frac{\theta_1-\theta_0}{2}\sin\frac{\theta_2-\theta_1}{2}\cos\frac{\theta_3-\theta_2}{2}...\cos\frac{\theta_{n-1}-\theta_{n-2}}{2}\begin{pmatrix} \cos\frac{\theta_{n-1}}{2} \\ \sin\frac{\theta_{n-1}}{2} \end{pmatrix},$$

....

$$Q_{n-1}\cdot P_{n-2}\cdot ...P_1\cdot P_0c_0 = -\cos\frac{\theta_0}{2}\cos\frac{\theta_1-\theta_0}{2}\cos\frac{\theta_2-\theta_1}{2}...\sin\frac{\theta_{n-1}-\theta_{n-2}}{2}\begin{pmatrix} \sin\frac{\theta_{n-1}}{2} \\ -\cos\frac{\theta_{n-1}}{2} \end{pmatrix},$$

$$Q_{n-1}\cdot Q_{n-2}\cdot ...Q_1\cdot Q_0c_0 = \sin\frac{\theta_0}{2}\cos\frac{\theta_1-\theta_0}{2}\cos\frac{\theta_2-\theta_1}{2}...\cos\frac{\theta_{n-1}-\theta_{n-2}}{2}\begin{pmatrix} \sin\frac{\theta_{n-1}}{2} \\ -\cos\frac{\theta_{n-1}}{2} \end{pmatrix},$$

similarly,

$$P_{n-1}\cdot P_{n-2}\cdot ...P_1\cdot P_0s_0 = \sin\tfrac{\theta_0}{2}\cos\tfrac{\theta_1-\theta_0}{2}\cos\tfrac{\theta_2-\theta_1}{2}...\cos\tfrac{\theta_{n-1}-\theta_{n-2}}{2}\begin{pmatrix} \cos\frac{\theta_{n-1}}{2} \\ \sin\frac{\theta_{n-1}}{2} \end{pmatrix},$$

$$P_{n-1}\cdot P_{n-2}\cdot ...P_1\cdot Q_0s_0 = \cos\frac{\theta_0}{2}\sin\frac{\theta_1-\theta_0}{2}\cos\frac{\theta_2-\theta_1}{2}...\cos\frac{\theta_{n-1}-\theta_{n-2}}{2}\begin{pmatrix} \cos\frac{\theta_{n-1}}{2} \\ \sin\frac{\theta_{n-1}}{2} \end{pmatrix},$$

$$P_{n-1}\cdot P_{n-2}\cdot .....Q_1\cdot P_0s_0 = -\sin\frac{\theta_0}{2}\sin\frac{\theta_1-\theta_0}{2}\sin\frac{\theta_2-\theta_1}{2}\cos\frac{\theta_3-\theta_2}{2}.....\cos\frac{\theta_{n-1}-\theta_{n-2}}{2}\begin{pmatrix} \cos\frac{\theta_{n-1}}{2} \\ \sin\frac{\theta_{n-1}}{2} \end{pmatrix},$$

...

$$Q_{n-1}\cdot P_{n-2}\cdot .....P_1\cdot P_0 s_0=\sin\frac{\theta_0}{2}\cos\frac{\theta_1-\theta_0}{2}\cos\frac{\theta_2-\theta_1}{2}.....\sin\frac{\theta_{n-1}-\theta_{n-2}}{2}\begin{pmatrix}\sin\frac{\theta_{n-1}}{2}\\-\cos\frac{\theta_{n-1}}{2}\end{pmatrix},$$

$$Q_{n-1}\cdot Q_{n-2}\cdot .....Q_1\cdot Q_0 s_0=\cos\frac{\theta_0}{2}\cos\frac{\theta_1-\theta_0}{2}\cos\frac{\theta_2-\theta_1}{2}.....\cos\frac{\theta_{n-1}-\theta_{n-2}}{2}\begin{pmatrix}-\sin\frac{\theta_{n-1}}{2}\\\cos\frac{\theta_{n-1}}{2}\end{pmatrix},$$

we obtain the explicit forms of the polynomials $T_n(\lambda)$, $S_n(\lambda)$, $B_n(\lambda)$, $U_n(\lambda)$

$$T_n=\lambda^n-(tg\frac{\theta_0}{2}tg\frac{\theta_1-\theta_0}{2}+...+tg\frac{\theta_{n-2}-\theta_{n-3}}{2}tg\frac{\theta_{n-1}-\theta_{n-2}}{2}-tg\frac{\theta_{n-1}-\theta_{n-2}}{2}tg\frac{\theta_{n-1}}{2})\lambda^{n-1}+ \\ +...+tg\frac{\theta_0}{2}tg\frac{\theta_{n-1}}{2} \tag{2.14}$$

$$S_n=tg\frac{\theta_{n-1}}{2}\lambda^n-((tg\frac{\theta_0}{2}tg\frac{\theta_1-\theta_0}{2}+...+tg\frac{\theta_{n-2}-\theta_{n-3}}{2}tg\frac{\theta_{n-1}-\theta_{n-2}}{2})tg\frac{\theta_{n-1}}{2}+tg\frac{\theta_{n-1}-\theta_{n-2}}{2})\lambda^{n-1}+ \\ +...-tg\frac{\theta_0}{2} \tag{2.15}$$

$$B_n=tg\frac{\theta_0}{2}\lambda^n-(-tg\frac{\theta_1-\theta_0}{2}+tg\frac{\theta_0}{2}(tg\frac{\theta_1-\theta_0}{2}tg\frac{\theta_2-\theta_1}{2}+...+tg\frac{\theta_{n-2}-\theta_{n-3}}{2}tg\frac{\theta_{n-1}-\theta_{n-2}}{2}- \\ -tg\frac{\theta_{n-1}-\theta_{n-2}}{2}tg\frac{\theta_{n-1}}{2}))\lambda^{n-1}+...+(-tg\frac{\theta_{n-1}}{2}) \tag{2.16}$$

$$U_n=tg\frac{\theta_0}{2}tg\frac{\theta_{n-1}}{2}\lambda^n+(tg\frac{\theta_1-\theta_0}{2}tg\frac{\theta_{n-1}}{2}-tg\frac{\theta_0}{2}tg\frac{\theta_{n-1}-\theta_{n-2}}{2}-tg\frac{\theta_0}{2}tg\frac{\theta_{n-1}}{2}(tg\frac{\theta_1-\theta_0}{2}tg\frac{\theta_2-\theta_1}{2}+ \\ +...+tg\frac{\theta_{n-2}-\theta_{n-3}}{2}tg\frac{\theta_{n-1}-\theta_{n-2}}{2}))\lambda^{n-1}+...+1 \tag{2.17}$$

The discriminant of the discrete system (2.1) is defined by

$$\Delta(\lambda)=c_{N,1}+s_{N,2}.$$

Substituting the representations given in (2.2) yields

$$\Delta(z)=\frac{\gamma_N}{z^N}\left[T_N(\lambda)+U_N(\lambda)\right],\qquad \lambda=z^2. \tag{2.18}$$

Since $W_n(z)=1$, the following relation

$$\Delta^2(z)-4=(s_{N,2}-c_{N,1})^2+4s_{N,1}c_{N,2} \tag{2.19}$$

is valid. Combining the representation (2.18) with the polynomial representation of the solutions given in (2.2), we obtain the following equivalent form of (2.19):

$$(U_N(\lambda)+T_N(\lambda))^2-4\lambda^N\frac{1}{\gamma_N^2}=(T_N(\lambda)-U_N(\lambda))^2+4B_N(\lambda)S_N(\lambda). \tag{2.20}$$

Denoting the left-hand side of (2.20) by R_{2N} and writing

$$K_N(\lambda)=T_N(\lambda)-U_N(\lambda),$$

equation (2.20) becomes

$$R_{2N}(\lambda)=(T_N(\lambda)-U_N(\lambda))^2+4B_N(\lambda)S_N(\lambda) \tag{2.21}$$

or its explicite polynomial form

$$R_{2N}(\lambda) = \left(1 + tg\frac{\theta_0}{2}tg\frac{\theta_{N-1}}{2}\right)^2 \lambda^{2N} + 2\left(1 + tg\frac{\theta_0}{2}tg\frac{\theta_{N-1}}{2}\right)\Bigg[\left(tg\frac{\theta_{N-1}}{2} - tg\frac{\theta_0}{2}\right)tg\frac{\theta_1 - \theta_0}{2} + \\ + \left(tg\frac{\theta_{N-1}}{2} - tg\frac{\theta_0}{2}\right)tg\frac{\theta_{N-1} - \theta_{N-2}}{2} - \left(1 + tg\frac{\theta_0}{2}tg\frac{\theta_{N-1}}{2}\right)(tg\frac{\theta_1 - \theta_0}{2}tg\frac{\theta_2 - \theta_1}{2} + \cdots + \\ + tg\frac{\theta_{N-2} - \theta_{N-3}}{2}tg\frac{\theta_{N-1} - \theta_{N-2}}{2})\Bigg]\lambda^{2N-1} + \cdots + \left(1 + tg\frac{\theta_0}{2}tg\frac{\theta_{N-1}}{2}\right)^2 \\ - 4\lambda^N\left(1 + tg^2\frac{\theta_0}{2}\right)\left(1 + tg^2\frac{\theta_1 - \theta_0}{2}\right)\cdots\left(1 + tg^2\frac{\theta_{N-1}}{2}\right). \tag{2.22}$$

Let $\{\lambda_j\}_{j=1}^{2N}$ be the set of roots of the polynomial $R_{2N}(\lambda)$, and let $\{\xi_k\}_{k=1}^{N}$ be the set of roots of the polynomial $B_N(\lambda)$. Then,

$$K_N(\xi_j) = \sigma_j\sqrt{R_{2N}(\xi_j)}, \qquad \sigma_j \in \{\pm 1\}, \qquad j = 1, \ldots, N. \tag{2.23}$$

The quantities σ_j determine the sheet of the hyperelliptic spectral curve

$$\Gamma: \qquad \mu^2 = R_{2N}(\lambda)$$

on which the divisor points lie. Indeed, defining

$$\mu_j := K_N(\xi_j),$$

relation (2.23) implies

$$(\xi_j, \mu_j) \in \Gamma, \qquad \mu_j = \sigma_j\sqrt{R_{2N}(\xi_j)}, \qquad \sigma_j \in \{\pm 1\}.$$

Thus the spectral data of the discrete system (2.1) are given by the branch points

$$\lambda_1, \ldots, \lambda_{2N}$$

of the hyperelliptic curve Γ together with the divisor

$$D = \sum_{j=1}^{N}(\xi_j, \mu_j), \qquad \mu_j = \sigma_j\sqrt{R_{2N}(\xi_j)}.$$

The signs σ_j specify on which sheet of the curve Γ the divisor points (ξ_j, μ_j) lie and therefore form an essential part of the spectral data in the finite-gap description of the system.

Proposition 2.1. *Let the spectral curve associated with the discrete system* (2.1) *be*

$$\Gamma: \qquad \mu^2 = R_{2N}(\lambda).$$

For each zero ξ_j of the polynomial $B_N(\lambda)$ define

$$\mu_j := K_N(\xi_j).$$

Then the point (ξ_j, μ_j) lies on the curve Γ and can be written in the form

$$(\xi_j, \mu_j) = \left(\xi_j, \sigma_j\sqrt{R_{2N}(\xi_j)}\right), \qquad \sigma_j \in \{\pm 1\}.$$

Thus the quantities σ_j determine on which sheet of the hyperelliptic curve Γ the divisor point lies.

Proof Recall the identity

$$R_{2N}(\lambda) = K_N(\lambda)^2 + 4B_N(\lambda)S_N(\lambda).$$

Since ξ_j is a zero of $B_N(\lambda)$, we have

$$B_N(\xi_j) = 0.$$

Substituting $\lambda = \xi_j$ into the above identity yields

$$R_{2N}(\xi_j) = K_N(\xi_j)^2.$$

By definition, $\mu_j = K_N(\xi_j)$, hence

$$\mu_j^2 = R_{2N}(\xi_j).$$

Therefore the point (ξ_j, μ_j) belongs to the spectral curve

$$\Gamma : \qquad \mu^2 = R_{2N}(\lambda).$$

Now, for each fixed ξ_j, the equation

$$\mu^2 = R_{2N}(\xi_j)$$

has two possible values

$$\mu = \pm\sqrt{R_{2N}(\xi_j)}.$$

Hence there exists a sign $\sigma_j \in \{\pm 1\}$ such that

$$\mu_j = K_N(\xi_j) = \sigma_j\sqrt{R_{2N}(\xi_j)}.$$

This sign specifies which of the two sheets of the hyperelliptic curve contains the point (ξ_j, μ_j).

Lemma 2.1. *For each zero ξ_j of $B_N(\lambda)$ there are two possible points of the spectral curve lying over ξ_j, namely*

$$\left(\xi_j, \pm\sqrt{R_{2N}(\xi_j)}\right).$$

Equivalently, the sheet sign $\sigma_j \in \{\pm 1\}$ is not determined by the branch points $\lambda_1, \ldots, \lambda_{2N}$ and the zeros $\xi_1, \ldots, \xi_N$ alone.

Thus the divisor

$$D = \sum_{j=1}^{N} (\xi_j, \mu_j), \qquad \mu_j = \sigma_j\sqrt{R_{2N}(\xi_j)},$$

is part of the spectral data only after an admissible choice of signs $\sigma_1, \ldots, \sigma_N$ has been fixed.

Proof For each j, the relation

$$\mu_j^2 = R_{2N}(\xi_j)$$

follows from Proposition 2.1. Therefore, over the point $\lambda = \xi_j$ there are exactly two points on the hyperelliptic curve:

$$\left(\xi_j, \sqrt{R_{2N}(\xi_j)}\right), \qquad \left(\xi_j, -\sqrt{R_{2N}(\xi_j)}\right).$$

Hence the sign σ_j may take either value $+1$ or -1, and it is not fixed by the values of $\lambda_1, \ldots, \lambda_{2N}$ and $\xi_1, \ldots, \xi_N$ alone.

Therefore, in order to define the divisor on the curve uniquely, one must choose for each ξ_j one of the two points lying above it, i.e. one must specify the corresponding sign σ_j. The proof is complete.

The spectral data of equation (2.1) consist of the roots

$$\lambda_1, \lambda_2, \ldots, \lambda_{2N},$$

the spectral parameters

$$\xi_j, \qquad j = 1, \ldots, N,$$

and the corresponding sheet signs

$$\sigma_1, \ldots, \sigma_N.$$

These data determine the factorized forms of the polynomials $R_{2N}(\lambda)$ and $B_N(\lambda)$,

$$\begin{aligned} R_{2N}(\lambda) &= (1 + tg\frac{\theta_{N-1}}{2} tg\frac{\theta_0}{2})^2 \Pi_{j=1}^{2N}(\lambda - \lambda_j), \\ B_N(\lambda) &= tg\frac{\theta_0}{2} \Pi_{j=1}^{N}(\lambda - \xi_j), \end{aligned} \tag{2.24}$$

and together with the interpolation conditions

$$K_N(\xi_j) = \sigma_j \sqrt{R_{2N}(\xi_j)}, \qquad j = 1, \ldots, N, \tag{2.25}$$

they determine the divisor on the spectral curve and therefore provide the finite-gap spectral characterization of the system.

Using the representations (2.14)–(2.17) in (2.21), together with the polynomial identity (2.24), we derive the following identities for ξ_k, $k = \overline{1, N}$

$$\xi_1 \xi_2 \ldots \xi_N = (-1)^{N+1} \frac{tg\frac{\theta_{N-1}}{2}}{tg\frac{\theta_0}{2}}. \tag{2.26}$$

We now present the main result concerning the trace formula for the discrete system (2.1), which is the key ingredient in the reconstruction of the solution

$$\{\theta_n\}_{n \in \mathbb{Z}}.$$

Let

$$a := tg\frac{\theta_0}{2}, \qquad b := tg\frac{\theta_{N-1}}{2}.$$

Then

$$\kappa = 1 - ab.$$

By Vieta's theorem applied to $B_N(\lambda)$,

$$\prod_{j=1}^{N} \xi_j = (-1)^{N+1} \frac{b}{a}.$$

Hence

$$ab = (-1)^{N+1} a^2 \prod_{j=1}^{N} \xi_j,$$

and therefore

$$tg^2\frac{\theta_0}{2} = a^2 = (-1)^{N+1} \frac{1 - \kappa}{\prod_{j=1}^{N} \xi_j}.$$

Define

$$\widetilde{B}_N(\lambda) := \prod_{j=1}^{N}(\lambda - \xi_j), \qquad \widetilde{B}'_N(\xi_j) = \prod_{\substack{m=1 \\ m\neq j}}^{N}(\xi_j - \xi_m).$$

Since that the roots $\xi_1, \ldots, \xi_N$ are simple and that

$$1 + (-1)^N \prod_{j=1}^{N} \xi_j \neq 0.$$

Choose the branch on the curve

$$\mu^2 = R_{2N}(\lambda)$$

by fixing

$$\nu_j := K_N(\xi_j) = \sigma_j \sqrt{R_{2N}(\xi_j)}, \qquad \sigma_j \in \{\pm 1\}.$$

Then

$$K_N(\lambda) = \kappa\, \widetilde{B}_N(\lambda) + \sum_{j=1}^{N} \nu_j \frac{\widetilde{B}_N(\lambda)}{(\lambda - \xi_j)\widetilde{B}'_N(\xi_j)}.$$

Evaluating at $\lambda = 0$ and using $K_N(0) = -\kappa$, we obtain

$$\kappa = \frac{(-1)^N \left(\prod_{j=1}^{N} \xi_j\right) \sum_{j=1}^{N} \dfrac{\sigma_j \sqrt{R_{2N}(\xi_j)}}{\xi_j \prod_{m\neq j}(\xi_j - \xi_m)}}{1 + (-1)^N \prod_{j=1}^{N} \xi_j}.$$

Substituting this into the previous formula yields

$$A_0 = (-1)^N \left(\prod_{j=1}^{N} \xi_j\right),$$

$$\sum_{j=1}^{N} \frac{\sigma_j \sqrt{R_{2N}(\xi_j)}}{\xi_j \prod_{m\neq j}(\xi_j - \xi_m)} = \left(1 + \operatorname{tg} \frac{\theta_{N-1}}{2} \operatorname{tg} \frac{\theta_0}{2}\right) \left(\sum_{j=1}^{N} \frac{\sigma_j \sqrt{\widetilde{R}_{2N}(\xi_j)}}{\xi_j \prod_{m\neq j}(\xi_j - \xi_m)}\right)$$

$$D_0 = \sum_{j=1}^{N} \frac{\sigma_j \sqrt{\widetilde{R}_{2N}(\xi_j)}}{\xi_j \prod_{m\neq j}(\xi_j - \xi_m)}$$

Substituting this into the previous formula yields

$$\cos\theta_0 = \frac{1 - tg^2 \frac{\theta_0}{2}}{1 + tg^2 \frac{\theta_0}{2}}, \tag{2.27}$$

where

$$tg^2 \frac{\theta_0}{2} = \frac{-1 - A_0 + A_0 D_0}{A_0(1 + A_0 + A_0 D_0)} \tag{2.28}$$

3. Trace formula

To reconstruct the solution θ_n, $n \in \mathbb{Z}$, we shift the index in (2.1) by an arbitrary integer k and consider the translated system

$$y_{n+k+1}(z) = L_{n+k}(z)y_{n+k}(z), \tag{3.1}$$

where

$$L_{n+k}(z) = zP_{n+k} + \frac{1}{z}Q_{n+k}.$$

Let $\varphi_n(z,k)$ and $\phi_n(z,k)$ denote the fundamental solutions of the system (3.1) satisfying the initial conditions

$$\varphi_0(z,k) = \begin{pmatrix}1\\0\end{pmatrix}, \qquad \phi_0(z,k) = \begin{pmatrix}0\\1\end{pmatrix}.$$

Lemma 3.1. *The fundamental solutions $\varphi_n(z,k)$ and $\phi_n(z,k)$ satisfy the relations*

$$\varphi_n(z,k) = s_{k,2}(z)c_{n+k}(z) - c_{k,2}(z)s_{n+k}(z), \tag{3.2}$$

and

$$\phi_n(z,k) = -s_{k,1}(z)c_{n+k}(z) + c_{k,1}(z)s_{n+k}(z). \tag{3.3}$$

Proof. Let

$$\Phi_k(z) = \big(c_k(z),\ s_k(z)\big).$$

Since the potential is N-periodic, i.e.,

$$\theta_{n+N} = \theta_n,$$

the solutions of (3.1) satisfy the monodromy relation

$$\Phi_{k+N}(z) = \Phi_k(z)M_N(z),$$

where the monodromy matrix is given by

$$M_N(z) = \begin{pmatrix}c_{N,1}(z) & s_{N,1}(z)\\ c_{N,2}(z) & s_{N,2}(z)\end{pmatrix}.$$

Consequently,

$$\begin{aligned} c_{N+k}(z) &= c_{N,1}(z)c_k(z) + c_{N,2}(z)s_k(z),\\ s_{N+k}(z) &= s_{N,1}(z)c_k(z) + s_{N,2}(z)s_k(z).\end{aligned}$$

As the solution Φ_{k+n} satisfy the system

$$\Phi_{k+n} = L_{n+k-1}\Phi_{k+n-1} = L_{n+k-1}L_{n+k-2}L_{n+k-3}\cdots L_k\Phi_k$$

According to the definition of the monodromy matrix, we introduce the monodromy matrix associated with the shifted system (3.1) by

$$M_n^k(z) = \begin{pmatrix}\varphi_{n,1}(z,k) & \phi_{n,1}(z,k)\\ \varphi_{n,2}(z,k) & \phi_{n,2}(z,k)\end{pmatrix}, \tag{3.4}$$

which admits the representation

$$M_n^k(z) = L_{n+k-1}(z)L_{n+k-2}(z)\cdots L_k(z). \tag{3.5}$$

Consequently,

$$M_n^k(z) = \Phi_{k+n}(z)\Phi_k^{-1}(z),$$

from which relations (3.2) and (3.3) follow immediately. □

According to the shifted monodromy matrix, the discriminant of the shifted system is defined as

$$\Delta(z,k) = \varphi_{N,1}(z,k) + \phi_{N,2}(z,k). \tag{3.6}$$

Lemma 3.2. *The discriminant $\Delta(z,k)$ defined by (3.6) is independent of the shift parameter k.*

Proof. By the periodicity of the coefficients, the monodromy matrices satisfy the similarity relation

$$M_N^k = M_k M_N M_k^{-1}$$

and thus

$$\Delta(z,k) = \operatorname{tr}(M_k M_N M_k^{-1}) = \operatorname{tr}(M_N) = \Delta(z).$$

Hence, the roots of the equations

$$\Delta(z) = \pm 2$$

are independent of the shift parameter k. □

The inverse spectral problem for (2.1) consists in reconstructing the solution

$$\{\theta_n\}_{n\in\mathbb{Z}}$$

from the corresponding spectral data.

