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The Inverse Spectral Problem Method for the Periodic Differential–Difference Sine–Gordon Equation

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Abstract

This paper is devoted to the investigation of the periodic discrete sine–Gordon equation by means of the inverse spectral problem method. The associated discrete Lax pair is studied, and the spectral theory of the corresponding periodic linear difference problem is established. Explicit representations of fundamental solutions are obtained in terms of polynomial structures, and the spectral data are characterized via a hyperelliptic curve.

Using these results, trace formulas are derived that provide an explicit reconstruction of the solution in terms of the spectral parameters. An algorithm for solving the inverse spectral problem is formulated based on the evolution of the spectral data. Finally, we derive a system of evolution equations for the spectral parameters, which may be regarded as a discrete analogue of the Dubrovin equations.

Keywords: *Discrete sine–Gordon equation; inverse spectral problem; Lax pair; spectral data; trace formula; hyperelliptic curve; finite-gap integration; Dubrovin equations.*

AMS Subject Classification (2020): *Primary: 34A55 ; Secondary: 37K15; 39A14; 35Q55.*

1. Introduction

The Sine–Gordon equation appears in various physical and dynamical processes, such as a chain of coupled pendulums [1], Josephson tunnel junctions between superconductors [4], dislocations in crystals [2, 3], and biological cellular structures [5]. The SG equation, as well as its discrete version (dSG), has been extensively studied in the context of the inverse scattering transform (IST) [6, 8, 7] of which application to dSG equations with self-consistent sources can be found in [9, 10]. Using another different approaches, interesting results have also been obtained for the discrete integrable SG equation, for instance, solutions of the discrete SG equation have been derived using a method based on Riccati equation expansion [11], while the variable-coefficient Jacobi elliptic function method has been used to construct two families of exact travelling-wave solutions [12]. The dSG equation also arises naturally in various physical models; for example, its formulation on metric graphs provides an effective framework for describing the propagation of deformations in branched solid structures [13].

In this paper, we focus on the investigation of the dSG equation with periodic boundary conditions and develop the spectral theory for the associated linear spectral difference equation. Our approach follows the methods developed in [14, 15, 16, 17] to integrate the dSG equation within the class of periodic functions.

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We consider the differential-difference sine-Gordon equation

$$\dot{\theta}_{n+1}(t) - \dot{\theta}_n(t) = 2sin\,\theta_{n+1}(t) + 2sin\theta_n(t), \quad n \in \mathbb{Z}, \ t > 0 \tag{1.1}$$

in the class of N-periodic functions $\theta_{n+N}(t) = \theta_n(t), \quad n \in Z, \ t \geq 0$, with the initial data

$$\theta_n(0) = \theta_n^0, \ n \in Z. \tag{1.2}$$

where the sequence $\{\theta_n^0\}_{n\in\mathbb{Z}}$ is assumed to be N-periodic and N is natural number.

The Lax pair representation associated with Eq. (1.1) is given as

$$y_{n+1}(z,t) = L_n(z,t)y_n(z,t), \tag{1.3}$$

$$\dot{y}_n(z,t) = H_n(z,t)y_n(z,t)$$

where $L_n(z,t)$ and $H_n(z,t)$ are the following matrix

$$\begin{gathered} L_n(z,t) = zP_n(t) + \frac{1}{z}Q_n(t), \\ H_n(z,t) = \frac{z^2+1}{z^2-1}\sigma_3 + i\omega_n\sigma_2, \\ P_n(t) = \frac{1}{2}\begin{pmatrix} 1+\cos\theta_n(t) & \sin\theta_n(t) \\ \sin\theta_n(t) & 1-\cos\theta_n(t) \end{pmatrix}, \\ Q_n(t) = \frac{1}{2}\begin{pmatrix} 1-\cos\theta_n(t) & -\sin\theta_n(t) \\ -\sin\theta_n(t) & 1+\cos\theta_n(t) \end{pmatrix}, \\ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \end{gathered} \tag{1.4}$$

Here z is the non-zero spectral parameter and $\omega_n(t)$ satisfy the compatibility equations [9]

$$\dot{\theta}_n(t) = -(\omega_{n+1}(t) + \omega_n(t)),$$

$$\omega_{n+1}(t) - \omega_n(t) = -2\sin\theta_n(t). \tag{1.5}$$

Using (1.5) together with the periodicity of $\omega_n(t)$, we obtain

$$\sum_{n=0}^{N-1} \sin\theta_n(t) = 0. \tag{1.6}$$

The identity (1.6) is the closure condition for the periodic chain: the total increment over one period must vanish, and hence the forcing terms $\sin\theta_n(t)$ must balance each other. Since (1.5) determines $\omega_n(t)$ only up to an additive function of t, we impose the normalization condition

$$\omega_0(t) = f(t).$$

This condition fixes the additive freedom in $\omega_n(t)$ and ensures uniqueness.