Corollary 3.1. *By formulas (2.27) and (2.28), the trace formula for the solution θ_k, $k \in \mathbb{Z}$, of equation (2.1) is given by*

$$\tan^2\frac{\theta_k}{2} = \frac{-1 - A_k + A_k D_k}{A_k(1 + A_k + A_k D_k)}, \tag{3.7}$$

where $\xi_{j,k}$, $j = 1,\dots,N$, are the roots of the equation

$$\varphi_{N,1}(z,k) = 0.$$

Lemma 3.3. *The spectral problem (2.1) can be uniquely reconstructed using its spectral data.*

Proof. Since the inverse spectral reconstruction relies on the spectral data at the kth step, we first derive recurrence relations for the polynomials $B_{N,k}(\lambda)$, $K_{N,k}(\lambda)$, and $S_{N,k}(\lambda)$.

According to the expansion (3.5) of the monodromy matrix M_n^k and periodicity of θ_n, it holds

$$M_N^k = L_{k-1} M_N^{k-1} L_{k-1}^{-1}, \tag{3.8}$$

and taking account (2.2) and (3.4) we have

$$\frac{z^N}{\gamma_N} M_N^k = \Psi_N^k \tag{3.9}$$

where

$$\Psi_N^k = \begin{pmatrix} T_{N,k} & B_{N,k} \\ S_{N,k} & U_{N,k} \end{pmatrix}.$$

According to the equalities (3.8) and (3.9), it follows that

$$\Psi_N^k = L_{k-1} \Psi_N^{k-1} L_{k-1}^{-1}, \tag{3.10}$$

$$L_{k-1} = \begin{pmatrix} \frac{z}{2}(1+\cos\theta_{k-1}) + \frac{1}{2z}(1-\cos\theta_{k-1}) & (\frac{z}{2} - \frac{1}{2z})\sin\theta_{k-1} \\ (\frac{z}{2} - \frac{1}{2z})\sin\theta_{k-1} & \frac{z}{2}(1-\cos\theta_{k-1}) + \frac{1}{2z}(1+\cos\theta_{k-1}) \end{pmatrix},$$

$$L_{k-1}^{-1} = \begin{pmatrix} \frac{z}{2}(1-\cos\theta_{k-1}) + \frac{1}{2z}(1+\cos\theta_{k-1}) & (\frac{1}{2z} - \frac{z}{2})\sin\theta_{k-1} \\ (\frac{1}{2z} - \frac{z}{2})\sin\theta_{k-1} & \frac{z}{2}(1+\cos\theta_{k-1}) + \frac{1}{2z}(1-\cos\theta_{k-1}) \end{pmatrix}.$$

Consequently, the equality (3.10) gives the following relations

$$\begin{aligned} T_{N,k} = ((\frac{z}{2}(1+\cos\theta_{k-1}) + \frac{1}{2z}(1-\cos\theta_{k-1}))T_{N,k-1} + S_{N,k-1}(\frac{z}{2} - \frac{1}{2z})\sin\theta_{k-1})(\frac{z}{2}(1-\cos\theta_{k-1}) + \\ + \frac{1}{2z}(1+\cos\theta_{k-1})) + ((\frac{z}{2}(1+\cos\theta_{k-1}) + \frac{1}{2z}(1-\cos\theta_{k-1}))B_{N,k-1} + \\ + U_{N,k-1}(\frac{z}{2} - \frac{1}{2z})\sin\theta_{k-1})(\frac{1}{2z} - \frac{z}{2})\sin\theta_{k-1} \end{aligned}$$

$$\begin{aligned} S_{N,k} = (T_{N,k-1}(\frac{z}{2} - \frac{1}{2z})\sin\theta_{k-1} + (\frac{z}{2}(1-\cos\theta_{k-1}) + \frac{1}{2z}(1+\cos\theta_{k-1}))S_{N,k-1})(\frac{z}{2}(1-\cos\theta_{k-1}) + \\ + \frac{1}{2z}(1+\cos\theta_{k-1})) + (B_{N,k-1}(\frac{z}{2} - \frac{1}{2z})\sin\theta_{k-1} + (\frac{z}{2}(1-\cos\theta_{k-1}) + \\ + \frac{1}{2z}(1+\cos\theta_{k-1}))U_{N,k-1})(\frac{1}{2z} - \frac{z}{2})\sin\theta_{k-1} \end{aligned}$$

$$\begin{aligned} B_{N,k} = ((\frac{z}{2}(1+\cos\theta_{k-1}) + \frac{1}{2z}(1-\cos\theta_{k-1}))T_{N,k-1} + S_{N,k-1}(\frac{z}{2} - \frac{1}{2z})\sin\theta_{k-1})(\frac{1}{2z} - \frac{z}{2})\sin\theta_{k-1} + \\ ((+\frac{z}{2}(1+\cos\theta_{k-1}) + \frac{1}{2z}(1-\cos\theta_{k-1}))B_{N,k-1} + U_{N,k-1}(\frac{z}{2} - \frac{1}{2z})\sin\theta_{k-1})(\frac{z}{2}(1+\cos\theta_{k-1}) + \\ + \frac{1}{2z}(1-\cos\theta_{k-1})) \end{aligned} \tag{3.11}$$

$$\begin{aligned} U_{N,k} = (T_{N,k-1}(\frac{z}{2} - \frac{1}{2z})\sin\theta_{k-1} + (\frac{z}{2}(1-\cos\theta_{k-1}) + \frac{1}{2z}(1+\cos\theta_{k-1}))S_{N,k-1})(\frac{1}{2z} - \frac{z}{2})\sin\theta_{k-1} + \\ (B_{N,k-1}(\frac{z}{2} - \frac{1}{2z})\sin\theta_{k-1} + (\frac{z}{2}(1-\cos\theta_{k-1}) + \frac{1}{2z}(1+\cos\theta_{k-1}))U_{N,k-1})(\frac{z}{2}(1-\cos\theta_{k-1}) \\ + \frac{1}{2z}(1-\cos\theta_{k-1})) \end{aligned}$$

Employing the equality (2.21) we obtain

$$S_{N,0}(\lambda) = \frac{R_{2N}(\lambda) - K_{N,0}^2(\lambda)}{4B_{N,0}(\lambda)}. \tag{3.12}$$

We have thus obtained the following algorithm for solving the inverse problem for the (2.1).

We have the system of spectral parameters $\{\xi_j, \sigma_j\}_{j=1}^{N}$ and the numbers $\lambda_1,\ \lambda_2, ..., \lambda_{2N}$.

1. Applying (2.27), we deduce θ_0.
2. Using relations (2.14)-(2.17), and (2.22), one constructs the polynomials $R_{2N}(\lambda)$, $B_{N,0}(\lambda)$, and $K_{N,0}(\lambda)$.
3. The polynomial $S_{N,0}(\lambda)$ is then determined by the formula (3.12).
4. From equality (3.11) we obtain $B_{N,k}(\lambda)$, whose roots $\xi_{j,k}$, $j = 1, \ldots, N$, are subsequently computed.
5. Applying the trace formula (3.7), we determine the value of (θ_k).

4. Evolution of spectral parameters

Theorem 4.1. *Let $\{\theta_n(t)\}_{n\in\mathbb{Z}}$ be a solution of (1.1)–(1.2). Then the periodic and antiperiodic spectra of the associated spectral problem (1.3) remain invariant under the time evolution. Furthermore, the spectral parameters $\xi_j(t)$, $j = 1, \dots, N$, satisfy the following evolution equations:*

$$\dot{\xi}_j(t) = \frac{\omega_0(1 + (-1)^{N+1}\prod_{j=1}^{N}\xi_j)tg^2\frac{\theta_0}{2})\sigma_j\sqrt{\widetilde{R}_{2N}}}{tg\frac{\theta_0}{2}\prod_{k\neq j}^{N}(\xi_j - \xi_k)} \tag{4.1}$$

Proof Using the expression (2.13) for the monodromy matrix M_N, he derivative of M_N with respect to t takes

$$\dot{M}_N = \frac{d(L_{N-1}L_{N-2}L_{N-3}\cdots L_0)}{dt}. \tag{4.2}$$

Using the Lax representation

$$\dot{L}_n = H_{n+1}L_n - L_nH_n.$$

for the derivative of L_j, $j = \overline{1,\ N}$ in equality (4.2) and after simplifications we have

$$\dot{M}_n = \frac{d(L_{n-1}\cdot L_{n-2}\cdot \dots L_1\cdot L_0)}{dt} = \frac{dL_{n-1}}{dt}L_{n-2}\cdot \dots L_1\cdot L_0 + L_{n-1}\frac{dL_{n-2}}{dt}\cdot \dots L_1\cdot L_0 + \\ + \dots + L_{n-1}L_{n-2}\cdot \dots L_1\cdot \frac{dL_0}{dt}$$

Due to the periodicity in N, we get

$$\dot{M}_N = (H_NL_{N-1} - L_{N-1}H_{N-1})\cdot L_{N-2}\cdot \dots L_1\cdot L_0 + L_{N-1}\cdot (H_{N-1}L_{N-2} - L_{N-2}H_{N-2})\cdot \dots L_1\cdot L_0 + \\ + L_{N-1}\cdot L_{N-2}\dots L_1\cdot (H_0L_0 - L_0H_0) = H_NL_{N-1}\cdot L_{N-2}\cdot \dots L_1\cdot L_0 - L_{N-1}\cdot L_{N-2}\cdot \dots L_1\cdot L_0\cdot H_0.$$

As a result, we have the equality

$$\dot{M}_N = H_NM_N - M_NH_0 \tag{4.3}$$

and we find that

$$\dot{s}_{N,1} = \omega_N(s_{N,2} - c_{N,1}). \tag{4.4}$$

Using (2.2), (2.26), (2.24) and (2.25) in the equality (4.4) we get (4.1).

Lemma 4.1. *$\lambda_k(t)$ is independent of the parameter t for all $k = \overline{1,\ 2N}$.*

Proof Demonstrating that $\lambda_k(t)$ is independent of the parameter t, is equivalent to proving that the derivative of $\Delta(z)$ with respect to t is zero. Since the equalities (4.3) and

$$\Delta(z,t) = \mathrm{tr}(M_N)$$

hold, the derivative of $\Delta(z)$ with respect to t gives

$$\dot{\Delta}(z) = \mathrm{tr}\,\dot{M}_N = 0. \tag{4.5}$$

Relation (4.5) implies that the zeros of $\Delta^2(z) - 4 = 0$ remain constant in t.

Conclusion

In this paper, we investigated the periodic differential–difference sine–Gordon equation by means of the inverse spectral problem method. We established the spectral theory for the associated periodic discrete linear system and obtained explicit polynomial representations of its fundamental solutions. Based on these representations, we derived trace formulas that reconstruct the potential from the spectral parameters.

Furthermore, by introducing the shifted monodromy matrix, we developed a constructive algorithm for solving the inverse spectral problem and recovering the periodic solution from the spectral data. Finally, we proved the invariance of the periodic and antiperiodic spectra under the time evolution and derived a discrete analogue of the Dubrovin equations governing the evolution of the auxiliary spectral parameters.

The obtained results provide a complete inverse spectral characterization of the periodic differential–difference sine–Gordon equation and establish a finite-gap integration framework for this discrete integrable system.

Acknowledgements

The author would like to express his sincere gratitude to professor Yakhshimuratov Alisher for the fruitful discussions and to professor Aknazar Khasanov for his valuable suggestions and comments that helped improve this work.

Availability of data and materials

Not applicable.

Competing interests

The authors declare that they have no competing interests.

Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

References

[1] Scott, A.C.: *A nonlinear Klein–Gordon equation.* Am. J. Phys. **37(1)**, 52–61 (1969). https://doi.org/10.1119/1.1975404
[2] Frenkel, J., Kontorova, T.: *On the theory of plastic deformation and twinning.* Izv. Akad. Nauk SSSR, Ser. Fiz. 1, 137–149 (1939).
[3] Borisov, A.B., Kiseliev, V.V.: *Topological defects in incommensurate magnetic and crystal structures and quasi-periodic solutions of the elliptic sine–Gordon equation.* Physica D **31(1)**, 49–64 (1988).
[4] Ustinov, A.V.: *Solitons in Josephson junctions.* Physica D **123(1)**, 315–329 (1998). https://doi.org/10.1016/S0167-2789(98)00131-6
[5] Ivancevic, V.G., Ivancevic, T.T.: *Sine–Gordon solitons, kinks and breathers as physical models of nonlinear excitations in living cellular structures.* J. Geom. Symmetry Phys. **31**, 1–56 (2013). https://doi.org/10.7546/jgsp-31-2013-1-56
[6] Ablowitz, M.J., Kaup, D.J., Newell, A.C., Segur, H.: *Method for solving the sine–Gordon equation.* Phys. Rev. Lett. **30**, 1262–1264 (1973). https://doi.org/10.1103/PhysRevLett.30.1262
[7] Levi, D., Ragnisco, O., Bruschi, M.: *Extension of the Zakharov–Shabat generalized inverse method to solve differential–difference and difference–difference equations.* Nuovo Cimento A **58**, 56–66 (1980). https://doi.org/10.1007/BF02730220
[8] Pilloni, L., Levi, D.: *The inverse scattering transform for solving the discrete sine–Gordon equation.* Phys. Lett. A **92**, 5–8 (1982).
[9] Babajanov, B.A., Babadjanova, A.K., Azamatov, A.Sh.:*Integration of the differential–difference sine–Gordon equation with a self-consistent source.* Teor. Mat. Fiz. **210(3)**, 375–386 (2022); https://doi.org/10.4213/tmf10152
[10] Babajanov, B., Fečkan, M., Babadjanova, A.: *On the differential–difference sine–Gordon equation with an integral type source.* Math. Slovaca **73(6)**, 1499–1510 (2023).
[11] Xie, F., Ji, M., Zhao, H.: *Some solutions of the discrete sine–Gordon equation.* Chaos, Solitons & Fractals **33**, 1791–1795 (2007).
[12] Ding, D.-J., Jin, D.-Q., Dai, C.-Q.: *Analytical solutions of differential–difference sine–Gordon equation.* Thermal Science **21(4)**, 1701–1705 (2017).

[13] Akramov, M.E., Yusupov, J.R., Askerzade, I.N., Matrasulov, D.U.: *Discrete sine–Gordon equation on metric graphs: A simple model for Josephson junction networks.* Physica Scripta **98(11)** (2023). https://doi.org/10.1088/1402-4896/acff4e
[14] Babajanov, B., Fečkan, M., Urazboev, G.: *On the periodic Toda lattice hierarchy with an integral source.* Commun. Nonlinear Sci. Numer. Simul. **52**, 110–123 (2017).
[15] Babajanov, B., Fečkan, M., Urazboev, G.: *On the periodic Toda lattice with a self-consistent source.* Commun. Nonlinear Sci. Numer. Simul. **22(1–3)**, 1223–1234 (2015).
[16] Babadjanova, A., Kriecherbauer, T., Urazboev, G.: *Periodic solutions of the discrete modified KdV equation with a self-consistent source.* Appl. Math. Comput. **376**, 125136 (2020).
[17] Babadjanova, A.K.: *The periodic solutions of the discrete modified KdV equation.* Uzbek Mathematical Journal **64**(1), 21–30 (2020). https://doi.org/10.29229/uzmj.2020-1-3

Affiliations

Bazar A. Babajanov
Address: Urgench State University, Urgench, 220100, Uzbekistan.
e-mail: a.murod@mail.ru
ORCID ID: https://orcid.org/0000-0001-6878-791X

Aygul K. Babadjanova
Address: Khorezm branch of V. I. Romanovski Institute of Mathematics, Uzbekistan Academy of Sciences; Urgench State University, Urgench, 220100, Uzbekistan.
e-mail: aygul.babadjanova1987@gmail.com
ORCID ID: https://orcid.org/0000-0003-3458-3913

Bekhzod Fayzullaev
Address: Urgench State University, Urgench, 220100, Uzbekistan.
e-mail: olimpmatem@gmail.com
ORCID ID: https://orcid.org/0009-0006-7985-4149

Uzbekistan Journal of Mathematics and Computer Science
Volume-2, Issue-3-special, Page 160–169 (2026)
DOI: https://doi.org/10.56143/ujmcs.v2i3s.13
Research article

Краевые задачи для вязко-трансзвукового уравнения в полуограниченных областях

Ю.П. Апаков * **Х.К. Иброхимов**

Аннотация

В настоящем исследовании изучены краевые задачи вязко-трансзвукового уравнений в полубесконечных областях. Единственность решения данных задач обоснована с использованием метода энергетических интегралов. При этом показано, что в случае нарушения условий единственности соответствующие однородные задачи обладают нетривиальными решениями. Существование решения обосновано методом разделения переменных; искомое решение построено в виде бесконечного ряда, для которого полностью доказана возможность почленного дифференцирования по всем переменным.

Ключевые слова: *Дифференциальные уравнения третьего порядка, вязко-трансзвуковое уравнение, краевые задачи, теорема единственности, существование решения, метод разделения переменных, спектральная задача (собственные значения), бесконечные функциональные ряды, абсолютная и равномерная сходимость.*

Предметная классификацие AMS (2020): *Основная: 35C10; Дополнительная: 35J15.*

1. Введение

Одним из наиболее перспективных направлений современной математической физики является исследование дифференциальных уравнений в частных производных третьего порядка. Данный класс уравнений служит важным теоретическим инструментом при решении задач нелинейной акустики, изучении гидродинамической теории космической плазмы, а также при математическом моделировании процессов фильтрации жидкости в пористых средах [1].

В газодинамических процессах резкие скачки параметров потока локализуются в узких зонах вблизи ударных волн. Наличие существенных градиентов в рассматриваемых средах требует обязательного учета влияния вязкости и теплопроводности газа, а также нелинейных свойств самого течения. В контексте теории коротких волн данные физические процессы находят свое математическое описание через вязкое трансзвуковое уравнение (ВТ-уравнение), выступающее основным предметом изучения в этой области [2-3].

Прикладное значение ВТ-уравнения наиболее отчетливо проявляется при анализе обтекания воздушным потоком крыльев самолета, когда число Маха $M \approx 1$. Как показано в фундаментальных работах Н.Н. Кузнецова и О.С. Рыжова [4-5], введение члена с третьей производной позволяет математически обосновать внутреннее трение газа и «размытие» структуры ударной волны (structure of the shock wave).

Исходя из системы уравнений Навье-Стокса, с учетом диссипативных свойств газа (вязкости и теплопроводности), выведено уравнение третьего порядка, обладающее кратными характеристиками. Отличительной особенностью данной модели является наличие второй производной по временной координате

Received : *28–april–2026,* ***Accepted :*** *31–july–2026*
**** Corresponding author***

[6]:

$$u_{xxx} + u_{yy} - \frac{\nu}{y} u_y = u_x u_{xx}, \quad \nu = \text{const}.$$

В частности, случай $\nu = 1$ соответствует осесимметричному течению, тогда как при $\nu = 0$ уравнение описывает плоскопараллельный поток [7].

Основополагающие принципы теории уравнений третьего порядка с кратными характеристиками были разработаны в трудах [8-9]. Согласно монографии [10], уравнения дисперсионно-диссипативного типа играют ключевую роль в анализе динамики нелинейных волн.

В исследованиях [11-12] осуществлено построение фундаментальных решений для уравнений третьего порядка с кратными характеристиками и наличием вторых производных по времени. Данные решения представлены в терминах вырожденных гипергеометрических функций, при этом детально проанализированы их свойства и получены асимптотические оценки для случая $|t| \to \infty$.

В работах [13-14] представлен анализ краевых задач для уравнений третьего порядка, рассматриваемых в полуограниченных трехмерных областях.

Вопросы постановки и исследования краевых задач для уравнений третьего порядка со второй временной производной в ограниченных областях отражены в публикациях [15–17].

2. Постановка задач

В областях $D^+ = \{(x,y) : x \in (0,+\infty),\ y \in (0,q)\}$ и $D^- = \{(x,y) : x \in (-\infty,0),\ y \in (0,q)\}$, рассмотрим уравнения в виде

$$L[u] = \left(\frac{\partial^3}{\partial x^3} + \frac{\partial^2}{\partial y^2} + \mu \right) u = 0, \tag{2.1}$$

При этом μ и q являются действительными параметрами ($\mu, q \in \mathbb{R}$), для которых нами рассматриваются нижеследующие задачи

Задача B_1. Найти регулярное решение уравнения (2.1), в области D^+ из класса $C^{3,2}_{x,y}(D^+) \cap C^{2,1}_{x,y}(D^+ \cup \Gamma_1)$, такое что при $x \to +\infty$ производная первого порядка по y и производная второго порядка по x ограничены и равномерны по переменной y, и $u_y \in L_2(D^+)$, удовлетворяющее следующим граничным условиям:

$$u(x,0) = u(x,q) = 0, \quad 0 \le x < +\infty, \tag{2.2}$$

$$\begin{cases} au(0,y) + bu_{xx}(0,y) = \psi_1(y), \\ u_x(0,y) = \psi_2(y), \\ \lim\limits_{x \to +\infty} u(x,y) = 0, \end{cases} \qquad 0 \le y \le q, \tag{2.3}$$

где $\Gamma_1 = \partial D^+$ - граница области D^+, $a, b \in \mathbb{R}$, такие что $a^2 + b^2 \neq 0$, а $\psi_i(y)\,(i = 1,2)$ -заданные достаточно гладкие функции, удовлетворяющие условиям:

$$\psi_i(0) = \psi_i(q) = 0,\ (i = 1,2). \tag{2.4}$$

Задача B_2. Найти регулярное решение уравнения (2.1), в области D^- из класса $C^{3,2}_{x,y}(D^-) \cap C^{2,1}_{x,y}(D^- \cup \Gamma_2)$, такое что при $x \to -\infty$ производная первого порядка по y и производная второго порядка по x ограничены и равномерны по переменной y, и $u_y \in L_2(D^-)$, удовлетворяющее граничным условиям (2.2) при $-\infty < x \le 0$ и

$$\begin{cases} cu(0,y) + du_{xx}(0,y) = \psi_3(y), \\ \lim\limits_{x \to -\infty} u(x,y) = \lim\limits_{x \to -\infty} u_x(x,y) = 0, \end{cases} \qquad 0 \le y \le q, \tag{2.5}$$

где $\Gamma_2 = \partial D^-$ - граница области D^-, $c, d \in \mathbb{R}$, такие что $c^2 + d^2 \neq 0$, а $\psi_3(y)$ — заданная достаточно гладкая функция, удовлетворяющее условиям:

$$\psi_3(0) = \psi_3(q) = 0. \tag{2.6}$$

3. Единственность решения

Теорема 3.1. *Если задача B_1 (B_2) имеет решение, то при выполнении условий $ab \le 0,\ \ \mu \le 0$ $(cd \ge 0,\ \mu \le 0)$ оно единственно.*

Доказательство. Предположим обратное. Пусть задача B_1 (B_2) имеет два решения $u_1(x,y)$ и $u_2(x,y)$. Тогда функция $u(x,y) = u_1(x,y) - u_2(x,y)$ удовлетворяет уравнению (2.1) с однородными краевыми условиями. Докажем, что $u(x,y) \equiv 0$ в $D^+ \cup \Gamma_1$ $(D^- \cup \Gamma_2)$.

В области D^+ (D^-) справедливо следующее тождество:

$$u\,L[u] \equiv \frac{\partial}{\partial x}\left(u\,u_{xx} - \frac{1}{2}\,u_x^2\right) + \frac{\partial}{\partial y}\left(u\,u_y\right) - u_y^2 + \mu u^2 = 0. \tag{3.1}$$

Интегрируя тождество (3.1) по области $D_p^+ = \{(x,y) : 0 < x < p,\ 0 < y < q\}$, где $p > 0$, имеем

$$\int\limits_0^p \int\limits_0^q \left[\frac{\partial}{\partial x}\left(u\,u_{xx} - \frac{1}{2}\,u_x^2\right) + \frac{\partial}{\partial y}\left(u\,u_y\right)\right] dxdy - \int\limits_0^p \int\limits_0^q u_y^2 dxdy + \mu \int\limits_0^p \int\limits_0^q u^2 dxdy = 0.$$

При стремлении параметра p к бесконечности $(p \to +\infty)$ область D_p^+ переходит в D^+. Учитывая это предельное расширение области, а также опираясь на однородность краевых условий задачи B_1, условие $a \ne 0$, асимптотическое поведение функции $u(x,y)$ на бесконечности $(x \to +\infty)$ и $u_y \in L_2(D^+)$, получаем

$$\frac{1}{2}\lim_{p\to+\infty} \int\limits_0^q u_x^2(p,y)\,dy - \frac{b}{a}\int\limits_0^q u_{xx}^2(0,y)\,dy + \int\limits_0^p \int\limits_0^q u_y^2(x,y)\,dxdy - \mu \int\limits_0^p \int\limits_0^q u^2(x,y)\,dxdy = 0$$

Рассмотрим случай, когда параметр $\mu = 0$, с учётом условий Теоремы 3.1, т.е. $ab \le 0$, приходим к равенству $u_y(x,y) = 0$. Решение данного уравнения имеет вид: $u(x,y) = f(x)$. Исходя из граничного условия (2.2), согласно которому $u(x,0) = 0$, заключаем, что $f(x) \equiv 0$. Следовательно, $u(x,y) \equiv 0$ в $D^+ \cup \Gamma_1$. Аналогично, при $\mu < 0$ из пятого слагаемого непосредственно следует, что $u(x,y) \equiv 0$ в $D^+ \cup \Gamma_1$.

Теперь интегрируя тождество (3.1) по области $D_l^- = \{(x,y) : l < x < 0,\ 0 < y < q\}$, где $l < 0$ получаем:

$$\int\limits_l^0 \int\limits_0^q \left[\frac{\partial}{\partial x}\left(u u_{xx} - \frac{1}{2}u_x^2\right) + \frac{\partial}{\partial y}\left(u u_y\right)\right] dxdy - \int\limits_l^0 \int\limits_0^q u_y^2 dxdy + \mu \int\limits_l^0 \int\limits_0^q u^2 dxdy = 0.$$

Рассмотрим предельный случай при стремлении параметра l к минус бесконечности $(l \to -\infty)$, что соответствует расширению подобласти D_l^- до границ области D^-. Принимая во внимание однородность краевых условий, заданных в задаче B_2, и условие $c \ne 0$, а также опираясь на асимптотическое поведение функции $u(x,y)$ при $x \to -\infty$ и $u_y \in L_2(D^-)$, получаем

$$\frac{d}{c}\int\limits_0^q u_{xx}^2(0,y)\,dy + \frac{1}{2}\int\limits_0^q u_x^2(0,y)\,dy + \int\limits_l^0 \int\limits_0^q u_y^2(x,y)\,dxdy - \mu \int\limits_l^0 \int\limits_0^q u^2(x,y)\,dxdy = 0.$$

Отсюда также легко следует, что $u(x,y) \equiv 0$ в $D^- \cup \Gamma_2$.

В случае $b \ne 0$ $(d \ne 0)$ аналогичным образом доказывается тождество $u(x,y) \equiv 0$ в области $D^+ \cup \Gamma_1$ $(D^- \cup \Gamma_2)$.

Теоема 3.1 доказана.

Замечание 3.1. Стоит подчеркнуть, что в случае невыполнения условий Теоремы 3.1 однородная задача B_1 допускает существование нетривиальных решений. В частности, при $a = bk_n^2$ (где $b \in \mathbb{R}\backslash\{0\}$) и $k_n = \sqrt[3]{\lambda_n - \mu}$,

задача:

$$\begin{cases} u_{xxx}+u_{yy}+\mu u=0, & \\ u(x,0)=u(x,q)=0, & 0\le x<+\infty, \\ k_n^2 u(0,y)+u_{xx}(0,y)=0, & \\ u_x(0,y)=0, & 0\le y\le q, \\ \lim\limits_{x\to\infty} u(x,y)=0, & \end{cases}$$

обладает ненулевым решением вида:

$$u_n(x,y)=A_1 e^{-\frac{k_n}{2}x}\cos\left(\frac{\sqrt{3}}{2}k_n x-\frac{\pi}{6}\right)\sin\frac{n\pi}{q}y, \quad A_1\in\mathbb{R}, \quad n\in N.$$

Замечание 3.2. Необходимо заметить, что нарушение условий Теоремы 3.1 приводит к тому, что однородная задача B_2 может обладать нетривиальными решениями. В частности, при $c=-dk_n^2$ (где $d\in\mathbb{R}\backslash\{0\}$) и $k_n=\sqrt[3]{\lambda_n-\mu}$, задача:

$$\begin{cases} u_{xxx}+u_{yy}+\mu u=0, & \\ u(x,0)=u(x,q)=0, & -\infty<x\le 0, \\ -k_n^2 u(0,y)+u_{xx}(0,y)=0, & \\ \lim\limits_{x\to-\infty} u(x,y)=\lim\limits_{x\to-\infty} u_x(x,y)=0, & 0\le y\le q, \end{cases}$$

существует ненулевое решение следующего вида:

$$u_n(x,y)=A_2 e^{k_n x}\sin\frac{n\pi}{q}y, \quad A_2\in\mathbb{R}, \quad n\in N.$$

4. Существование решения

Теорема 4.1. *Если* $\psi_i(y)\in C^2[0,q], \quad (i=1,2)$ *и выполняются условия* $a\ne bk_n^2$ *и (2.4) то решение задачи* B_1 *существует.*

Доказательство. Решение задачи B_1 представим в следующем виде:

$$u(x,y)=X(x)Y(y). \tag{4.1}$$

После подстановки выражения (4.1) в уравнение (2.1) и применения метода разделения переменных для функции $Y(y)$ приходим к классической задаче Штурма-Лиувилля:

$$\begin{cases} Y''+\lambda Y=0, \\ Y(0)=Y(q)=0, \end{cases} \tag{4.2}$$

где λ - параметр разделения.

Нетривиальные решения рассматриваемой задачи (4.2) возможны лишь при выполнении условия:

$$\lambda_n=\left(\frac{n\pi}{q}\right)^2, \quad n\in N.$$

Указанные числа представляют собой спектр собственных значений задачи (4.2), при этом соответствующие им собственные функции определяются выражением:

$$Y_n(y)=\sqrt{\frac{2}{q}}sin\left(\frac{n\pi}{q}y\right), \quad n\in N. \tag{4.3}$$

Следует подчеркнуть, что система собственных функций (4.3) задачи (4.2) обладает свойствами полноты и ортонормированности в пространстве $L_2(D^+)$, составляя тем самым базис [18].

Для $X_n(x)$ приходим к следующей задаче:

$$\begin{cases} \mathrm{X}'''_n(x) - (\lambda_n - \mu)\,\mathrm{X}_n(x) = 0, \\ aX_n(0) + bX''_n(0) = \psi_{1n}, \\ X'_n(0) = \psi_{2n}, \\ \lim\limits_{x\to+\infty} \mathrm{X}_n(x) = 0, \end{cases} \tag{4.4}$$

где

$$\psi_{in} = \sqrt{\frac{2}{q}} \int_0^q \psi_i(y) sin\left(\frac{n\pi}{q} y\right) dy, \quad (i = 1, 2), \quad n \in N.$$

Решение уравнения в задаче (4.4) имеет вид

$$X_n(x) = C_{1n} e^{k_n x} + e^{-\frac{1}{2}k_n x}\left(C_{2n}\cos\frac{\sqrt{3}}{2}k_n x + C_{3n}\sin\frac{\sqrt{3}}{2}k_n x\right) \tag{4.5}$$

где

$$k_n = \sqrt[3]{\lambda_n - \mu} = \sqrt[3]{\left(\frac{n\pi}{q}\right)^2 - \mu}, \; n \in N.$$

Исходя из условий задачи B_1, на бесконечности должно выполняться соотношение $\lim\limits_{x\to+\infty} X_n(x) = 0$. Данное требование обуславливает обнуление коэффициента C_{1n} в выражении (4.5). В результате подстановки $C_{1n} = 0$ функция (4.5) преобразуется к следующему виду:

$$X_n(x) = e^{-\frac{1}{2}k_n x}\left(C_{2n}\cos\frac{\sqrt{3}}{2}k_n x + C_{3n}\sin\frac{\sqrt{3}}{2}k_n x\right) \tag{4.6}$$

Опираясь на условия, сформулированные в задаче (4.4), приходим к следующему результату:

$$\begin{cases} \left(a - \frac{bk_n^2}{2}\right) C_{2n} - \frac{\sqrt{3}bk_n^2}{2} C_{3n} = \psi_{1n}, \\ -\frac{k_n}{2} C_{2n} + \frac{\sqrt{3}k_n}{2} C_{3n} = \psi_{2n}. \end{cases} \tag{4.7}$$

Определитель системы (4.7), принимает следующий вид:

$$\Delta = \begin{vmatrix} a - \frac{bk_n^2}{2} & -\frac{\sqrt{3}bk_n^2}{2} \\ -\frac{k_n}{2} & \frac{\sqrt{3}k_n}{2} \end{vmatrix} = \frac{\sqrt{3}k_n}{2}\left(a - bk_n^2\right) = \frac{\sqrt{3}k_n^3}{2}\overline{\Delta}$$

где

$$\overline{\Delta} = \left(\frac{a}{k_n^2} - b\right).$$

На основании условий Теоремы 4.1 (в частности, при $a \neq bk_n^2$), главный определитель системы отличен от нуля ($\Delta \neq 0$). Следовательно, система (4.7) имеет единственное решение вида:

$$C_{2n} = \frac{1}{\overline{\Delta}}\left(\frac{\psi_{1n}}{k_n^2} + b\frac{\psi_{2n}}{k_n}\right),$$

$$C_{3n} = \frac{1}{\sqrt{3}\,\overline{\Delta}}\left(\frac{\psi_{1n}}{k_n^2} + \frac{\psi_{2n}}{k_n}\left(\frac{2a}{k_n^2} - b\right)\right).$$

С учётом (4.1), решение задачи B_1 имеет вид:

$$u(x,y) = \sqrt{\frac{2}{q}} \sum_{n=1}^{\infty} e^{-\frac{1}{2}k_n x}\left(C_{2n}\cos\frac{\sqrt{3}}{2}k_n x + C_{3n}\sin\frac{\sqrt{3}}{2}k_n x\right) \sin\left(\frac{n\pi}{q} y\right). \tag{4.8}$$

В дальнейшем, для упрощения записи, максимальное из всех найденных положительных констант в полученных оценках будет обозначаться символом M.

Далее установим условия, обеспечивающие абсолютную и равномерную сходимость ряда (4.8) в пределах области $D^+ \cup \Gamma_1$. Опираясь на выражение (4.8), запишем:

$$|u(x,y)| \le \sqrt{\tfrac{2}{q}} \sum_{n=1}^{+\infty} \left| \left[e^{-\frac{1}{2}k_n x} \left(C_{2n} \cos \tfrac{\sqrt{3}}{2} k_n x + C_{3n} \sin \tfrac{\sqrt{3}}{2} k_n x \right) \right] \sin \left(\tfrac{n\pi}{q} y \right) \right| \le \\ \le M \sum_{n=1}^{+\infty} [|C_{2n}| + |C_{3n}|]. \tag{4.9}$$

Выполним оценку коэффициентов C_{in}, $(i = 2, 3)$ в следующем виде:

$$|C_{2n}| \le M \left[\frac{|\psi_{1n}|}{k_n^2} + \frac{|\psi_{2n}|}{k_n} \right], \quad |C_{3n}| \le M \left[\frac{|\psi_{1n}|}{k_n^2} + \frac{|\psi_{2n}|}{k_n} \right].$$

В результате подстановки полученных оценок коэффициентов C_{in}, $(i = 2, 3)$ в выражение (4.9), получим

$$|u(x,y)| \le M \sum_{n=1}^{+\infty} \left[\frac{|\psi_{1n}|}{k_n^2} + \frac{|\psi_{2n}|}{k_n} \right]. \tag{4.10}$$

Применяя метод интегрирования по частям к выражениям ψ_{in} $(i = 1, 2)$ и принимая во внимание условие (2.4), получим

$$\psi_{in} = \left(\frac{q}{\pi} \right)^2 \frac{\Psi_{in}}{n^2}, \quad (i = 1, 2),$$

где

$$\Psi_{in} = -\sqrt{\frac{2}{q}} \int_0^q \psi''_i(y) \sin \left(\frac{n\pi}{q} y \right) dy, \quad (i = 1, 2).$$

Тогда

$$|\psi_{in}| \le M \frac{|\Psi_{in}|}{n^2}, \quad (i = 1, 2), \quad M = const > 0.$$

С учетом полученных выше оценок, из соотношения (4.10) имеем:

$$|u(x,y)| \le M \sum_{n=1}^{+\infty} \left[\frac{|\Psi_{1n}|}{n^{\frac{10}{3}}} + \frac{|\Psi_{2n}|}{n^{\frac{8}{3}}} \right] < \infty.$$

Таким образом, вышеизложенные оценки подтверждают абсолютную и равномерную сходимость ряда (4.8) в замкнутой области $D^+ \cup \Gamma_1$.