The solution of the inverse spectral problem is based on the trace formulae and the evolution equations for the spectral data, which together provide an algorithm for reconstructing the periodic solution $\{\theta_n(t)\}_{n=-\infty}^{\infty}$ of (1.1)–(1.2).

In Section 2, we study the discrete spectral theory for the linear difference equation associated with the dSG equation. Section 3 is devoted to obtaining the trace formula for the shifted discrete linear system and presenting an algorithm for solving the inverse spectral problem. In the final section, we derive a discrete analogue of the Dubrovin system that describes the time evolution of the spectral data associated with the underlying linear problem.

2. Discrete spectral problem

Let us study the following discrete system, temporally omitting its dependence on time t

$$y_{n+1}(z) = L_n(z) y_n(z), \tag{2.1}$$
$$L_n = zP_n + \frac{1}{z} Q_n$$

where z is the is the non-zero spectral parameter and

$$P_n = \frac{1}{2}\begin{pmatrix} 1+\cos\theta_n & \sin\theta_n \\ \sin\theta_n & 1-\cos\theta_n \end{pmatrix},$$
$$Q_n = \frac{1}{2}\begin{pmatrix} 1-\cos\theta_n & -\sin\theta_n \\ -\sin\theta_n & 1+\cos\theta_n \end{pmatrix}.$$

Here $\theta_{n+N} = \theta_n$ and $y_n = (y_{n,1},\ y_{n,2})^T$, $n \in Z$. Let $c_n(z)$, $n \in Z$ and $s_n(z)$, $n \in Z$, be the solutions of system (2.1) determined by the initial conditions $c_0(z) = (1,\ 0)^T$ and $s_0(z) = (0,\ 1)^T$. Then these solutions admit the representation

$$c_n(z) = \frac{\gamma_n}{z^n}\begin{pmatrix} T_n(z^2) \\ S_n(z^2) \end{pmatrix}, \quad s_n(z) = \frac{\gamma_n}{z^n}\begin{pmatrix} B_n(z^2) \\ U_n(z^2) \end{pmatrix}. \tag{2.2}$$

Here γ_n is given by

$$\gamma_n = \cos\frac{\theta_0}{2}\prod_{j=1}^{n-1}\cos\frac{\theta_j-\theta_{j-1}}{2}\cdot\cos\frac{\theta_{n-1}}{2}, \qquad n \geq 1,$$

with the convention

$$\gamma_0 = 1.$$

The polynomial sequences $\{T_n(z^2)\}$, $\{S_n(z^2)\}$, $\{B_n(z^2)\}$, and $\{U_n(z^2)\}$, where each polynomial is of degree n, satisfy the following recurrence relations:

$$T_{n+1}(\lambda) = \frac{\cos\frac{\theta_{n-1}}{2}\left[\lambda(1+\cos\theta_n)+(1-\cos\theta_n)\right]}{2\cos\frac{\theta_n-\theta_{n-1}}{2}\cos\frac{\theta_n}{2}} T_n(\lambda) + \frac{(\lambda-1)\sin\frac{\theta_n}{2}\cos\frac{\theta_{n-1}}{2}}{\cos\frac{\theta_n-\theta_{n-1}}{2}} S_n(\lambda), \tag{2.3}$$

$$S_{n+1}(\lambda) = \frac{\cos\frac{\theta_{n-1}}{2}\left(\lambda(1-\cos\theta_n)+(1+\cos\theta_n)\right)}{2\cos\frac{\theta_n-\theta_{n-1}}{2}\cos\frac{\theta_n}{2}} S_n(\lambda) + \frac{(\lambda-1)\sin\frac{\theta_n}{2}\cos\frac{\theta_{n-1}}{2}}{\cos\frac{\theta_n-\theta_{n-1}}{2}} T_n(\lambda), \tag{2.4}$$

$$B_{n+1}(\lambda) = \frac{\cos\frac{\theta_{n-1}}{2}\left(\lambda(1+\cos\theta_n)+(1-\cos\theta_n)\right)}{2\cos\frac{\theta_n-\theta_{n-1}}{2}\cos\frac{\theta_n}{2}} B_n(\lambda) + \frac{(\lambda-1)\sin\frac{\theta_n}{2}\cos\frac{\theta_{n-1}}{2}}{\cos\frac{\theta_n-\theta_{n-1}}{2}} U_n(\lambda), \tag{2.5}$$

$$U_{n+1}(\lambda) = \frac{\cos\frac{\theta_{n-1}}{2}\left(\lambda(1-\cos\theta_n)+(1+\cos\theta_n)\right)}{2\cos\frac{\theta_n-\theta_{n-1}}{2}\cos\frac{\theta_n}{2}} U_n(\lambda) + \frac{(\lambda-1)\sin\frac{\theta_n}{2}\cos\frac{\theta_{n-1}}{2}}{\cos\frac{\theta_n-\theta_{n-1}}{2}} B_n(\lambda), \tag{2.6}$$

where $\lambda = z^2$, and the initial values are

$$T_1(\lambda) = \lambda + tg^2\frac{\theta_0}{2}, \tag{2.7}$$
$$S_1(\lambda) = (\lambda-1)tg\frac{\theta_0}{2}, \tag{2.8}$$
$$B_1(\lambda) = (\lambda-1)tg\frac{\theta_0}{2}, \tag{2.9}$$
$$U_1(\lambda) = \lambda tg^2\frac{\theta_0}{2} + 1. \tag{2.10}$$

We denote the fundamental matrix of the system (2.1) by

$$M_n(z) = \begin{pmatrix} c_{n,1}(z) & s_{n,1}(z) \\ c_{n,2}(z) & s_{n,2}(z) \end{pmatrix}. \tag{2.11}$$

The corresponding Wronskian is defined as

$$W_n(z) = \det M_n(z) = c_{n,1}(z)s_{n,2}(z) - c_{n,2}(z)s_{n,1}(z). \tag{2.12}$$

It follows that the Wronskian (2.12) is independent of both the discrete variable n and the spectral parameter z. Moreover, the matrix (2.11) can be rewritten in the form

$$M_n = L_{n-1} \cdot L_{n-2} \cdot .....L_1 \cdot L_0 \tag{2.13}$$

since

$$W_n(z) = det(M_n(z)) = \Pi_{j=0}^{n-1} det(L_j).$$

Since

$$\det(L_j) = 1,$$

it follows that

$$W_n(z) = W_0(z) = 1.$$

According to the recurrence relations (2.3)–(2.10) and the following multiplication properties of the matrices P_j and Q_j, for $j = 1, \dots, n-1$, we obtain

$$P_0 c_0 = \frac{1}{2}\begin{pmatrix} 1+\cos\theta_0 \\ \sin\theta_0 \end{pmatrix} = \cos\frac{\theta_0}{2}\begin{pmatrix} \cos\frac{\theta_0}{2} \\ \sin\frac{\theta_0}{2} \end{pmatrix},$$

....

$$P_{n-1}P_{n-2}...P_1P_0c_0 = \cos\frac{\theta_0}{2}\cos\frac{\theta_1-\theta_0}{2}...\cos\frac{\theta_{n-1}-\theta_{n-2}}{2}\begin{pmatrix} \cos\frac{\theta_{n-1}}{2} \\ \sin\frac{\theta_{n-1}}{2} \end{pmatrix},$$

$$P_{n-1}\cdot P_{n-2}\cdot ...P_1\cdot Q_0c_0 = -\sin\frac{\theta_0}{2}\sin\frac{\theta_1-\theta_0}{2}\cos\frac{\theta_2-\theta_1}{2}...\cos\frac{\theta_{n-1}-\theta_{n-2}}{2}\begin{pmatrix} \cos\frac{\theta_{n-1}}{2} \\ \sin\frac{\theta_{n-1}}{2} \end{pmatrix},$$

$$P_{n-1}\cdot P_{n-2}\cdot ...Q_1\cdot P_0c_0 = -\cos\frac{\theta_0}{2}\sin\frac{\theta_1-\theta_0}{2}\sin\frac{\theta_2-\theta_1}{2}\cos\frac{\theta_3-\theta_2}{2}...\cos\frac{\theta_{n-1}-\theta_{n-2}}{2}\begin{pmatrix} \cos\frac{\theta_{n-1}}{2} \\ \sin\frac{\theta_{n-1}}{2} \end{pmatrix},$$