Перейдем к обоснованию абсолютной и равномерной сходимости частных производных ряда (4.8), входящих в уравнение (2.1), в замкнутой области $D^+ \cup \Gamma_1$. С этой целью продифференцируем ряд (4.8) по переменной y до второго порядка включительно, в результате чего получим следующее выражение:

$$\frac{\partial^2 u}{\partial y^2} = -\sum_{n=1}^{+\infty} \lambda_n \left[e^{-\frac{1}{2}k_n x} \left(C_{2n} \cos \frac{\sqrt{3}}{2} k_n x + C_{3n} \sin \frac{\sqrt{3}}{2} k_n x \right) \right] Y_n(y).$$

Принимая во внимание полученную оценку для функции $u(x, y)$, имеем

$$\left| \frac{\partial^2 u}{\partial y^2} \right| \le M \sum_{n=1}^{+\infty} \left[\frac{|\Psi_{1n}|}{n^{\frac{4}{3}}} + \frac{|\Psi_{2n}|}{n^{\frac{2}{3}}} \right].$$

Применяя неравенства Коши–Буняковского и Бесселя к полученным выражениям, приходим к следующей оценке:

$$\left|\frac{\partial^2 u}{\partial y^2}\right| \leq M\left[\sum_{n=1}^{+\infty}\frac{|\Psi_{1n}|}{n^{\frac{4}{3}}}+\sqrt{\sum_{n=1}^{+\infty}|\Psi_{2n}|^2}\sqrt{\sum_{n=1}^{+\infty}\frac{1}{n^{\frac{4}{3}}}}\right]\leq$$
$$\leq M\left[\sum_{n=1}^{+\infty}\frac{|\Psi_{1n}|}{n^{\frac{4}{3}}}+\|\psi''_{2n}(y)\|_{L_2(0,q)}\sqrt{\sum_{n=1}^{+\infty}\frac{1}{n^{\frac{4}{3}}}}\right]<\infty,$$

где

$$\sum_{n=1}^{+\infty}|\Psi_{2n}|^2\leq\|\psi''_2(y)\|^2_{L_2(0,q)}.$$

Таким образом, функциональный ряд, представляющий частную производную $\frac{\partial^2 u}{\partial y^2}$, сходится абсолютно и равномерно в замкнутой области $D^+\cup\Gamma_1$.

Далее выполним дифференцирование ряда (4.8) по переменной x до третьего порядка:

$$\frac{\partial^3 u}{\partial x^3}=\sum_{n=1}^{+\infty}k_n^3\left[e^{-\frac{1}{2}k_n x}\left(C_{2n}\cos\frac{\sqrt{3}}{2}k_n x-C_{3n}\sin\frac{\sqrt{3}}{2}k_n x\right)\right]Y_n(y).$$

С учетом оценки $u(x,y)$ имеем:

$$\left|\frac{\partial^3 u}{\partial x^3}\right|\leq M\sum_{n=1}^{+\infty}\left[\frac{|\Psi_{1n}|}{n^{\frac{4}{3}}}+\frac{|\Psi_{2n}|}{n^{\frac{2}{3}}}\right].$$

Отсюда также как, u_{yy} следует абсолютная и равномерная сходимость ряда в области $D^+\cup\Gamma_1$.

Теорема 4.1 доказана.

Теорема 4.2. *Если $\psi_3(y)\in C^2[0,q]$ и выполняются условия $c\neq -dk_n^2$ и (2.6) то решение задачи B_2 существует.*

Доказательство. Решение задачи B_2 также имеет вид (4.1), и при подстановке в (2.1) для функции $X(x)$ в области D^- получаем следующую задачу:

$$\begin{cases}\mathrm{X}'''_n(x)-(\lambda_n-\mu)\mathrm{X}_n(x)=0,\\ cX_n(0)+dX''_n(0)=\psi_{3n},\\ \lim\limits_{x\to-\infty}X_n(x)=\lim\limits_{x\to-\infty}X'_n(x)=0,\end{cases}\tag{4.11}$$

где

$$\psi_{3\,n}=\sqrt{\frac{2}{q}}\int_0^q\psi_3(y)\,sin\left(\frac{n\pi}{q}y\right)dy,\ \ n\in N.$$

Общее решение уравнения в задаче (4.11) имеет вид (4.5). В силу условий $\lim\limits_{x\to-\infty}X_n(x)=\lim\limits_{x\to-\infty}X'_n(x)=0$ следует, что $C_{2n}=C_{3n}=0$, тогда (4.5) принимает следующий вид:

$$X_n(x)=C_{1n}e^{k_n x}.\tag{4.12}$$

С учетом заданных граничных условий, коэффициент C_{1n} определяется следующим образом:

$$C_{1n}=\frac{\psi_{3n}}{c+dk_n^2},$$

где

$$\psi_{3n}=\sqrt{\frac{2}{q}}\int_0^q\psi_3(y)\sin\left(\frac{n\pi}{q}y\right)dy,\ n\in N.\tag{4.13}$$

На основании соотношений (4.3) и (4.12) решение задачи B_2 имеет в следующем виде:

$$u(x,y)=\sqrt{\frac{2}{q}}\sum_{n=1}^{+\infty}C_{1n}e^{k_nx}\sin\left(\frac{n\pi}{q}y\right),\ n\in N. \tag{4.14}$$

Перейдем к обоснованию абсолютной и равномерной сходимости ряда (4.14). Непосредственно из выражения (4.14) получим следующую оценку:

$$|u(x,y)|\le\sqrt{\frac{2}{q}}\sum_{n=1}^{+\infty}\left|C_{1n}e^{k_nx}\sin\left(\frac{n\pi}{q}y\right)\right|\le M\sum_{n=1}^{+\infty}|C_{1n}|. \tag{4.15}$$

Оценка C_{1n} имеет вид

$$|C_{1n}|\le M\frac{|\psi_{3n}|}{k_n^2},\quad M=const>0.$$

Применяя метод интегрирования по частям к выражению (4.13) и принимая во внимание условие (2.6), приходим к следующей оценке:

$$|\psi_{3n}|\le M\frac{|\Psi_{3n}|}{n^2},$$

где

$$\Psi_{3n}=-\sqrt{\frac{2}{q}}\int_0^q\psi''_3(y)\sin\left(\frac{n\pi}{q}y\right)dy.$$

Тогда из (4.15) получаем

$$|u(x,y)|\le M\sum_{n=1}^{+\infty}\frac{|\Psi_{3n}|}{n^{\frac{10}{3}}}<\infty.$$

Следовательно, ряд (4.14) сходится абсолютно и равномерно в области $D^-\cup\Gamma_2$. Далее обоснуем, что частные производные функционального ряда (4.14), входящие в исходное уравнение (2.1), также обладают свойствами абсолютной и равномерной сходимости в указанной области $D^-\cup\Gamma_2$. Для проверки этого факта вычислим почленную производную ряда (4.14) по переменной y:

$$\frac{\partial^2u}{\partial y^2}=-\sum_{n=1}^{+\infty}\lambda_nC_{1n}e^{k_nx}Y_n(y).$$

С учетом оценку (4.15) имеем:

$$\left|\frac{\partial^2u}{\partial y^2}\right|\le M\sum_{n=1}^{+\infty}\frac{|\Psi_{3n}|}{n^{\frac{4}{3}}}<\infty.$$

Таким образом, ряд, соответствующий функции $\frac{\partial^2u}{\partial y^2}$, сходится абсолютно и равномерно в $D^-\cup\Gamma_2$.

Теперь вычислим частную производную третьего порядка по переменной x от ряда (4.14) получаем

$$\frac{\partial^3u}{\partial x^3}=\sum_{n=1}^{+\infty}k_n^3C_{1n}e^{k_nx}Y_n(y).$$

С учетом оценки для $u(x,y)$ получаем:

$$\left|\frac{\partial^3u}{\partial x^3}\right|\le M\sum_{n=1}^{+\infty}\frac{|\Psi_{3n}|}{n^{\frac{4}{3}}}<\infty.$$

Таким образом, установлено, что функциональный ряд (4.14) и все частные производные, входящие в состав уравнения (2.1), сходятся абсолютно и равномерно в замкнутой области $D^-\cup\Gamma_2$. На этом доказательство Теоремы 4.2 считается завершенным.

5. Заключение

В настоящей работе проведено комплексное исследование краевых задач для вязко-трансзвукового уравнения в полубесконечных областях. Единственность решения сформулированных задач была строго обоснована с применением метода энергетических интегралов. Ключевым результатом исследования является установление того факта, что при нарушении условий единственности соответствующие однородные задачи обладают нетривиальными решениями, что подчеркивает высокую чувствительность рассматриваемых математических моделей к заданным граничным параметрам.

Существование решения обосновано методом разделения переменных. Искомое решение было построено в виде бесконечного функционального ряда, для которого проведено детальное доказательство абсолютной и равномерной сходимости. Кроме того, была подтверждена законность почленного дифференцирования полученного ряда по всем переменным, что обеспечивает обоснованность его применения в качестве решения исходного уравнения в частных производных в исследуемой области.

Предложенный аналитический подход представляет собой эффективный инструментарий для решения широкого класса дифференциальных уравнений третьего порядка. Перспективы дальнейших исследований заключаются в расширении данных результатов на более сложные нелинейные уравнения, нестационарные задачи, а также системы дифференциальных уравнений с различными спектральными характеристиками.

Благодарность

Авторы хотели бы выразить искреннюю благодарность редактору и рецензентам за их полезные комментарии и предложения.

Финансирование

Для этой работы нет финансирования.

Список литературы

[1] Юлдашев Т. К.: *Обратная задача для одного интегро-дифференциального уравнения Фредгольма третьего порядка.* Вестн. Сам. гос. техн. ун-та. Сер. Физ.-мат. науки. – 2014. – № 1(34). – С. 56-65.

[2] Диесперов В.Н., Ломакин Л.А.: *Об одной краевой задаче для линеаризованного осесимметрического ВТ уравнения.* ЖВМ и МФ. – 1974. – Т. 14. – № 5. – С. 1244-1260.

[3] Alymkulov K., Tursunov D.A., Azimov B.A.: *Generalized method of boundary layer function for bisingularly perturbed differential cole equation.* (2017) Far East Journal of Mathematical Sciences, 101 (3), pp. 507 - 516. DOI: 10.17654/MS101030507

[4] Кузнецов Н.Н., Рыжов О.С.: *Об уравнениях вязкого трансзвукового течения.* ЖВМ и МФ. – 1969. – Т. 9. – №. 1. – С. 164-180.

[5] Ryzhov, O. S.: *Viscous Transonic Flows*. Annual Review of Fluid Mechanics.10(1), pp. 66-73. (1978).

[6] Рыжов О. С.: *Асимптотическая картина обтекания тел вращения со звуковым потоком вязкого и теплопроводящего газа.* ПММ. – 1965. – Т. 29. – Вып. 6. – С. 1004-1014.

[7] Диесперов В.Н.: *О функции Грина линеаризованного вязкого трансзвукового уравнения.* ЖВМ и МФ. - 1972. - Т. 12. - No 5. - С. 1265-1279.

[8] Block H.: *Sur les equations lineaires aux derives parielles a carateristiques multiples.*Ark. Mat. Astron. Fus. – 1912. – 7(13). – pp. 1-34.

[9] Del Vecchio E.: *Sulle equazioni.* Memorie R. Accad. Sci. Ser.2. – Torino. – 1915. – 66. – pp. 1-41.

[10] Whitham G.B.: *Linear and Nonlinear Waves.* John Wiley Sons, 1974.

[11] Джураев Т. Д., Апаков Ю.П.: *Об автомодельном решении одного уравнения третьего порядка.* Вестник Сам. гос. техн. ун-та. – 2007. – № 2(15). – С. 18-26.

[12] Джураев Т. Д., Апаков Ю.П.: *К теории уравнения третьего порядка, содержащего вторую производную по времени.* Украинский математический журнал. – 2010. – Т. 62. – № 1. – С. 40-51.

[13] Apakov Yu.P., Hamitov A.A.: *On solution of the boundary value problems posed for an equation with the third-order multiple characteristics in semi-bounded domains in three dimensional space.* Boletin de la Sociedad Matematica Mexicana. 2023. 29:58. – pp.2-14. https://doi.org/10.1007/s40590-023-00523-1

[14] Apakov Yu.P., Hamitov A.A.: *Boundary value problems for a third order equation with multiple characteristics in three-dimensional space in semi-bounded domains.* Filomat. 2025. 39:8. - pp. 2603-2612. https://doi.org/10.2298/FIL2508603A

[15] Апаков Ю.П., Иброхимов Х. К.: *О разрешимости краевой задачи для вязко-трансзвукового уравнения.* Научный вестник НамГУ. – 2024. – №10. – С. 43-48.
[16] Apakov Yu. P., Ibrokhimov Kh. K.: *On solving the second boundary value problem for the Viscous Transonic Equation.* Bulletin of the Karaganda University. – 2025. – 3(19). – pp. 34-45.
[17] Apakov Yu. P., Ibrokhimov Kh. K.: *Bitsadze-Samarskii Type Problem For The Viscous-Transonic Equation.* Journal of Mathematical Sciences. Published: 13 February 2026. https://doi.org/10.1007/s10958-026-08224-6.
[18] Тихонов, А.Н., Самарский, А.А.:*Уравнения математической физики.* М.:"Наука". -724 с. (1966).

Boundary value problems for the viscous-transonic equation in semi-bounded domains

Apakov Yusupjon Pulatovich,* Ibrokhimov Khusniddin Komiljon ugli

Abstract

This research examines boundary value problems associated with the viscous-transonic equation within semi-infinite domains. The uniqueness of the solutions is established by employing the energy integral method. Furthermore, it is demonstrated that when uniqueness conditions are not satisfied, the corresponding homogeneous problems admit non-trivial solutions. The existence of the solution is addressed through the method of separation of variables; the required solution is represented as an infinite series, and the validity of its term-by-term differentiation with respect to all variables is rigorously proven.

Keywords

Third-order partial differential equation, viscous-transonic equation, boundary value problems, uniqueness theorem, existence of solution, method of separation of variables, spectral problem (eigenvalue), infinite functional series, absolute and uniform convergence.

Affiliations

Apakov Yusupjon Pulatovich
Address: Namangan State Technical University, Dept. of Higher Mathematics, 160103, Namangan, Uzbekistan and Institute of Mathematics named after V.I.Romanovsky Academy of Sciences of the Republic of Uzbekistan.
e-mail: yusupjonapakov@gmail.com
ORCID ID: https://orcid.org/0000-0001-9611-1466

Ibrokhimov Khusniddin Komiljon ugli
Address: Namangan State Technical University, Dept. of Higher Mathematics, 160103, Namangan, Uzbekistan.
e-mail: xusniddin571@mail.ru
ORCID ID: https://orcid.org/0009-0001-7142-0956

Uzbekistan Journal of Mathematics and Computer Science
Volume-2, Issue-3-special, Page 170–178 (2026)
DOI: https://doi.org/10.56143/ujmcs.v2i3s.14
Research article

UZJMCS

Uniqueness of 3-adic periodic Gibbs measures for the Ising model on a (2,3)-ary tree

A.M. Tukhtabaev

Abstract

In this paper, translation-invariant 3-adic Gibbs measures for the Ising model on a $(2,3)$-ary tree are studied. The dynamics of the function corresponding to this model is investigated, and the existence of a unique global attracting fixed point is proved.

Keywords: *p-adic numbers; Ising model; Gibbs measure; (i,k)-ary tree.*

AMS Subject Classification (2020): *Primary: 37B05; Secondary: 37B10; 12J12; 39A70.*

1. Introduction

The Ising model was invented by the physicist Wilhelm Lenz, who gave it as a problem to his student Ernst Ising. The one-dimensional Ising model was solved by Ising in [1] and he proved that the model does not exhibit the phase transition. Using geometric-probabilistic approach, the contour argument was introduced by Peierls for the two-dimensional square-lattice Ising model to show the existence of phase transitions at low temperatures [2]. For this model an analytic description was given by Lars Onsager [3]. The existence of a phase transition for the Ising model on the Cayley trees was established independently by Katsura, Takizawa [4] and Preston [5] (see also [6] for more details).

In the p-adic context, during the last few decades great interest has been devoted to the investigation of various properties of some classical models of statistical mechanics (see, for example, [7, 8, 9, 10, 11]). These investigations lead the development of the p-adic stochastic processes [12, 13]. The p-adic Ising model on a Cayley tree was considered in [15, 14], and for this model the uniqueness of p-adic Gibbs measure was proved. To enlarge the set of p-adic Gibbs measures, the notion of p-adic generalized Gibbs measures for the Ising model were introduced in [14]. In [16] the authors proved the existence of infinitely many periodic p-adic generalized Gibbs measures for the Ising model on the semi-infinite Cayley tree. Furthermore, in [17] $G_k^{(2)}$-periodic p-adic generalized Gibbs measures for the same model on the Cayley trees have been studied. In [18] the authors studied translation-invariant p-adic generalized Gibbs measures for the Ising model on Cayley trees of order k. Moreover, the set of fixed points of the Ising-Potts mapping was described, which allows to detect the existence of the phase transition.

In present paper, we study 3-adic quasi-Gibbs measures for the Ising model on a $(2,3)$-ary tree. It should be noted that for the case $p > 3$, this problem was studied in paper [19].

Received : *28–april–2026,* ***Accepted :*** *31–july–2026*
**** Corresponding author***

2. Preliminaries

2.1. *p-adic numbers*

For the field of rational numbers $\mathbb{Q}$, and a predetermined prime p, any $x \in \mathbb{Q} \setminus \{0\}$ possesses a representation $x = p^r \frac{n}{m}$, where $r \in \mathbb{Z}$, $m \in \mathbb{Z}^+$, and n, m are integers such that $\gcd(n, p) = \gcd(m, p) = 1$, where $\gcd(a, b)$ is the greatest common divisor of a and b.

The p-adic norm on $\mathbb{Q}$, defined as $| x |_p = p^{-r}$ for non-zero $x = p^r \frac{n}{m}$ and $| 0 |_p = 0$, is a non-Archimedean valuation. Consequently, it adheres to the strong triangle inequality, which states that $| x + y |_p \leq \max\{| x |_p , | y |_p\}$ (see [24]).

The field of rational numbers $\mathbb{Q}$ is incomplete with respect to the p-adic norm. The completion of $\mathbb{Q}$ under this norm results in the construction of the field of p-adic numbers $\mathbb{Q}_p$.

Later we use the following important properties of the p-adic norm:
1) if $| x |_p \neq | y |_p$, then $| x \pm y |_p = \max\{| x |_p, | y |_p\}$;
2) if $| x |_p = | y |_p$, then $| x \pm y |_p \leq | x |_p$.

Every non-zero p-adic number y possesses a unique canonical expansion:

$$y = p^{\gamma(y)}(y_0 + y_1 p + y_2 p^2 + ...),$$

where the p-adic valuation $\gamma(y) \in \mathbb{Z}$ and the coefficients y_j, $j = 0, 1, 2, ...$ satisfy $0 < y_0 \leq p - 1$ and $0 \leq y_j \leq p - 1$ for $j \geq 1$. The p-adic norm is extended to $\mathbb{Q}_p$ as $| y |_p = p^{-\gamma(y)}$, where $\gamma(y)$ is also denoted as $\mathrm{ord}_p(y)$ (see e.g., [24]).

Let a be an element of the p-adic field $\mathbb{Q}_p$ and let $r > 0$. We define the open ball $B(a, r)$, the closed ball $\overline{B(a, r)}$, and the sphere $S(a, r)$ centered at a with radius r as the sets:

$$B(a, r) = \{x \in \mathbb{Q}_p : | x - a |_p < r\},$$
$$\overline{B(a, r)} = \{x \in \mathbb{Q}_p : | x - a |_p \leq r\},$$
$$S(a, r) = \{x \in \mathbb{Q}_p : | x - a |_p = r\}.$$

The sets $\mathbb{Z}_p := \overline{B(0, 1)}$ and $\mathbb{Z}_p^* = S(0, 1)$ are respectively called *p-adic integers* and *p-adic units*.

p-adic *exponential* is defined by

$$\exp_p(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!},$$

which converges for $x \in B(0, p^{-1/(p-1)})$.

Put

$$\mathcal{E}_p = \left\{x \in \mathbb{Q}_p :| x - 1 |_p < p^{-1/(p-1)}\right\}.$$

The set $\mathcal{E}_p$ is a range of exponential. A more detailed see [20], [21].

The symbol $o[x]$ was adopted in the work [22] to facilitate simplified computations. By definition, $z = o[x]$ means that $| z |_p < | x |_p$.

Lemma 2.1. *(Hensel's lemma)[23, 24] Let $F(x)$ be a polynomial whose coefficients are p-adic integers and $F'(x)$ be its formal derivative. If there exists an element $a \in \mathbb{Z}_p$ that satisfies the following strict inequality:*

$$| F(a) |_p < | F'(a) |_p^2,$$

then there exists a unique root α in the ring $\mathbb{Z}_p$ satisfying the following properties:

- $F(\alpha) = 0$;
- $| \alpha - a |_p \leq \frac{|F(a)|_p}{|F'(a)|_p} < | F'(a) |_p$, *i.e.* $\alpha = a + o[F'(a)]$.

2.2. *p-adic measure*

Let $(X, \mathcal{B})$ be a measurable space, where $\mathcal{B}$ represents an algebra of subsets of X. A function $\mu : \mathcal{B} \to \mathbb{Q}_p$ is termed a p-adic measure if it satisfies the property of finite additivity: for any finite collection of pairwise disjoint sets $A_1, A_2, \ldots, A_n \in \mathcal{B}$, the following equality holds:

$$\mu\left(\bigcup_{j=1}^{n} A_j\right) = \sum_{j=1}^{n} \mu(A_j).$$

Furthermore, if the total measure is unity ($\mu(X) = 1$), the p-adic measure is specifically referred to as a p-adic probability measure. A p-adic measure μ is defined as bounded if the supremum of the p-adic norm of the measures of all sets in the algebra is finite:

$$\sup\{|\ \mu(A)\ |_p : A \in \mathcal{B}\} < \infty.$$

2.3. *(i, k)-ary tree*

Let $T = (V,\ E)$ be a rooted infinite tree, i.e.,

1. T is a connected graph without cycles;
2. $|V|$ is infinite;
3. one vertex (e.g. x_0) has been assigned as the root of the tree.

Definition 2.1. Let k be a positive integer. A *k-ary tree* is the rooted infinite tree such that each vertex has at most $k + 1$ edges.

Definition 2.2. The k-ary tree is called a *regular tree* of order k if every vertex except the root has exactly $k + 1$ edges, the root has k edges.

Two vertices x and y are called *nearest neighbors* if there exists an edge $l \in L$ connecting them $l = \langle x, y\rangle$.
A collection of nearest neighbor pairs $\langle x, x_1\rangle, \langle x_1, x_2\rangle, \ldots, \langle x_m, y\rangle$ is called a *path* from x to y.
The *distance* between the vertices x and y ($d(x, y)$) is the number of edges of the shortest path from x to y.
Let us fix $x_0 \in V$ vertex as a root of the tree. Then for any nonnegative integer n, we introduce the following sets

$$W_n = \{x \in V : d(x, x_0) = n\}, V_n = \bigcup_{m=0}^{n} W_m,$$

$$V_e = \bigcup_{n=0}^{+\infty} W_{2n},\ V_o = \bigcup_{n=0}^{+\infty} W_{2n+1},$$

$$L_n = \{\langle x, y\rangle \in L : \forall x, y \in V_n\}.$$

For a given $x \in W_n$, we denote

$$S(x) = \{y \in W_{n+1} : d(x, y) = 1\}.$$

Definition 2.3. Let $1 \le i < k$. A rooted k-ary tree is called an *(i, k)-ary tree* if each vertex x has exactly $\kappa_{i,k}$ edges, where

$$\kappa_{i,k} = \begin{cases} i, \text{ if } x = x_0, \\ i + 1, \text{ if } x \in V_e \setminus \{x_0\}, \\ k + 1, \text{ if } x \in V_o. \end{cases}$$

2.4. A level function

Let $\mathbb{Q}_p$ be field of p-adic numbers.

Definition 2.4. A function $s : V \to \mathbb{Q}_p$ is called *level function* if $s^{(n)} := s(u) = s(v)$ for all $u, v \in W_n$, $n \in \mathbb{N}$. Any level function has the following form

$$s^{(n)} = \begin{cases} s_e^{(n)}, \text{ if } n \text{ is even,} \\ s_o^{(n)}, \text{ if } n \text{ is odd,} \end{cases}$$

Definition 2.5. The level function $s = \left(s_e^{(2n)}, s_o^{(n+1)}\right)$ is called *d-periodic* if there exists $d \in \mathbb{N}$ such that

$$s_e^{(2n+d)} = s_e^{(2n)}, \; s_o^{(2n+1+d)} = s_o^{(2n+1)},$$

for all $n \in \mathbb{N}$.

Definition 2.6. Any 1-periodic function is called *translation-invariant*.

It should be noted that the definitions in this section are presented in [19].

2.5. p-adic Ising model

Let $\mathbb{Q}_p$ be the field of p-adic numbers, and let $\Phi = \{-1, 1\}$. We consider the p-adic Ising model on a $(2, 3)$-ary tree. A *configuration* σ on V is a mapping $x \in V \mapsto \sigma(x) \in \Phi$. The set of all configurations on V is denoted by $\Omega := \Phi^V$. By Ω_n we denote the set of all configurations on V_n.

The p-adic Ising model's Hamiltonian on Ω_n is defined by

$$\mathcal{H}_n(\sigma_n) = J \sum_{\langle x,y\rangle \in L_n} \sigma_n(x)\sigma_n(y), \quad \forall \sigma_n \in \Omega_n, \tag{2.1}$$

where $J \in B(0, p^{\frac{-1}{p-1}})$ is a constant.

For a given function $\mathbf{h}$ we define a probability measure $\mu_{\mathbf{h}}^{(n)}$ on Ω_n as follows:

Let $\mathbf{h} : V \to \mathbb{Q}_p^2$ be a vector function such that $\mathbf{h} = \{(h_{x,-1}, h_{x,1}) \in \mathbb{Q}_p^2 : \forall x \in V\}$.

$$\mu_{\mathbf{h}}^{(n)}(\sigma_n) = \frac{\exp_p(\mathcal{H}_n(\sigma_n)) \prod\limits_{x \in W_n} h_{x,\sigma_n(x)}}{Z_{n,\mathbf{h}}}, \quad \forall \sigma_n \in \Omega_n, \tag{2.2}$$

here $Z_{n,\mathbf{h}}$ is the corresponding normalizing factor.

If the p-adic probability measures (1.2) are compatible, then by the p-adic analogue of Kolmogorov's theorem there exists a unique limiting measure $\mu_{\mathbf{h}}$ on the set Ω such that $\mu_{\mathbf{h}}\left(\left\{\sigma|_{V_n} \equiv \sigma_n\right\}\right) = \mu_{\mathbf{h}}^{(n)}(\sigma_n)$ for all $n \in \mathbb{N}$ and $\sigma_n \in \Omega_{V_n}$.

Such a measure $\mu_{\mathbf{h}}$ is called *p-adic quasi Gibbs measure* Moreover, if $h_{x,\sigma(x)} \in \mathcal{E}_p$ for every $x \in V$ and every configuration $\sigma \in \Omega$, then corresponding measure $\mu_{\mathbf{h}}$ is called *p-adic Gibbs measure*.

Proposition 2.1. *[15, 14] The sequence of p-adic probability measures $\{\mu_h^{(n)}\}_{n\geq 1}$, determined by formula (1.2) is consistent if and only if for any $x \in V\backslash\{x_0\}$, the following equation holds*

$$h_x = \prod_{y \in S(x)} \frac{\theta h_y + 1}{h_y + \theta}, \tag{2.3}$$

where $\theta = \exp_p(2J)$, $\theta \neq 1$.

3. Main results

Using the eqution (1.3), the problem of finding p-adic quasi-Gibbs measures for the Ising model on a $(2,3)$-ary tree leads to the analysis of the dynamics of the following function:

$$f(u) = \frac{au^6 + 2u^3 + 1}{u^6 + 2u^3 + a}, \tag{3.1}$$

where $a = \frac{\theta^2-\theta+1}{\theta} \in \mathcal{E}_3 \setminus \{1\}$.

Remark 3.1. Note that $\sqrt{a-1} = \frac{\theta-1}{\sqrt{\theta}} \in \mathbb{Q}_3$.

We are going to find fixed points of $f(u)$. We consider the equation $f(u) = u$. The last equation follows that

$$(u-1)\left(u^6 - (a-1)u^5 - (a-1)u^4 - (a-3)u^3 - (a-1)u^2 - (a-1)u + 1\right) = 0. \tag{3.2}$$

Clear that, one of the solution of (3.2) is $u_0 = 1$.

Now we consider the follow equation

$$u^6 - (a-1)u^5 - (a-1)u^4 - (a-3)u^3 - (a-1)u^2 - (a-1)u + 1 = 0. \tag{3.3}$$

We divide u^3 ($u = 0$ is not solution of (3.3)) both sides of (3.3), and using the notation

$$t = u + \frac{1}{u}, \tag{3.4}$$

we get

$$t^3 - (a-1)t^2 - (a+2)t + a + 1 = 0. \tag{3.5}$$

Lemma 3.1. *If t^* is a solution of (3.5) then $t^* \in \mathcal{E}_3$.*

Proof. Clear that $t = 0$ is not solution of (3.5). Let t^* be solution of (3.5).

Case 1. Assume that $0 < |t^*|_3 < 1$. Then

$$\left|t^{*3} - (a-1)t^{*2} - (a+2)t^* + a + 1\right|_3 = 1 \neq 0.$$

This case will not occur.

Case 2. Assume that $|t^*|_3 > 1$. It follows that

$$|t^{*3}|_3 = |t^*|_3^3 > |t^{*2}|_3 > |(a-1)t^{*2}|_3,$$

$$|t^{*3}|_3 > |t^*|_3 > |(a+2)t^*|_3,$$

$$|t^{*3}|_3 > 1 = |a+1|_3.$$

The strong triangle inequality yields that

$$\left|t^{*3} - (a-1)t^{*2} - (a+2)t^* + a + 1\right|_3 = |t^{*3}|_3 > 1 \neq 0.$$

This case also will not occur.

Case 3. Assume that $|t^*|_3 = 1$ and $t^* \notin \mathcal{E}_3$. It gives t^* has $2 + o[1]$ form, i.e., $t^* = 2 + o[1]$. Substituting this expression into (3.5), and after some algebras, we get

$$1 + o[1] = 0,$$

which is impossible. This contradiction proves the assumption of the lemma. □

Now we are going to solve (3.5).

Lemma 3.2. *The equation* (3.5) *is solvable iff* $|a-1|_3 < \frac{1}{27}$. *Moreover, the equation* (3.5) *has a unique solution and the solution has the following form*

$$t = -2 + 3^{ord_3(a-1)-2} + o[3^{ord_3(a-1)-2}].$$

Proof. Thanks to Lemma 3.1, the solution of (3.5) has the form $1 + o[1]$. Denote

$$g(t) = t^3 - (a-1)t^2 - (a+2)t + a + 1. \tag{3.6}$$

Then $g(1) = 1 - a$, $g'(1) = 3(1-a)$. Since $0 < |1-a|_3 < 1$, clear that

$$|g'(1)|_3^2 = \frac{1}{9}|1-a|_3^2 = \frac{1}{9}|g(1)|_3^2 < |g(1)|_3,$$

i.e. the condition of Hensel's lemma does not hold.

We rewrite (3.5) as follows

$$(t-1)(t-a)(t+2) = a - 1. \tag{3.7}$$

According to Lemma 3.1, we have

$$t = 1 + 3^r T,\ a = 1 + 3^l A, \tag{3.8}$$

where $r, l \in \mathbb{N}$, $T, A \in \mathbb{Z}_p^*$. According to Remark 3.1, we have that l is even and $A \equiv 1 (\mathrm{mod}\, 3)$.

Substituting (3.8) into (3.7), we have

$$3^r T(3^r T - 3^l A)(3 + 3^r T) = 3^l A. \tag{3.9}$$

Case 1. Suppose that $l \le r$. Then 3^{r+l+1} divides LHS of (3.9), however 3^{r+l+1} does not divide RHS of (3.9). It means that (3.5) does not have any solution in closed ball $\overline{B}(1, |a-1|_3)$.

Case 2. Suppose that $1 < r < l$. Then (3.9) gives that

$$3^{2r+1}T(T - 3^{l-r}A)(1 + 3^{r-1}T) = 3^l A.$$

Last equation is solvable iff $l = 2r + 1$ and

$$3^{r-1}T^3 + (1 - 9^r A)T^2 - 3^{r+1}AT - A = 0. \tag{3.10}$$

This contrary to l is even.

Case 3. Suppose that $r = 1 < l$. Then (3.9) gives that

$$3^3 T(T - 3^{l-1}A)(1 + T) = 3^l A$$

i.e.

$$T(T - 3^{l-1}A)(1 + T) = 3^{l-3}A. \tag{3.11}$$

The equation (3.11) follows that $l \ge 3$ and

$$H(T) := T^3 + (1 - 3^{l-1}A)T^2 - 3^{l-1}AT - 3^{l-3}A = 0. \tag{3.12}$$

Since l is even. It yields that $l > 3$. The (3.11) is solvable iff

$$|T + 1|_3 = 3^{3-l}.$$

It gives that

$$T = -1 + 3^{l-3}T_1,\ T_1 \in \mathbb{Z}_3^*.$$

Substituting the last expression into (3.12) and after some algebras we have

$$H(T_1) := 3^{2l-6}T_1^3 + 3^{l-3}(-2 - 3^{l-1}A)T_1^2 + (3^{l-1}A + 1)T_1 - A = 0. \tag{3.13}$$

Since $A \equiv 1 (\mathrm{mod}\, 3)$, the equation (3.13) gives that $|H(2)|_3 = 1$, $|H(1)|_3 < 1$, $|H'(1)|_3 = 1$. By Hensel's lemma, there exists a unique solution of (3.13), such that $T_1 \equiv 1 (\mathrm{mod}\, 3)$. It gives that the equation (3.12) has a unique solution such that

$$T = -1 + 3^{l-3} + o[3^{l-3}],$$

i.e.

$$t = -2 + 3^{l-2} + o[3^{l-2}].$$

□

Theorem 3.1. *The equation* (3.2) *has a unique solution* $u_0 = 1$.

Proof. Clear that the equation (3.2) always has a solution $u_0 = 1$.

Using notation (3.4), we have

$$u_{1,2} = \frac{t \pm \sqrt{t^2 - 4}}{2}. \tag{3.14}$$

The solution $u_{1,2}$ exist in $\mathbb{Q}_3$ if and only if $\sqrt{t^2 - 4}$ exists in $\mathbb{Q}_3$.

Assume that $|a - 1|_3 < \frac{1}{27}$. According to Lemma 3.2, $t = -2 + 3^{l-2} + o[3^{l-2}]$. It follows that

$$\sqrt{t^2 - 4} = \sqrt{3^{l-2}(2 + o[1])} \notin \mathbb{Q}_3.$$

Thus, the equation (3.2) has a unique solution $u_0 = 1$. □

Theorem 3.2. *Let* f_a *be a function defined by* (3.1). *Then* $u_0 = 1$ *is a global attractor for* f_a.

Proof. It is easy to see that

$$f_a'(u) = \frac{6u^2(a-1)(u^6 + (a+1)u^3 + 1)}{(u^6 + 2u^3 + a)^2}$$

and

$$|f_a'(1)|_3 = \left|\frac{6(a-1)}{a+3}\right|_3 < 1.$$

Thus $u_0 = 1$ is attractive fixed point.

Let us assume that $u \in \mathbb{Q}_3 \setminus \mathbb{Z}_3^*$. Then

$$f_a(u) = \begin{cases} \frac{a+o[1]}{1+o[1]}, & \text{if } u \in \mathbb{Q}_3 \setminus \mathbb{Z}_3, \\ \frac{1+o[1]}{a+o[1]}, & \text{if } u \in 3\mathbb{Z}_3, \end{cases}$$

which implies $f_a(u) \in \mathcal{E}_3$.

Moreover, for all $u \in \mathcal{E}_3$ the following holds:

$$|f_a(u) - f(1)|_p = |f_a(u) - 1|_p = \frac{1}{3} \cdot |a-1|_3 |u-1|_3.$$

This relation shows that f acts as a contraction mapping on $\mathcal{E}_3$. Since $\mathcal{E}_3$ is compact, the contraction mapping principle guarantees the existence of a unique fixed point in $\mathcal{E}_3$, and consequently, we have

$$\lim_{n\to\infty} f_a^n(u) = 1,$$

for every $u \in \mathbb{Q}_3 \setminus B(2,1)$.

Assume that $u \in B(2,1)$, i.e. $u = 2 + o[1]$. Then

$$|f_a(u) - 1|_3 = \left|\frac{(a-1)(u^3-1)(u^3+1)}{(u^3+1)^2 + a - 1}\right|_3 =$$

$$= \begin{cases} |u^3+1|_3, & \text{if } |(u^3+1)^2|_3 \le |a-1|_3 \\ \frac{|a-1|_3}{|u^3+1|_3}, & \text{if } |(u^3+1)^2|_3 > |a-1|_3 \end{cases}$$

$$= \begin{cases} |u^3+1|_3, & \text{if } |(u^3+1)^2|_3 \le |a-1|_3 \\ \le |u^3+1|_3, & \text{if } |(u^3+1)^2|_3 > |a-1|_3 \end{cases}$$

$$< 1.$$

From the last one, we get $f_a(u) \in \mathcal{E}_3$. This completes the proof. □

Corollary 3.1. *Any periodic measure* 3*-adic quasi Gibbs measure for the Ising model on* $(2,3)$*-ary tree is translation-invariant. Moreover, there is a unique translation-invariant measure.*

Proof. The proof follows directly from Theorem 3.2. □

Conclusion

As we previously mentioned, this problem was resolved for the case $p > 3$ in [19]. In that work, it was proven that if the condition $p \equiv 1 \pmod 4$ holds, there exist at least 2^d distinct d-periodic measures; whereas if the condition $p \not\equiv 1 \pmod 4$ holds, any d-periodic measure is translation-invariant, and furthermore, there exists a unique translation-invariant p-adic quasi Gibbs measure. In the present paper, a result similar to that obtained under the condition $p \not\equiv 1 \pmod 4$ in [19] is established for the case $p = 3$. The case $p = 2$ has been omitted from this paper as it requires extensive computations.

Funding

No funds, grants, or other support was received.

Availability of data and materials

Not applicable.

Competing interests

The author declares that they have no conflict of interest.