. . . .

$$Q_{n-1}\cdot P_{n-2}\cdot ...P_1\cdot P_0c_0 = -\cos\frac{\theta_0}{2}\cos\frac{\theta_1-\theta_0}{2}\cos\frac{\theta_2-\theta_1}{2}...\sin\frac{\theta_{n-1}-\theta_{n-2}}{2}\begin{pmatrix} \sin\frac{\theta_{n-1}}{2} \\ -\cos\frac{\theta_{n-1}}{2} \end{pmatrix},$$

$$Q_{n-1}\cdot Q_{n-2}\cdot ...Q_1\cdot Q_0c_0 = \sin\frac{\theta_0}{2}\cos\frac{\theta_1-\theta_0}{2}\cos\frac{\theta_2-\theta_1}{2}...\cos\frac{\theta_{n-1}-\theta_{n-2}}{2}\begin{pmatrix} \sin\frac{\theta_{n-1}}{2} \\ -\cos\frac{\theta_{n-1}}{2} \end{pmatrix},$$

similarly,

$$P_{n-1}\cdot P_{n-2}\cdot ...P_1\cdot P_0s_0 = \sin\tfrac{\theta_0}{2}\cos\tfrac{\theta_1-\theta_0}{2}\cos\tfrac{\theta_2-\theta_1}{2}...\cos\tfrac{\theta_{n-1}-\theta_{n-2}}{2}\begin{pmatrix} \cos\frac{\theta_{n-1}}{2} \\ \sin\frac{\theta_{n-1}}{2} \end{pmatrix},$$

$$P_{n-1}\cdot P_{n-2}\cdot ...P_1\cdot Q_0s_0 = \cos\frac{\theta_0}{2}\sin\frac{\theta_1-\theta_0}{2}\cos\frac{\theta_2-\theta_1}{2}...\cos\frac{\theta_{n-1}-\theta_{n-2}}{2}\begin{pmatrix} \cos\frac{\theta_{n-1}}{2} \\ \sin\frac{\theta_{n-1}}{2} \end{pmatrix},$$

$$P_{n-1}\cdot P_{n-2}\cdot .....Q_1\cdot P_0s_0 = -\sin\frac{\theta_0}{2}\sin\frac{\theta_1-\theta_0}{2}\sin\frac{\theta_2-\theta_1}{2}\cos\frac{\theta_3-\theta_2}{2}......\cos\frac{\theta_{n-1}-\theta_{n-2}}{2}\begin{pmatrix} \cos\frac{\theta_{n-1}}{2} \\ \sin\frac{\theta_{n-1}}{2} \end{pmatrix},$$

...

$$Q_{n-1}\cdot P_{n-2}\cdot .....P_1\cdot P_0 s_0=\sin\frac{\theta_0}{2}\cos\frac{\theta_1-\theta_0}{2}\cos\frac{\theta_2-\theta_1}{2}......\sin\frac{\theta_{n-1}-\theta_{n-2}}{2}\begin{pmatrix}\sin\frac{\theta_{n-1}}{2}\\ -\cos\frac{\theta_{n-1}}{2}\end{pmatrix},$$

$$Q_{n-1}\cdot Q_{n-2}\cdot .....Q_1\cdot Q_0 s_0=\cos\frac{\theta_0}{2}\cos\frac{\theta_1-\theta_0}{2}\cos\frac{\theta_2-\theta_1}{2}......\cos\frac{\theta_{n-1}-\theta_{n-2}}{2}\begin{pmatrix}-\sin\frac{\theta_{n-1}}{2}\\ \cos\frac{\theta_{n-1}}{2}\end{pmatrix},$$

we obtain the explicit forms of the polynomials $T_n(\lambda)$, $S_n(\lambda)$, $B_n(\lambda)$, $U_n(\lambda)$

$$\begin{aligned}T_n=\lambda^n-(tg\frac{\theta_0}{2}tg\frac{\theta_1-\theta_0}{2}+...+tg\frac{\theta_{n-2}-\theta_{n-3}}{2}tg\frac{\theta_{n-1}-\theta_{n-2}}{2}-tg\frac{\theta_{n-1}-\theta_{n-2}}{2}tg\frac{\theta_{n-1}}{2})\lambda^{n-1}+\\ +...+tg\frac{\theta_0}{2}tg\frac{\theta_{n-1}}{2}\end{aligned}\tag{2.14}$$