References

[1] Ising E.: Beitrag zur Theorie des Ferromagnetismus. Z. Physik (1925). https://doi.org/10.1007/BF02980577
[2] Peierls R.: On Ising's model of ferromagnetism. Math. Proc. Camb. Philos. Soc. (1936). https://doi.org/10.1017/S0305004100019174
[3] Onsager L.: Crystal statistics. I. A two-dimensional model with an order-disorder transition. Physical Review Series II. (1944). https://doi.org/10.1103/PhysRev.65.117
[4] Katsura S., Takizawa M.: Bethe lattice and the Bethe approximation. Progress of Theoretical Physics (1974). https://doi.org/10.1143/PTP.51.82
[5] Preston C.J.: Gibbs states on countable sets. Cambridge University Press, London (1974). https://doi.org/10.1017/CBO9780511897122
[6] Georgii H.-O.: Gibbs Measures and Phase Transitions. W. de Gruyter, Berlin, (1988). https://doi.org/10.1515/9783110250329
[7] Areféva I.Ya.,Dragovic B.,Frampton P.H.,Volovich I.V.: The wave function of the Universe and p-adic gravity. Int. J. Modern Phys. A. 6, 4341-4358 (1991)
[8] Avetisov V.A.,Bikulov A.H.,Kozyrev S.V.: Application of p-adic analysis to models of spontaneous breaking of the replica symmetry. J. Phys. A: Math. Gen. 32, 8785-8791 (1999)
[9] Khrennikov A.Yu., Kozyrev S.V., Zuniga-Galindo W.A.: Ultrametric Pseudodifferential Equations and Applications. Cambridge Univ. Press, (2018)
[10] Marinari E., Parisi G.: On the p-adic five-point function. Physics Letters B. 203(1-2), 52-54 (1988)
[11] Zuniga-Galindo W.A., Torba S.M.: Non-Archimedean Coulomb gases. J. Math. Phys. 61, 013504 (2020)
[12] Khrennikov A.Yu.: Generalized probabilities taking values in non-Archimedean fields and in topological groups. Russian J. Math. Phys. 14, 142-159 (2007)
[13] Khrennikov A.Yu., Ludkovsky S.: Stochastic processes on non-Archimedean spaces with values in non-Archimedean fields. Markov Process. Related Fields. 9, 131-162 (2003)
[14] Khakimov O.N.: On a Generalized p-adic Gibbs Measure for Ising model on trees. p-Adic Numbers Ultrametric Anal. Appl. 6(3), 207-217 (2014)
[15] Khamraev M.,Mukhamedov F., Rozikov U.: On uniqueness of Gibbs measure for p-adic λ-model on the Cayley tree. Lett. Math. Phys. 70, 17-28 (2004)
[16] Mukhamedov F., Khakimov O.: On Julia set and chaos in p-adic Ising model on the Cayley tree. Math. Phys. Anal. Geom. 20(23), (2017)
[17] Rahmatullaev M.M., Tukhtabaev A.M.: On periodic p-adic generalized Gibbs measures for Ising model on a Cayley tree. Lett. Math. Phys. 112 (2022)
[18] Mukhamedov F., Khakimov O.: Translation-invariant generalized p-adic Gibbs measures for the Ising model on Cayley trees, Math. Methods Appl. Sci. 44(16), 12302-12316 (2021)
[19] Mukhamedov F.M., Khakimov O.N.: Chaotic behavior of renormalization group of the p-adic Ising model on $(2,3)$-ary trees,*JMAA*, Vol. 560(1), pp. 1–18, (2026)
[20] Schikhof W.H.: Ultrametric Calculus. Cambridge Univ. Press, Cambridge (1984)
[21] Vladimirov V.S., Volovich I.V., Zelenov E.V.: p-Adic Analysis and Mathematical Physics. World Sci. Publ., Singapore (1994)
[22] Mukhamedov F., Khakimov O.: p-adic monomial equations and their perturbations. Izvestiya Math. 84:2, 348-360 (2020)
[23] Borevich Z.I., Shafarevich I.R.: Number Theory, Academic Press, New York, (1966)
[24] Koblitz N.: p-Adic Numbers, p-Adic Analysis, and Zeta-Functions. Springer, Berlin (1977)

Affiliations

Akbarkhuja Tukhtabaev
Address: Namangan state university, Dept. of Mathematics, 160107, Bobursho street, Namangan, Uzbekistan.
e-mail: akbarxoja.toxtaboyev@mail.ru
ORCID ID: https://orcid.org/0000-0001-8855-9199

Uzbekistan Journal of Mathematics and Computer Science
Volume-2, Issue-3-special, Page 179–190 (2026)
DOI: https://doi.org/10.56143/ujmcs.v2i3s.15
Research article

Краевая задача с интегральным условием для уравнения смешанного типа с операторами Бесселя

К.Т. Каримов * **Д.С. Олимова**

Аннотация

В статье исследуется нелокальная краевая задача с интегральным условием для трехмерного уравнения смешанного типа с операторами Бесселя в параллелепипеде. Для обоснования единственности решения применяется метод спектрального анализа. На основе разложения по собственным функциям получено аналитическое представление решения в виде двойного ряда Фурье по тригонометрическим и бесселевым функциям. Предложенный подход обеспечивает конструктивное построение решения рассматриваемой задачи.

Ключевые слова: *уравнения смешанного типа; сингулярный коэффициент; оператор Бесселя; функция Бесселя; функция Макдональда; интегральное условие.*

Предметная классификацие AMS (2020): *35M10.*

1. Введение. Постановка задачи

Рассмотрим уравнение

$$\left(B^x_{-1/2} + \operatorname{sgn} y \cdot B^y_{\beta-1/2} + B^z_{\gamma-1/2}\right) u(x,y,z) = 0 \tag{1.1}$$

в области $\Omega \setminus \{y=0\}$, где $\Omega = \{(x,y,z) : x \in (0,a),\ y \in (-b_0,b),\ z \in (0,c)\}$, $a, b_0, b, c \in R^+$, $\beta, \gamma \in (0,1/2)$, а $B^y_q \equiv \frac{\partial^2}{\partial y^2} + \frac{2q+1}{y}\frac{\partial}{\partial y}-$ оператор Бесселя введенным в работе [1].

В области Ω уравнение (1.1) является уравнением смешанного типа. В подобласти $\Omega_+ = \Omega \cap \{y > 0\}$ оно относится к эллиптическому типу, а в подобласти $\Omega_- = \Omega \cap \{y < 0\}-$ к гиперболическому. Коэффициенты уравнения имеют сингулярности на плоскостях $y = 0$ и $z = 0$, причем плоскость $y = 0$ является также поверхностью, на которой происходит смена типа уравнения.

Задача В. Найти функцию $u(x,y,z)$, удовлетворяющую уравнению (1.1) в области $\Omega \setminus \{y=0\}$ и условиям

$$u(x,y,z) \in C^{2,2,2}_{x,y,z}(\Omega),\ \ u, u_x,\ z^{2\gamma}u_z \in C\left(\bar{\Omega}\right), \tag{1.2}$$

$$u(x,y,0) = 0,\ \ u(x,y,c) = 0,\ \ x \in [0,a],\ \ y \in [-b_0,b], \tag{1.3}$$

$$u(x,-b_0,z) = f_1(x,z),\ \ u(x,b,z) = f_2(x,z),\ \ x \in [0,a],\ \ z \in [0,c]; \tag{1.4}$$

$$\lim_{y\to-0}(-y)^{2\beta} u_y(x,y,z) = \lim_{y\to+0} y^{2\beta} u_y(x,y,z),\ \ x \in (0,a),\ \ z \in (0,c), \tag{1.5}$$

$$u(0,y,z) = 0,\ \ y \in [-b_0,b],\ z \in [0,c], \tag{1.6}$$

Received : *28–april–2026,* ***Accepted :*** *31–july–2026*
*** Corresponding author***

а также нелокальным интегральным условиям

$$\int_0^a u(x,y,z)\,dx = 0, \quad y \in [-b_0, b], \quad z \in [0,c]. \tag{1.7}$$

Условие (1.7) обычно называется интегральным условием первого рода. Сведем его к нелокальному условию другого вида.

Если проинтегрируем уравнение (1.1) при фиксированном $y \in (-b_0, b)$ и $z \in (0,c)$ по переменной x на промежутке от ε до $a-\varepsilon$, где $\varepsilon > 0-$ достаточно малое число, то получим

$$\int_\varepsilon^{a-\varepsilon} \left(B^x_{-1/2} + \operatorname{sgn} y \cdot B^y_{\beta-1/2} + B^z_{\gamma-1/2}\right) u(x,y,z)\,dx =$$

$$= \int_\varepsilon^{a-\varepsilon} u_{xx}(x,y,z)\,dx + \int_\varepsilon^{a-\varepsilon} \left(\operatorname{sgn} y \cdot B^y_{\beta-1/2} + B^z_{\gamma-1/2}\right) u(x,y,z)\,dx = 0$$

или

$$u_x(a-\varepsilon,y,z) - u_x(\varepsilon,y,z) + \left(\operatorname{sgn} y \cdot B^y_{\beta-1/2} + B^z_{\gamma-1/2}\right) \int_\varepsilon^{a-\varepsilon} u(x,y,z)\,dx = 0.$$

В силу условий (1.7), в последнем равенстве перейдем к пределу при $\varepsilon \to 0$, в результате получим условие вида $u_x(a,y,z) - u_x(0,y,z) = 0$. Таким образом, если функция $u(x,y,z)$ удовлетворяет уравнению (1.1) и условию (1.7), то эта функция удовлетворяет условию

$$u_x(a,y,z) - u_x(0,y,z) = 0, \quad y \in [-b_0, b], \quad z \in [0,c], \tag{1.8}$$

выражающее равенство потоков через боковые стороны $x = 0$ и $x = 1$ параллелепипеда Ω.

Нелокальные краевые задачи с интегральными условиями возникают при математическом моделировании физических процессов, в которых непосредственное измерение величин на границе исследуемой области невозможно или существенно затруднено. К числу таких процессов относятся, в частности, задачи теплопроводности [2], [3] и влагопереноса в капиллярно-пористых средах [4]. В связи с этим исследование нелокальных задач с интегральными условиями представляет значительный теоретический и прикладной интерес. В последние годы данное направление активно развивается: изучаются вопросы существования и единственности решений, а также разрабатываются эффективные методы доказательства разрешимости таких задач (см., например, [5], [6], [7], [8]).

Особенностью рассматриваемой нелокальной задачи является несамосопряженность соответствующего дифференциального оператора. Вследствие этого система его собственных функций, вообще говоря, не является полной, что требует ее дополнения системой присоединенных функций. Фундаментальные результаты, посвященные исследованию базисных свойств систем собственных и присоединенных функций, получены в работах [9], [10]. Следует также отметить исследования [11], [12], [13], [14], [15], в которых рассмотрен ряд близких по постановке и методам исследования нелокальных задач, главным образом в двумерных областях.

Нелокальные краевые задачи для эллиптических уравнений с сингулярными коэффициентами в трехмерных областях исследованы в работах [16], [17], [18], [19]. Однако аналогичные нелокальные задачи для трехмерных уравнений смешанного типа с сингулярными коэффициентами до настоящего времени методом спектрального анализа практически не изучены, что определяет актуальность настоящего исследования.

В настоящей работе вместо задачи {(1.1)-(1.7)} рассматривается задача {(1.1)-(1.6), (1.8)}.

2. Построение собственных и присоединенных функций

Для получения решения задач {(1.1)-(1.6), (1.8)}, формально применим метод Фурье. Сначала найдем нетривиальные решения задачи {(1.1)-(1.3), (1.5), (1.6), (1.8)}. С этой целью, разделив переменные по формуле $u(x,y,z) = X(x)\,Q(y)\,Z(z)$, из уравнения (1.1) и краевых условий (1.3), (1.5), (1.6), (1.8) получим уравнение

$$\operatorname{sgn} y \cdot B^{y}_{\beta-1/2} Q(y) - \lambda Q(y) = 0,\ y \in (-b_0, 0) \cup (0, b) \tag{2.1}$$

и задачи на собственные значения

$$B^{z}_{\gamma-1/2} Z(z) + (\lambda - \mu) Z(z) = 0,\ Z(0) = 0,\ Z(c) = 0; \tag{2.2}$$

$$X''(x) + \mu X(x) = 0, \tag{2.3}$$

$$X(0) = 0,\ \ X'(0) = X'(a), \tag{2.4}$$

где $\mu, \lambda \in R-$ константа разделения.

Сначала рассмотрим задачу {(2.3), (2.4)}. Нетрудно убедиться, что при $\mu < 0$ задача не имеет нетривиальных решений. При $\mu = 0$ решением задачи является функция $X(x) = c_1 x$, где $c_1 = const \neq 0$. При $\mu > 0$, решение задачи {(2.3), (2.4)} будем искать в виде $X(x) = c_2 \sin\sqrt{\mu}x + c_3 \cos\sqrt{\mu}x$, где $c_2,\ c_3-$ произвольные постоянные. Определим константы $c_2,\ c_3$ из условий (2.4). Из первого условия (2.4) следует, что $c_3 = 0$. Из второго условия (2.4) получим $c_2\sqrt{\mu}\left[1 - \cos\left(\sqrt{\mu}a\right)\right] = 0$, $c_2 \neq 0$. Решая это уравнение, находим $\mu_n = (2\pi n/a)^2,\ n \in N$.

Итак, задача {(2.3), (2.4)} имеет собственные значения вида $\mu_n = (2\pi n)^2$, $n = 0, 1, 2, \ldots$. Собственные функции, соответствующие этим собственным значениям, имеют вид: $X_0(x) = c_1 x$, $X_n(x) = c_2 \sin(2\pi n x/a)$, $n \in N$. Для удобства дальнейших вычислений положим $c_1 = 1,\ c_2 = 1$:

$$X_0(x) = x,\ \ X_n(x) = \sin(2\pi n x/a),\ \ n \in N. \tag{2.5}$$

Отметим, что собственные функции (2.5) не полны и не образуют базис в пространстве $L_2(0,a)$. Дополним собственные функции $X_n(x)$ до полной системы присоединенными функциями краевой задачи {(2.3), (2.4)}.

Присоединенную функцию $\tilde{X}_n(x)$, отвечающую тому же μ_n, что и собственной функции $X_n(x)$, определим как решение следующей задачи:

$$\tilde{X}_n''(x) + \mu_n \tilde{X}_n(x) = p_n X_n(x),\ \ \tilde{X}_n(0) = 0,\ \ \tilde{X}_n'(0) = \tilde{X}_n'(a),\ \ n = 0, 1, \ldots, \tag{2.6}$$

где $p_n \neq 0-$ произвольная постоянная.

При $n = 0$ задача (2.6) не имеет решения. Если положить $p_n = -4\pi n/a$, то для $n \in N$ получим $\tilde{X}_n(x) = x\cos(2\pi n x/a)$.

Систему собственных и присоединенных функций задачи {(2.3), (2.4)} обозначим следующим образом:

$$X_0(x) = x,\ \ X_n(x) = \sin(2\pi n x/a),\ \ \tilde{X}_n(x) = x\cos(2\pi n x/a),\ \ n \in N. \tag{2.7}$$

Каждому собственному значению μ_n при $n \in N$ соответствует собственная функция $X_n(x)$ и присоединенная функция $\tilde{X}_n(x)$.

Как известно [20], задача {(2.3), (2.4)} является несамосопряженной. Сопряженной ей будет следующая задача:

$$T''(x) + \bar{\mu} T(x) = 0,\ \ x \in (0,a), \tag{2.8}$$

$$T'(a) = 0,\ \ T(0) = T(a). \tag{2.9}$$

В самом деле, $\int\limits_0^a X''(x)\,T(x)\,dx = \{X'(a)\,[T(a) - T(0)] - X(a)\,T'(a)\} + \int\limits_0^a X(x)\,T''(x)\,dx$.

Отсюда видно, что выражение в фигурной скобке будет равно нулю при $T(a) = T(0)$, $T'(a) = 0$.

Решая задачу {(2.8), (2.9)}, найдем собственные значения $\bar{\mu}_n = \mu_n = (2\pi n/a)^2$, $n = 0, 1, ...$ и собственные функции

$$T_0(x) = s_0, \;\; T_n(x) = s_n \cos(2\pi n x/a), \;\; n \in N,$$

где $s_n \neq 0-$ произвольные постоянные.

Из соображений биортонормировки, положим $s_0 = \frac{2}{a^2}$, $s_n = \frac{4}{a^2}$, $n \in N$.

Присоединенные функции сопряженной задачи {(2.8), (2.9)} определяются из задачи

$$\tilde{T}_n''(x) + \bar{\mu}_n \tilde{T}_n(x) = p_n T_n(x), \;\; \tilde{T}_n(0) = \tilde{T}_n(a), \;\; \tilde{T}_n'(a) = 0, \;\; n = 0, 1, ... \tag{2.10}$$

Они имеют вид $\tilde{T}_n(x) = \left(4/a^2\right)(a-x)\sin(2\pi n x/a)$, $n \in N$. При $n = 0$ задача (2.10) не имеет решения.

Систему собственных и присоединенных функций сопряженной задачи обозначим следующим образом:

$$T_0(x) = \frac{2}{a^2}, \;\; T_n(x) = \frac{4}{a^2}\cos\frac{2\pi n x}{a}, \;\; \tilde{T}_n(x) = \frac{4(a-x)}{a^2}\sin\frac{2\pi n x}{a}, \;\; n \in N. \tag{2.11}$$

При этом каждому $\mu_n = (2\pi n/a)^2$, $n \in N$ соответствует собственная функция $T_n(x)$ и присоединенная $\tilde{T}_n(x)$.

Лемма 2.1. *Последовательности функций (15) и (19), образуют биортогональную на интервале* $(0, a)$ *систему функций.*

Доказательство леммы 2.1 проводится непосредственно с помощью вычисления соответствующих интегралов.

В работе [13] доказано, что система функций (2.7) и (2.11) полна в пространстве $L_2(0, a)$ и образует базис в пространстве $L_2(0, a)$.

Теперь перейдем к исследованию задачи (2.2). Решение задачи (2.2) определяется формулой [17]

$$Z_m(z) = z^{1/2-\gamma} J_{1/2-\gamma}(\sigma_m z/c), \;\; m \in N. \tag{2.12}$$

Здесь $J_\nu(z) -$ функция Бесселя первого рода порядка ν , а $\sigma_m -$ m-ый положительный корень уравнения $J_{1/2-\gamma}(z) = 0$, $\lambda_{nm} = (2\pi n/a)^2 + (\sigma_m/c)^2$, $n + 1, m \in N$.

Согласно работе [21], система собственных функций (2.2) ортогональна и полна в пространстве $L_2(0, c)$ с весом $z^{2\gamma}$.

Полагая в уравнении (2.2) $\lambda = \lambda_{nm}$, найдем его общее решение:

$$Q_{nm}(y) = \begin{cases} a_{nm} y^{1/2-\beta} I_{1/2-\beta}\left(\sqrt{\lambda_{nm}} y\right) + b_{nm} y^{1/2-\beta} K_{1/2-\beta}\left(\sqrt{\lambda_{nm}} y\right), \;\; y \in (0, b], \\ c_{nm}(-y)^{1/2-\beta} J_{1/2-\beta}\left[\sqrt{\lambda_{nm}}(-y)\right] + d_{nm}(-y)^{1/2-\beta} Y_{1/2-\beta}\left[\sqrt{\lambda_{nm}}(-y)\right], \; y \in [-b_0, 0), \end{cases} \tag{2.13}$$

где a_{nm}, b_{nm}, c_{nm} и $d_{nm} -$ произвольные постоянные, $I_l(x)$ и $K_l(x) -$ функция Бесселя мнимого аргумента и функция Макдональда порядка l [21] соответственно.

Теперь в (2.13) на основании $u \in C\left(\bar{\Omega}\right)$ и условия склеивания (1.5) подберем постоянные a_{nm}, b_{nm}, c_{nm} и d_{nm} так, чтобы выполнялись условия

$$Q_{nm}(+0) = Q_{nm}(-0), \;\; \lim_{y\to-0}(-y)^{2\beta} Q_{nm}'(y) = \lim_{y\to+0} y^{2\beta} Q_{nm}'(y). \tag{2.14}$$

Из (2.13) в силу $\beta \in (0, 1/2)$ следует, что первое из равенств (2.14) выполнено, если $d_{nm} = -\pi b_{nm}/2$ при любых a_{nm} и c_{nm}, а второе равенство имеет место, при $c_{nm} = \frac{\pi b_{nm}}{2} ctg\left(\frac{\pi}{4} - \frac{\beta\pi}{2}\right) - a_{nm}$ и $d_{nm} = -\frac{\pi b_{nm}}{2}$. С учетом последних равенств, функция (2.13) перепишется в виде

$$Q_{nm}(y) = \begin{cases} a_{nm} y^{1/2-\beta} I_{1/2-\beta}\left(\sqrt{\lambda_{nm}} y\right) + b_{nm} y^{1/2-\beta} K_{1/2-\beta}\left(\sqrt{\lambda_{nm}} y\right), \;\; y \in [0, b], \\ -a_{nm}(-y)^{1/2-\beta} J_{1/2-\beta}\left[\sqrt{\lambda_{nm}}(-y)\right] + b_{nm}(-y)^{1/2-\beta} \bar{Y}_{1/2-\beta}\left[\sqrt{\lambda_{nm}}(-y)\right], \; y \in [-b_0, 0], \end{cases} \tag{2.15}$$

где $\bar{Y}_{1/2-\beta}\left[\sqrt{\lambda_{nm}}(-y)\right] = \frac{\pi}{2\cos\beta\pi}\left\{J_{1/2-\beta}\left[\sqrt{\lambda_{nm}}(-y)\right] + J_{\beta-1/2}\left[\sqrt{\lambda_{nm}}(-y)\right]\right\}$.

3. Единственность решения задачи B

Пусть $u(x,y,z)$ – решение задачи B. Рассмотрим следующие функции:

$$\omega_{0m}(y) = d_m \int_0^c \int_0^a u(x,y,z)\, T_0(x)\, z^{2\gamma} Z_m(z)\, dxdz, \quad m \in N, \tag{3.1}$$

$$\omega_{nm}(y) = d_m \int_0^c \int_0^a u(x,y,z)\, T_n(x)\, z^{2\gamma} Z_m(z)\, dxdz, \quad m,n \in N, \tag{3.2}$$

$$\tilde{\omega}_{nm}(y) = d_m \int_0^c \int_0^a u(x,y,z)\, \tilde{T}_n(x)\, z^{2\gamma} Z_m(z)\, dxdz, \quad n,m \in N, \tag{3.3}$$

где $\left\{T_0(x),\ T_n(x),\ \tilde{T}_n(x)\right\}$ и $Z_m(z)$ – функции, определенные равенствами (2.11) и (2.12) соответственно, а $d_m = 2/\left[cJ_{3/2-\gamma}(\sigma_m)\right]^2$.

Аналогично задаче A, изученными авторами [22], можно доказать, что функция $\omega_{nm}(y)$ удовлетворяет дифференциальному уравнению

$$\operatorname{sgn} y \cdot B^y_{\beta-1/2}\omega_{nm}(y) - \lambda_{nm}\omega_{nm}(y) = 0,\ y \in (-b_0,0)\cup(0,b),$$

т.е. уравнению (2.1) при $\lambda = \lambda_{nm}$.

Кроме того, в силу граничных условий (1.4), из (3.2) следует, что функция $\omega_{nm}(y)$ удовлетворяет следующим граничным условиям:

$$\omega_{nm}(-b_0) = f_{1nm},\ \ \omega_{nm}(b) = f_{2nm}, \tag{3.4}$$

где $f_{jnm} = d_m \int_0^c \int_0^a f_j(x,z)\, T_n(x)\, z^{2\gamma} Z_m(z)\, dxdz,\ j = \overline{1,2}$.

Отсюда следует, что функция $\omega_{nm}(y)$, которая имеет вид (3.2), является решением уравнения (2.1) при $\lambda = \lambda_{nm}$, удовлетворяющим условиям (3.4). Поэтому, удовлетворяя общее решение (2.15) уравнения (2.1) (при $\lambda = \lambda_{nm}$) условиям (3.4), получим систему уравнений

$$\begin{cases} a_{nm} I_{1/2-\beta}\left(\sqrt{\lambda_{nm}}b\right) + b_{nm} K_{1/2-\beta}\left(\sqrt{\lambda_{nm}}b\right) = b^{\beta-1/2} f_{2nm}, \\ -a_{nm} J_{1/2-\beta}\left[\sqrt{\lambda_{nm}}b_0\right] + b_{nm} \bar{Y}_{1/2-\beta}\left[\sqrt{\lambda_{nm}}b_0\right] = b_0^{\beta-1/2} f_{1nm}. \end{cases} \tag{3.5}$$

откуда при условии

$$\Delta_{nm}(b_0,b) = I_{1/2-\beta}\left(\sqrt{\lambda_{nm}}b\right)\bar{Y}_{1/2-\beta}\left(\sqrt{\lambda_{nm}}b_0\right) + K_{1/2-\beta}\left(\sqrt{\lambda_{nm}}b\right) J_{1/2-\beta}\left(\sqrt{\lambda_{nm}}b_0\right) \neq 0, \tag{3.6}$$

однозначно находим коэффициенты a_{nm} и b_{nm} :

$$a_{nm} = \frac{b^{\beta-1/2}\bar{Y}_{1/2-\beta}\left(\sqrt{\lambda_{nm}}b_0\right) f_{2nm} - b_0^{\beta-1/2} K_{1/2-\beta}\left(\sqrt{\lambda_{nm}}b\right) f_{1nm}}{\Delta_{nm}(b_0,b)},$$

$$b_{nm} = \frac{b_0^{\beta-1/2} I_{1/2-\beta}\left(\sqrt{\lambda_{nm}}b\right) f_{1nm} + b^{\beta-1/2} J_{1/2-\beta}\left(\sqrt{\lambda_{nm}}b_0\right) f_{2nm}}{\Delta_{nm}(b_0,b)}.$$

Подставляя эти коэффициенты в (2.15), получим

$$\omega_{nm}(y) = \begin{cases} \Big[(y/b)^{1/2-\beta}\,\Delta_{nm}(b_0,y)\, f_{2nm} + \\ +(y/b_0)^{1/2-\beta}\, A_{nm}(y,b)\, f_{1nm}\Big] \Big/ \Delta_{nm}(b_0,b),\ y \in [0,b]; \\ \Big[(-y/b_0)^{1/2-\beta}\,\Delta_{nm}(-y,b)\, f_{1nm} + \\ +(-y/b)^{1/2-\beta}\, B_{nm}(b_0,-y)\, f_{2nm}\Big] \Big/ \Delta_{nm}(b_0,b),\ y \in [-b_0,0], \end{cases} \tag{3.7}$$

где

$$A_{nm}(y,b)=I_{1/2-\beta}\left(\sqrt{\lambda_{nm}}b\right)K_{1/2-\beta}\left(\sqrt{\lambda_{nm}}y\right)-K_{1/2-\beta}\left(\sqrt{\lambda_{nm}}b\right)I_{1/2-\beta}\left(\sqrt{\lambda_{nm}}y\right),$$

$$B_{nm}(b_0,-y)=J_{1/2-\beta}\left(\sqrt{\lambda_{nm}}b_0\right)\bar{Y}_{1/2-\beta}\left(-\sqrt{\lambda_{nm}}y\right)-\bar{Y}_{1/2-\beta}\left(\sqrt{\lambda_{nm}}b_0\right)J_{1/2-\beta}\left(-\sqrt{\lambda_{nm}}y\right).$$

Аналогично $\omega_{nm}(y)$ получим, что функция $\omega_{0m}(y)$, определенная формулой (24), удовлетворяет условиям

$$\operatorname{sgn}y\cdot B^{y}_{\beta-1/2}\omega_{0m}(y)-\lambda_{0m}\omega_{0m}(y)=0,\ y\in(-b_0,0)\cup(0,b),\tag{3.8}$$

$$\omega_{0m}(-0)=\omega_{0m}(+0),\ \lim_{y\to-0}(-y)^{2\beta}\omega'_{0m}(y)=\lim_{y\to0}y^{2\beta}\omega'_{0m}(y),\tag{3.9}$$

$$\omega_{0m}(-b_0)=f_{10m},\quad \omega_{0m}(b)=f_{20m},\tag{3.10}$$

где $f_{j0m}=\frac{2d_m}{a^2}\int\limits_0^c\int\limits_0^a f_j(x,z)\,z^{2\gamma}Z_m(z)\,dxdz,\ \ j=\overline{1,2}.$

Единственное решение задачи (3.8)-(3.10) определяется формулой

$$\omega_{0m}(y)=\begin{cases}\Big[(y/b)^{1/2-\beta}\Delta_{0m}(b_0,y)\,f_{20m}+\\ +(y/b_0)^{1/2-\beta}A_{0m}(y,b)\,f_{10m}\Big]\Big/\Delta_{0m}(b_0,b),\ y\in[0,b];\\ \Big[(-y/b_0)^{1/2-\beta}\Delta_{0m}(-y,b)\,f_{10m}+\\ +(-y/b)^{1/2-\beta}B_{0m}(b_0,-y)\,f_{20m}\Big]\Big/\Delta_{0m}(b_0,b),\ y\in[-b_0,0],\end{cases}\tag{3.11}$$

где

$$A_{0m}(y,b)=I_{1/2-\beta}\left(\sqrt{\lambda_{0m}}b\right)K_{1/2-\beta}\left(\sqrt{\lambda_{0m}}y\right)-K_{1/2-\beta}\left(\sqrt{\lambda_{0m}}b\right)I_{1/2-\beta}\left(\sqrt{\lambda_{0m}}y\right),$$

$$B_{0m}(b_0,-y)=J_{1/2-\beta}\left(\sqrt{\lambda_{0m}}b_0\right)\bar{Y}_{1/2-\beta}\left(-\sqrt{\lambda_{0m}}y\right)-\bar{Y}_{1/2-\beta}\left(\sqrt{\lambda_{0m}}b_0\right)J_{1/2-\beta}\left(-\sqrt{\lambda_{0m}}y\right).$$

$$\Delta_{0m}(b_0,b)=I_{1/2-\beta}\left(\sqrt{\lambda_{0m}}b\right)\bar{Y}_{1/2-\beta}\left(\sqrt{\lambda_{0m}}b_0\right)+K_{1/2-\beta}\left(\sqrt{\lambda_{0m}}b\right)J_{1/2-\beta}\left(\sqrt{\lambda_{0m}}b_0\right)\neq0.\tag{3.12}$$

Повторяя те же действия над функцией $\omega_{nm}(y)$, заданной формулой (3.4), что и для функции $\tilde{\omega}_{nm}(y)$, получим неоднородное дифференциальное уравнение

$$[\tilde{\omega}_{nm}(y)]''+\frac{2\beta}{y}[\tilde{\omega}_{nm}(y)]'-\lambda_{nm}\tilde{\omega}_{nm}(y)=\frac{4\pi n}{a}\omega_{nm}(y),\ y\in(-b_0,0)\cup(0,b),\tag{3.13}$$

где $y\in(-b_0,0)\cup(0,b)$, с соответствующими граничными условиями

$$\tilde{\omega}_{nm}(-b_0)=\tilde{f}_{1nm},\ \ \tilde{\omega}_{nm}(b)=\tilde{f}_{2nm},\tag{3.14}$$

и условиями сопряжения

$$\tilde{\omega}_{nm}(0+0)=\tilde{\omega}_{nm}(0-0),\quad \lim_{y\to-0}(-y)^{2\beta}\tilde{\omega}'_{nm}(y)=\lim_{y\to0}y^{2\beta}\tilde{\omega}'_{nm}(y),\tag{3.15}$$

где $\tilde{f}_{jnm}=d_m\int\limits_0^c\int\limits_0^a f_j(x,z)\,\tilde{T}_n(x)\,z^{2\gamma}Z_m(z)\,dxdz,\ \ j=\overline{1,2}$, а функция $\omega_{nm}(y)$ определяется равенством (3.7).

На основании метода вариации произвольных постоянных найдем решение задачи (3.13)-(3.15), которое определяется по формуле

$$\tilde{\omega}_{nm}(y)=\begin{cases}\Big[(y/b)^{1/2-\beta}\Delta_{nm}(b_0,y)\,\tilde{f}_{2nm}+\\ +(y/b_0)^{1/2-\beta}A_{nm}(y,b)\,\tilde{f}_{1nm}\Big]\Big/\Delta_{nm}(b_0,b)+\tilde{\omega}^{+}_{nm}(y),\ y\in[0,b];\\ \Big[(-y/b_0)^{1/2-\beta}\Delta_{nm}(-y,b)\,\tilde{f}_{1nm}+\\ +(-y/b)^{1/2-\beta}B_{nm}(b_0,-y)\,\tilde{f}_{2nm}\Big]\Big/\Delta_{nm}(b_0,b)+\tilde{\omega}^{-}_{nm}(y),\ y\in[-b_0,0],\end{cases}\tag{3.16}$$

где функции $\tilde{\omega}_{nm}^{+}(y)$ и $\tilde{\omega}_{nm}^{-}(y)$ определены соответственно равенствами

$$\tilde{\omega}_{nm}^{+}(y)=\frac{4\pi n}{a\Delta_{nm}(b_0,b)}\left\{y^{1/2-\beta}A_{nm}(y,b)\left[\int\limits_{-b_0}^{0}(-t)^{1/2+\beta}B_{nm}(b_0,-t)\omega_{nm}(t)\,dt+\right.\right.$$

$$\left.\left.+\int\limits_{0}^{y}t^{1/2+\beta}A_{nm}(t,b)\,\omega_{nm}(t)\,dt\right]+y^{1/2-\beta}\Delta_{nm}(b_0,y)\int\limits_{y}^{b}t^{1/2+\beta}A_{nm}(t,b)\,\omega_{nm}(t)\,dt\right\},\tag{3.17}$$

$$\tilde{\omega}_{nm}^{-}(y)=\frac{4\pi n}{a\Delta_{nm}(b_0,b)}\left\{(-y)^{1/2-\beta}\Delta_{nm}(-y,b)\int\limits_{-b_0}^{y}(-t)^{1/2+\beta}B_{nm}(b_0,-t)\omega_{nm}(t)\,dt+\right.$$

$$+(-y)^{1/2-\beta}B_{nm}(b_0,-y)\left[\int\limits_{y}^{0}(-t)^{1/2+\beta}B_{nm}(b_0,-t)\,\omega_{nm}(t)\,dt+\right.$$

$$\left.\left.+\int\limits_{0}^{b}t^{1/2+\beta}A_{nm}(t,b)\,\omega_{nm}(t)\,dt\right]\right\}.\tag{3.18}$$

При условии (3.6) из формул (3.7), (3.11), (3.7) следует единственность решения задачи B, так как если $f_1(x,z)\equiv 0$, $f_2(x,z)\equiv 0$ на $x\in[0,a]$, $z\in[0,c]$, то $\omega_{nm}(y)\equiv 0$, $\omega_{0m}(y)\equiv 0$, $\tilde{\omega}_{nm}(y)\equiv 0$ для $n\in N$ на $[-b_0,b]$. Тогда из (3.1)-(3.3) имеем

$$\int\limits_0^c\int\limits_0^a u(x,y,z)\,T_0(x)\,z^{2\gamma}Z_m(z)\,dxdz=0,\quad \int\limits_0^c\int\limits_0^a u(x,y,z)\,T_n(x)\,z^{2\gamma}Z_m(z)\,dxdz=0,$$

$$\int\limits_0^c\int\limits_0^a u(x,y,z)\,\tilde{T}_n(x)\,z^{2\gamma}Z_m(z)\,dxdz=0,\quad n\in N.$$

Отсюда в силу полноты системы функций (2.11) в пространстве $L_2(0,a)$ и $u(x,y,z)\in C\left(\bar{\Omega}\right)$ следует, что $\int\limits_0^c u(x,y,z)\,z^{2\gamma}Z_m(z)\,dz=0$. В силу полноты системы функций (2.12) с весом $z^{2\gamma}$ в пространстве $L_2(0,c)$ и $u(x,y,z)\in C\left(\bar{\Omega}\right)$, следует, что $u(x,y,z)\equiv 0$ в $\bar{\Omega}$.

Если при некоторых b_0, b и $n=l$, $m=k\in N$ нарушено условие (3.6), т.е. $\Delta_{lk}(b_0,b)=0$, то однородная задача B (где $f_1(x,z)=f_2(x,z)=0$) имеет нетривиальное решение

$$u_{lk}(x,y,z)=\begin{cases}y^{1/2-\beta}\left[\Delta_{lk}(b_0,y)+A_{lk}(y,b)\right]X_l(x)\,Z_k(z), & y>0;\\ (-y)^{1/2-\beta}\left[\Delta_{lk}(-y,b)+B_{lk}(b_0,-y)\right]X_l(x)\,Z_k(z), & y<0,\end{cases}\tag{3.19}$$

где $Z_k(z)$ находятся по формуле (2.12), $X_l(x)=A_1x+A_2\sin(2\pi nx/a)+A_3x\cos(2\pi nx/a)$, $A_j=\overline{1,3}-$ произвольные постоянные.

Нетрудно убедиться, что построенная функция (3.19) относится к классу $u(x,y,z)\in C^{2,2,2}_{x,y,z}(\Omega)$ и является решением уравнения (1.1) во всей области Ω.

Естественно возникает вопрос о существовании значений параметра b_0, при которых выражение $\Delta_{nm}(b_0,b)$ обращается в нуль при фиксированных n и m. Представим выражение $\Delta_{nm}(b_0,b)$ в следующем виде:

$$\Delta_{nm}(b_0,b)=I_{1/2-\beta}\left(\sqrt{\lambda_{nm}}b\right)\delta_{nm}(b_0,b),$$

где

$$\delta_{nm}(b_0,b)=\bar{Y}_{1/2-\beta}\left(\sqrt{\lambda_{nm}}b_0\right)+\frac{K_{1/2-\beta}\left(\sqrt{\lambda_{nm}}b\right)}{I_{1/2-\beta}\left(\sqrt{\lambda_{nm}}b\right)}J_{1/2-\beta}\left(\sqrt{\lambda_{nm}}b_0\right).$$

Существование нулей $\delta_{nm}(b_0, b)$ относительно b_0 следует из того, что $J_{1/2-\beta}\left(\sqrt{\lambda_{nm}}y\right)$ и $\bar{Y}_{1/2-\beta}\left(\sqrt{\lambda_{nm}}y\right)$, $y = b_0$, являются линейно независимыми решениями уравнения Бесселя

$$y^2Q''(y) + yQ'(y) + \left[\lambda_{nm}y^2 - (1/2-\beta)^2\right]Q(y) = 0. \tag{3.20}$$

Поскольку функции $J_{1/2-\beta}\left(\sqrt{\lambda_{nm}}y\right)$ и $\delta_{nm}(y, b)$ являются линейно независимыми решениями уравнения (3.20), то, согласно общей теории линейных дифференциальных уравнений [23], их нули строго чередуются. Иными словами, на каждом интервале между двумя последовательными нулями одного из этих решений содержится ровно один нуль другого. Поскольку функция $J_{1/2-\beta}\left(\sqrt{\lambda_{nm}}y\right)$ имеет счётное множество положительных нулей, функция $\delta_{nm}(y, b)$ также обладает счётным множеством положительных нулей относительно переменной y при фиксированном b_0. Тем самым доказано следующее утверждение.

Теорема 3.1. *Если решение задачи B существует, то оно является единственным тогда и только тогда, когда* $\Delta_{nm}(b_0, b) \neq 0$ *при всех* $n, m \in N$.

4. Существование решения задачи B

Рассмотрим теперь вопрос о существовании решения задачи B. Поскольку b_0 и $b-$ любые положительные числа, при достаточно больших n и m выражение $\Delta_{nm}(b_0, b)$, которое входит в знаменатели формул (3.7), (3.11), (3.16), может стать достаточно малым из-за существования счетного множества нулей относительно b_0. Следовательно, возникает проблема малых знаменателей как и при изучении задачи A для уравнения (1) [22].

Как известно [24], асимптотические поведение функции $I_v(x)$ и $K_v(x)$ при больших x, имеет вид

$$I_\nu(x) \approx \frac{e^x}{(2\pi x)^{1/2}}, \quad K_\nu(x) \approx \left(\frac{\pi}{2x}\right)^{1/2} e^{-x}, \tag{4.1}$$

поэтому величина $\frac{K_{1/2-\beta}\left(\sqrt{\lambda_{nm}}b\right)}{I_{1/2-\beta}\left(\sqrt{\lambda_{nm}}b\right)} J_{1/2-\beta}\left(\sqrt{\lambda_{nm}}b_0\right)$ есть бесконечно малая более высокого порядка, чем $\bar{Y}_{1/2-\beta}\left(\sqrt{\lambda_{nm}}b_0\right)$ при больших n и m.

Поэтому достаточно рассмотреть выражение

$$\tilde{\delta}_{nm}(b_0) = \bar{Y}_{1/2-\beta}\left(\sqrt{\lambda_{nm}}b_0\right) = \frac{\pi}{2\cos\beta\pi}\left[J_{1/2-\beta}\left(\sqrt{\lambda_{nm}}b_0\right) + J_{\beta-1/2}\left(\sqrt{\lambda_{nm}}b_0\right)\right],$$

которое также имеет счетное множество положительных нулей относительно b_0.

Лемма 4.1. *Пусть* b_0- *любое натуральное число или* $b_0 = p/q-$ *любое дробное число, где* $(p, q) = 1$, $(4, q) = 1$, $p, q \in N$, *причем*

$$n^2 + (m - \gamma/2)^2/c^2 \neq (d - 1/4)^2 q^2/p^2, \ d \in Z, \ \forall n, m \in N,$$

то существуют положительное число C_0 *и* $\forall n_0, m_0 \in N$, *такие, что при всех* $n > n_0$ *и* $m > m_0$ *справедлива оценка*

$$\left|\tilde{\delta}_{nm}(b)\right| \geq C_0. \tag{4.2}$$

Доказательство. Согласно работе [21], при достаточно больших m, для $m-$ ного положительного корня уравнения $J_{1/2-\gamma}(x) = 0$, имеет место соотношение

$$\sigma_m \approx \pi(m - \gamma/2). \tag{4.3}$$

При достаточно больших n, справедлива следующая приблизительная равенства

$$\mu_n \approx (\pi n)^2. \tag{4.4}$$

Тогда в силу асимптотической формулы функции Бесселя при достаточно больших ξ [24]

$$J_\nu(\xi) \approx \left(\frac{2}{\pi\xi}\right)^{1/2} \cos\left(\xi - \frac{\nu\pi}{2} - \frac{\pi}{4}\right), \tag{4.5}$$

и с учетом (4.2) и (4.3) справедлива приблизительная равенства

$$\lambda_{nm}^{1/4}\Delta_{nm}(b_0) \approx A\sin\left(\sqrt{\lambda_{nm}}b_0 + \pi/4\right), \tag{4.6}$$

где $A = \sqrt{\pi}(2b_0)^{-1/2}\cos^{-1}(\beta\pi/2 + \pi/4)$.