$$\begin{aligned}S_n=tg\frac{\theta_{n-1}}{2}\lambda^n-((tg\frac{\theta_0}{2}tg\frac{\theta_1-\theta_0}{2}+...+tg\frac{\theta_{n-2}-\theta_{n-3}}{2}tg\frac{\theta_{n-1}-\theta_{n-2}}{2})tg\frac{\theta_{n-1}}{2}+tg\frac{\theta_{n-1}-\theta_{n-2}}{2})\lambda^{n-1}+\\ +...-tg\frac{\theta_0}{2}\end{aligned}\tag{2.15}$$

$$\begin{aligned}B_n=tg\frac{\theta_0}{2}\lambda^n-(-tg\frac{\theta_1-\theta_0}{2}+tg\frac{\theta_0}{2}(tg\frac{\theta_1-\theta_0}{2}tg\frac{\theta_2-\theta_1}{2}+...+tg\frac{\theta_{n-2}-\theta_{n-3}}{2}tg\frac{\theta_{n-1}-\theta_{n-2}}{2}-\\ -tg\frac{\theta_{n-1}-\theta_{n-2}}{2}tg\frac{\theta_{n-1}}{2}))\lambda^{n-1}+...+(-tg\frac{\theta_{n-1}}{2})\end{aligned}\tag{2.16}$$

$$\begin{aligned}U_n=tg\frac{\theta_0}{2}tg\frac{\theta_{n-1}}{2}\lambda^n+(tg\frac{\theta_1-\theta_0}{2}tg\frac{\theta_{n-1}}{2}-tg\frac{\theta_0}{2}tg\frac{\theta_{n-1}-\theta_{n-2}}{2}-tg\frac{\theta_0}{2}tg\frac{\theta_{n-1}}{2}(tg\frac{\theta_1-\theta_0}{2}tg\frac{\theta_2-\theta_1}{2}+\\ +...+tg\frac{\theta_{n-2}-\theta_{n-3}}{2}tg\frac{\theta_{n-1}-\theta_{n-2}}{2}))\lambda^{n-1}+...+1\end{aligned}\tag{2.17}$$

The discriminant of the discrete system (2.1) is defined by

$$\Delta(\lambda)=c_{N,1}+s_{N,2}.$$

Substituting the representations given in (2.2) yields

$$\Delta(z)=\frac{\gamma_N}{z^N}\left[T_N(\lambda)+U_N(\lambda)\right],\qquad \lambda=z^2.\tag{2.18}$$

Since $W_n(z)=1$, the following relation

$$\Delta^2(z)-4=(s_{N,2}-c_{N,1})^2+4s_{N,1}c_{N,2}\tag{2.19}$$

is valid. Combining the representation (2.18) with the polynomial representation of the solutions given in (2.2), we obtain the following equivalent form of (2.19):

$$(U_N(\lambda)+T_N(\lambda))^2-4\lambda^N\frac{1}{\gamma_N^2}=(T_N(\lambda)-U_N(\lambda))^2+4B_N(\lambda)S_N(\lambda).\tag{2.20}$$

Denoting the left-hand side of (2.20) by R_{2N} and writing

$$K_N(\lambda)=T_N(\lambda)-U_N(\lambda),$$

equation (2.20) becomes

$$R_{2N}(\lambda)=(T_N(\lambda)-U_N(\lambda))^2+4B_N(\lambda)S_N(\lambda)\tag{2.21}$$

or its explicite polynomial form

$$R_{2N}(\lambda) = \left(1 + tg\frac{\theta_0}{2}tg\frac{\theta_{N-1}}{2}\right)^2 \lambda^{2N} + 2\left(1 + tg\frac{\theta_0}{2}tg\frac{\theta_{N-1}}{2}\right)\Bigg[\left(tg\frac{\theta_{N-1}}{2} - tg\frac{\theta_0}{2}\right)tg\frac{\theta_1 - \theta_0}{2} +$$
$$+ \left(tg\frac{\theta_{N-1}}{2} - tg\frac{\theta_0}{2}\right)tg\frac{\theta_{N-1} - \theta_{N-2}}{2} - \left(1 + tg\frac{\theta_0}{2}tg\frac{\theta_{N-1}}{2}\right)(tg\frac{\theta_1 - \theta_0}{2}tg\frac{\theta_2 - \theta_1}{2} + \cdots +$$
$$+ tg\frac{\theta_{N-2} - \theta_{N-3}}{2}tg\frac{\theta_{N-1} - \theta_{N-2}}{2})\Bigg]\lambda^{2N-1} + \cdots + \left(1 + tg\frac{\theta_0}{2}tg\frac{\theta_{N-1}}{2}\right)^2 \tag{2.22}$$
$$- 4\lambda^N\left(1 + tg^2\frac{\theta_0}{2}\right)\left(1 + tg^2\frac{\theta_1 - \theta_0}{2}\right)\cdots\left(1 + tg^2\frac{\theta_{N-1}}{2}\right).$$