Предположим, что $\pi n,\ (m-\gamma/2)\pi/c$ и $\pi h-$ числа Пифагора, т.е. $(\pi n)^2 + \left[\frac{\pi}{c}\left(m - \frac{\gamma}{2}\right)\right]^2 = \pi^2 h^2,\ h \in R$. Для примера здесь покажем оценку (4.6), для случая, когда $h-$ натуральное число. Тогда (4.6) перепишется в виде $\sqrt{\pi h}\Delta_{nm}(b_0) \approx A\sin(\pi h b_0 + \pi/4)$.

Если, например, $b_0 = l \in N$, то получим

$$\left|\sqrt{\pi h}\Delta_{nm}(l)\right| = A\sin\frac{\pi}{4} = \frac{A}{\sqrt{2}} \geq \tilde{C}_0 > 0,\ \ \tilde{C}_0 = const. \tag{4.7}$$

Пусть $b_0 = p/q-$ рациональное число, где $p, q \in N$, $(p,q) = (4,q) = 1$. Тогда

$$\left|\sqrt{\pi h}\Delta_{nm}\left(\frac{p}{q}\right)\right| = A\left|\sin\pi\left(\frac{hp}{q} + \frac{1}{4}\right)\right|.$$

Разделим hp на q с остатком: $hp = sq + r,\ s = 1,2,...,r = 0,1,...,0 \leq r \leq q\breve{}1$. Если учесть это, то выражение $\sqrt{\pi h}\Delta_{nm}(p/q)$ примет вид

$$\sqrt{\pi h}\Delta_{nm}\left(\frac{p}{q}\right) = A\left|\sin\pi\left(\frac{r}{q} + \frac{1}{4}\right)\right| \geq \tilde{C}_0 > 0. \tag{4.8}$$

Потребуем, чтобы число $\tilde{C}_0$ было больше нуля, а это возможно только тогда, когда $hp/q + 1/4 \neq d,\ d \in Z$ или $n^2 + (m-\gamma/2)^2/c^2 \neq (d-1/4)^2 q^2/p^2,\ d \in Z$.

Из (4.7), (4.8) и $\lambda_{nm}^{-1/4} < 1$ при всех $n > n_0$ и $m > m_0$ следует справедливость оценки (4.2). Лемма 4.1 доказана.

Если $\delta_{nm}(b_0, b) \neq 0$ при $n = 1,2,...,n_0$ и выполнена оценка (4.2), то решение задачи В можно представить в виде суммы двойного ряда

$$u(x,y,z) = \sum_{m=1}^{\infty}\omega_{0m}(y)X_0(x)Z_m(z) + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\omega_{nm}(y)\tilde{X}_n(x)Z_m(z) + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\tilde{\omega}_{nm}(y)X_n(x)Z_m(z), \tag{4.9}$$

где функции $\omega_{0m}(y)$, $\omega_{nm}(y)$, $\tilde{\omega}_{nm}(y)$, $X_0(x)$, $X_n(x)$, $\tilde{X}_n(x)$ и $Z_m(z)$ находятся соответственно по формулам (3.11), (3.7), (3.16), (2.7), (2.7), (2.7) и (2.12).

Теперь покажем, что при определенных условиях относительно функций $f_1(x,z)$ и $f_2(x,z)$ ряд (4.9) и его производные $u_x,\ z^{2\gamma}u_z$ в замкнутой области $\bar{\Omega}$, а производные второго порядка по x и $B^y_{\beta-1/2}u,\ B^z_{\gamma-1/2}u$ соответственно в любом компакте $K \subset \Omega_+ \cup \Omega_-$ сходятся равномерно.

Лемма 4.2. *Для функций* (2.7) *и* (2.11) *справедливы следующие оценки*

$$|R_n(x)| \leq b_1, |R_n'(x)| \leq b_2 n, |R_n''(x)| \leq b_3 n^2, \tag{4.10}$$

где $b_j,\ j = \overline{1,3}-$ *положительные постоянные, а в качестве функции* $R_n(x)$ *можно взять одну из функций, приведенных в формулах* (2.7) *и* (2.11).

Справедливость оценки (4.10) легко следует из свойства тригонометрических функций.

В работе [22] доказаны следующие оценки:

$$|Z_m(z)| \leq b_4\frac{z^{1-2\gamma}}{\sigma_m^{\gamma-1/2}},\ \ \left|z^{2\gamma}Z_m'(z)\right| \leq b_5\sigma_m^{1/2},\ \ \left|B^z_{\gamma-1/2}Z_m(z)\right| \leq b_4\frac{z^{1-2\gamma}}{\sigma_m^{\gamma-5/2}}, \tag{4.11}$$

$$J^2_{3/2-\gamma}(\sigma_m) \geq b_6 (\sigma_m)^{-1}, \quad |d_m| \leq b_7 \sigma_m, \tag{4.12}$$

где $b_j,\ j = \overline{4,7}-$ положительные постоянные.

Аналогично леммам 4.7 работы [22], можно доказать, следующую лемму.

Лемма 4.3. *Если $\beta, \gamma \in (0, 1/2)$ и функция $f_1(x,z)$ и $f_2(x,z)$ удовлетворяют условиям*
I. $f_l(x,z) \in C^{4,5}_{x,z}\left(\bar{\Pi}\right), l = \overline{1,2}$, *где* $\Pi = \{(x,z) :\ x \in (0,a),\ z \in (0,c)\}$;
II. $\left(\partial^j/\partial x^j\right) f_l(x,z)\big|_{x=0} = 0, \quad \left(\partial^j/\partial x^j\right) f_l(x,z)\big|_{x=a} = 0,\ l = \overline{1,2},\ j = \overline{0,3}$;
III. $\left(\partial^j/\partial z^j\right) f_l(x,z)\big|_{z=0} = 0, \quad \left(\partial^j/\partial z^j\right) f_l(x,z)\big|_{z=c} = 0,\ l = \overline{1,2},\ j = \overline{0,4}$.
Тогда для коэффициентов $f_{lnm},\ l = \overline{1,2}$, определенных равенствами (21), справедлива следующая оценка:

$$|f_{lnm}| \leq b_8 n^{-4} (\sigma_m)^{-4,5},\ l = \overline{1,2}, \tag{4.13}$$

где b_8- некоторая положительная постоянная.

Теперь найдем оценки для функции $\omega_{nm}(y)$, определяемые равенствами (3.7). Для удобства перепишем её в виде

$$\omega_{nm}(y) = \begin{cases} P^1_{nm}(y) f_{2nm} + P^2_{nm}(y) f_{1nm}, & y \in [0,b]; \\ \\ P^3_{nm}(y) f_{1nm} + P^4_{nm}(y) f_{2nm}, & y \in [-b_0, 0], \end{cases} \tag{4.14}$$

где $f_{jnm} = d_m \int\limits_0^c \int\limits_0^a f_j(x,z) T_n(x) z^{2\gamma} Z_m(z)\, dxdz,\ j = \overline{1,2}$,

$$P^1_{nm}(y) = \frac{y^{1/2-\beta} \Delta_{nm}(b_0, y)}{b^{1/2-\beta} \Delta_{nm}(b_0, b)}, \quad P^2_{nm}(y) = \frac{y^{1/2-\beta} A_{nm}(y, b)}{b_0^{1/2-\beta} \Delta_{nm}(b_0, b)}, \quad y \in [0,b],$$

$$P^3_{nm}(y) = \frac{(-y)^{1/2-\beta} \Delta_{nm}(-y, b)}{b_0^{1/2-\beta} \Delta_{nm}(b_0, b)},\ P^4_{nm}(y) = \frac{(-y)^{1/2-\beta} B_{nm}(b_0, -y)}{b^{1/2-\beta} \Delta_{nm}(b_0, b)},\ y \in [-b_0, 0].$$

Лемма 4.4. *Если выполнена оценка (45) при $n > n_0$ и $m > m_0$, тогда для таких n и m справедливы следующие оценки:*

$$\left|P^j_{nm}(y)\right| \leq b_9, y \in [0,b], \quad \left|B^y_{\beta-1/2} P^j_{nm}(y)\right| \leq b_9 \lambda_{nm},\ y \in (0,b),\ j = \overline{1,2}, \tag{4.15}$$

$$\left|P^j_{nm}(y)\right| \leq b_{10},\ y \in [-b_0, 0], \quad \left|B^y_{\beta-1/2} P^j_{nm}(y)\right| \leq b_{10} \lambda_{nm},\ y \in (-b_0, 0),\ j = \overline{3,4}, \tag{4.16}$$

где $b_9, b_{10}-$ произвольные положительные постоянные.

Доказательство. Сначала рассмотрим функцию $P^1_{nm}(y)$ и ее перепишем в другом виде

$$P^1_{nm}(y) = \frac{y^{1-2\beta} b_0^{\beta-1/2} \bar{I}_{1/2-\beta}\left(\sqrt{\lambda_{nm}} y\right) \bar{J}_{\beta-1/2}\left(\sqrt{\lambda_{nm}} b_0\right)}{(1-2\beta) b^{1/2-\beta} \Delta_{nm}(b_0, b)} +$$

$$+ \frac{b_0^{1/2-\beta} \bar{I}_{\beta-1/2}\left(\sqrt{\lambda_{nm}} y\right) \bar{J}_{1/2-\beta}\left(\sqrt{\lambda_{nm}} b_0\right)}{(1-2\beta) b^{1/2-\beta} \Delta_{nm}(b_0, b)},$$

где $\bar{J}_\nu(z)$, $\bar{I}_\nu(z)-$ функции Бесселя-Клиффорда [25] и для них справедливы неравенства $\left|\bar{J}_\nu(z)\right| \leq 1$ и $\left|\bar{I}_\nu(z)\right| \leq e^z,\ z > 0$ при $\nu > -1/2$. В силу этих неравенств и оценок (4.2), вытекает первая оценка из (4.15) при $j = 1$.

Непосредственные вычисления показывают, что функция $P^1_{nm}(y)$ при $y \in (0,b)$ удовлетворяет уравнению (2.1). Тогда, справедливо равенство $B^y_{\beta-1/2} P^1_{nm}(y) = \lambda_{nm} P^1_{nm}(y)$. Отсюда, в силу первой оценки (4.15) при $j = 1$, следует справедливость второй оценки из (4.15) при $j = 1$.

Аналогично, доказываются оценки для функции $P^j_{nm}(y),\ j = \overline{2,4}$. Лемма 4.4 доказана.

Аналогичным методом доказывается оценки для функции $\omega_{0m}(y)$ и $\tilde{\omega}_{nm}(y)$.

В силу оценок (4.15), (4.16) и (4.13), функции (4.14) и $B^y_{\beta-1/2}\omega_{nm}(y)$ оцениваются следующим образом:

$$|\omega_{nm}(y)| \le b_{11} n^{-4} \sigma_m^{-4,5}, \quad y \in [-b_0, b], \tag{4.17}$$

$$\left|B^y_{\beta-1/2}\omega_{nm}(y)\right| \le b_{12}\lambda_{nm} n^{-4}\sigma_m^{-4,5}, \quad y \in (-b_0, 0)\cup(0, b). \tag{4.18}$$

Аналогичным образом функции (3.11), $B^y_{\beta-1/2}\omega_{0m}(y)$, (39) и $B^y_{\beta-1/2}\tilde{\omega}_{nm}(y)$ оцениваются следующим образом:

$$|\omega_{0m}(y)| \le b_{13}\sigma_m^{-4,5}, \quad y \in [-b_0, b], \tag{4.19}$$

$$\left|B^y_{\beta-1/2}\omega_{0m}(y)\right| \le b_{14}\lambda_{0m}\sigma_m^{-4,5}, \quad y \in (-b_0, 0)\cup(0, b). \tag{4.20}$$

$$|\tilde{\omega}_{nm}(y)| \le b_{15} n^{-4}\sigma_m^{-4,5}, \quad y \in [-b_0, b], \tag{4.21}$$

$$\left|B^y_{\beta-1/2}\tilde{\omega}_{nm}(y)\right| \le b_{16}\lambda_{nm} n^{-4}\sigma_m^{-4,5}, \quad y \in (-b_0, 0)\cup(0, b), \tag{4.22}$$

где $b_j,\ j = \overline{11, 16}$– положительные постоянные.

Согласно оценкам (4.3), (4.4), (4.10)-(4.12) и (4.19)-(4.22), ряды (4.9), $u_x, z^{2\gamma}u_z$ в $\bar{\Omega}$, оцениваются соответственно числовыми рядами

$$b_{17}\sum_{n=1}^{\infty} n^{-4}\sum_{m=1}^{\infty} m^{-4-\gamma}, \quad b_{18}\sum_{n=1}^{\infty} n^{-4}\sum_{m=1}^{\infty} m^{-4-\gamma}, \quad b_{19}\sum_{n=1}^{\infty} n^{-4}\sum_{m=1}^{\infty} m^{-4}, \tag{4.23}$$

а ряды u_{xx}, $B^y_{\beta-1/2}u$, $B^z_{\gamma-1/2}u$ в любом компакте $K \subset \Omega_+ \cup \Omega_-$ оцениваются соответственно числовыми рядами

$$b_{20}\sum_{n=1}^{\infty} n^{-2}\sum_{m=1}^{\infty} m^{-4-\gamma}, \quad b_{21}\sum_{n=1}^{\infty} n^{-2}\sum_{m=1}^{\infty} m^{-2-\gamma}, \quad b_{22}\sum_{n=1}^{\infty} n^{-4}\sum_{m=1}^{\infty} m^{-2-\gamma}, \tag{4.24}$$

где $b_j,\ j = \overline{17, 22}$– положительные постоянные.

Числовые ряды в (4.23)) и (4.24) сходятся, то абсолютно и равномерно сходится ряд (4.9) и ряды u_x, $z^{2\gamma}u_z$ в $\bar{\Omega}$, а ряды u_{xx}, $B^y_{\beta-1/2}u$ и $B^z_{\gamma-1/2}u$ сходятся на каждом компакте $K \subset \Omega_+ \cup \Omega_-$. Поэтому функция $u(x, y, z)$, определенная рядом (4.9), удовлетворяет всем условиям задачи В.

Список литературы

[1] Киприянов И.А.: Сингулярные эллиптические краевые задачи. Москва: Наука, Физматлит. (1997).

[2] Cannon J.R.: *The solution of the heat equation subject to the specification of energy.* Quart. Appl. Math. **21**, 155-160 (1963).

[3] Ионкин Н.И., Моисеев Е.И.: *О задаче для уравнения теплопроводности с двуточечными краевыми условиями.* Дифференциальные уравнения. **15** (7), 1284-1295 (1979).

[4] Нахушев А.М.: *Об одном приближенном методе решения краевых задач для дифференциальных уравнений и его приложения к динамике почвенной влаги и грунтовых вод.* Дифференциальные уравнения. **18** (1), 72-81 (1982).

[5] Дмитриев В.Б.: *Нелокальная задача с интегральными условиями для волнового уравнения.* Вестник СамГУ. (2), 15-17 (2006).

[6] Кожанов А.И., Пулькина Л.С.: *О разрешимости некоторых граничных задач со смещением для линейных гиперболических уравнений.* Матем. журнал ин-та матем., Алматы. **9** (32), 78-92 (2009).

[7] Пулькина Л.С.: *Нелокальная задача для гиперболического уравнения с интегральными условиями I рода с ядрами, зависящими от времени.* Известия вузов. Математика. (10), 32-44 (2012).

[8] Сабитова Ю.К.: *Краевая задача с нелокальным интегральным условием для уравнений смешанного типа с вырождением на переходной линии.* Математические заметки. **98** (3), 393-406 (2025). http://doi.org/10.4213/mzm9135

[9] Ильин В.А.: *О безусловной базисности на замкнутом интервале систем собственных и присоединенных функций дифференциального оператора второго порядка.* Доклады АН СССР. **273** (5), 1048-1053 (1983).

[10] Ильин В.А.: *Об абсолютной и равномерной сходимости разложений по собственным и присоединенным функциям несамосопряженного эллиптического оператора.* Доклады АН СССР. **274** (1), 19-22 (1984).

[11] Алимов Ш.А.: *Об одной спектральной задаче типа Бицадзе-Самарского.* Доклады АН СССР. **287** (6), 1289-1290 (1986).

[12] Ашуров Р.Р.: *О биортогональных разложениях одного несамосопряженного оператора Шредингера.* Дифференциальные уравнения. **27** (1), 156-158 (1991).

[13] Ионкин Н.И.: *Решение одной краевой задачи теории теплопроводности с неклассическим краевым условием.* Дифференциальные уравнения. **13** (2), 294-304 (1977).
[14] Моисеев Е.И.: *О разрешимости одной нелокальной краевой задачи.* Дифференциальные уравнения. **37** (11), 1565-1567 (2001). https://doi.org/10.1023/A:1017937403853
[15] Моисеев Е.И.: *О решении спектральным методом одной нелокальной краевой задачи.* Дифференциальные уравнения. **35** (8), 1094-1100 (1999).
[16] Каримов К.Т.: *Нелокальная задача с интегральным условием для трехмерного эллиптического уравнения с сингулярными коэффициентами.* Бюллетень Института математики. (6), 10-24 (2018).
[17] Karimov K.T.: *Nonlocal Problem for an Elliptic Equation with Singular Coefficients in a Semi-infinite Parallelepiped.* Lobachevskii Journal of Mathematics. **41** (1), 46-57 (2020). https://doi.org/10.1134/S1995080220010084
[18] Karimov K.T.: *Nonlocal Problem for a Three-dimensional Elliptic Equation with Singular Coefficients in a Rectangular Parallelepiped.* Journal of Siberian Federal University. Mathematics and Physics. **13** (5), 533-546 (2020). https://doi.org/10.17516/1997-1397-2020-13-5-533-546
[19] Urinov A.K., Karimov K.T.: *Nonlocal boundary value problems for a three-dimensional elliptic equation with singular coefficients in a semi-infinite parallelepiped.* Siberian Electronic Mathematical Reports. **17**, 161-178 (2020). https://doi.org/10.33048/semi.2020.17.012
[20] Наймарк М.А.: Линейные дифференциальные операторы. -М.:Наука. (1969).
[21] Ватсон Г. Н.: Теория бесселевых функций. -М.:Т.1.Изд. ИЛ. (1949).
[22] Karimov K., Olimova D.: *A non-local boundary value problem in a semi-infinite parallelepiped for a three-dimensional equation of mixed type with two singular coefficients.* Bull. Inst. Math. **8** (3), 80-92 (2025).
[23] Трикоми Ф.: Дифференциальные уравнения, -М.ИЛ, (1962).
[24] Лебедев Н.Н.: Специальные функции и их приложения. -М.: Физматлит. (1963).
[25] Капилевич М.Б.: *Об одном уравнении смешанного эллиптико-гиперболического типа.* Математический сборник. (1), 11-38 (1952).

Boundary value problem with integral condition for a mixed-type equation with Bessel operators

K.T.Karimov, D.S.Olimova

Abstract

In this paper, we study a nonlocal problem with an integral condition in a parallelepiped for a three-dimensional mixed-type equation with Bessel operators. The proof of the uniqueness of the solution and its construction are performed using spectral analysis. The solution is constructed as a double Fourier series of the sum of trigonometric and Bessel functions.

Keywords

mixed type equations; singular coefficient; Bessel operator; Bessel function; Macdonald function; integral condition.

Affiliations

K.T. Karimov
Address: 1) Fergana State University, Fergana, Uzbekistan; 2) V.I.Romanovskiy Institute of Mathematics, Uzbekistan Academy of Sciences
e-mail: karimovk80@mail.ru
ORCID ID: https://orcid.org/0000-0002-9098-4116

D.S. Olimova
Address: Fergana Military Academic Lyceum of the Ministry of Defense "Temurbekov School Fergana, Uzbekistan.
e-mail: dilfuzaolimova406@gmail.com
ORCID ID: https://orcid.org/0009-0006-3624-3065