Let $\{\lambda_j\}_{j=1}^{2N}$ be the set of roots of the polynomial $R_{2N}(\lambda)$, and let $\{\xi_k\}_{k=1}^{N}$ be the set of roots of the polynomial $B_N(\lambda)$. Then,

$$K_N(\xi_j) = \sigma_j\sqrt{R_{2N}(\xi_j)}, \qquad \sigma_j \in \{\pm 1\}, \qquad j = 1, \ldots, N. \tag{2.23}$$

The quantities σ_j determine the sheet of the hyperelliptic spectral curve

$$\Gamma: \qquad \mu^2 = R_{2N}(\lambda)$$

on which the divisor points lie. Indeed, defining

$$\mu_j := K_N(\xi_j),$$

relation (2.23) implies

$$(\xi_j, \mu_j) \in \Gamma, \qquad \mu_j = \sigma_j\sqrt{R_{2N}(\xi_j)}, \qquad \sigma_j \in \{\pm 1\}.$$

Thus the spectral data of the discrete system (2.1) are given by the branch points

$$\lambda_1, \ldots, \lambda_{2N}$$

of the hyperelliptic curve Γ together with the divisor

$$D = \sum_{j=1}^{N}(\xi_j, \mu_j), \qquad \mu_j = \sigma_j\sqrt{R_{2N}(\xi_j)}.$$

The signs σ_j specify on which sheet of the curve Γ the divisor points (ξ_j, μ_j) lie and therefore form an essential part of the spectral data in the finite-gap description of the system.

Proposition 2.1. *Let the spectral curve associated with the discrete system* (2.1) *be*

$$\Gamma: \qquad \mu^2 = R_{2N}(\lambda).$$

For each zero ξ_j of the polynomial $B_N(\lambda)$ define

$$\mu_j := K_N(\xi_j).$$

Then the point (ξ_j, μ_j) lies on the curve Γ and can be written in the form

$$(\xi_j, \mu_j) = \left(\xi_j, \sigma_j\sqrt{R_{2N}(\xi_j)}\right), \qquad \sigma_j \in \{\pm 1\}.$$

Thus the quantities σ_j determine on which sheet of the hyperelliptic curve Γ the divisor point lies.

Proof Recall the identity

$$R_{2N}(\lambda) = K_N(\lambda)^2 + 4B_N(\lambda)S_N(\lambda).$$

Since ξ_j is a zero of $B_N(\lambda)$, we have

$$B_N(\xi_j) = 0.$$

Substituting $\lambda = \xi_j$ into the above identity yields

$$R_{2N}(\xi_j) = K_N(\xi_j)^2.$$

By definition, $\mu_j = K_N(\xi_j)$, hence

$$\mu_j^2 = R_{2N}(\xi_j).$$

Therefore the point (ξ_j, μ_j) belongs to the spectral curve

$$\Gamma : \qquad \mu^2 = R_{2N}(\lambda).$$

Now, for each fixed ξ_j, the equation

$$\mu^2 = R_{2N}(\xi_j)$$

has two possible values

$$\mu = \pm\sqrt{R_{2N}(\xi_j)}.$$

Hence there exists a sign $\sigma_j \in \{\pm 1\}$ such that

$$\mu_j = K_N(\xi_j) = \sigma_j\sqrt{R_{2N}(\xi_j)}.$$

This sign specifies which of the two sheets of the hyperelliptic curve contains the point (ξ_j, μ_j).

Lemma 2.1. *For each zero ξ_j of $B_N(\lambda)$ there are two possible points of the spectral curve lying over ξ_j, namely*

$$\left(\xi_j, \pm\sqrt{R_{2N}(\xi_j)}\right).$$

Equivalently, the sheet sign $\sigma_j \in \{\pm 1\}$ is not determined by the branch points $\lambda_1, \ldots, \lambda_{2N}$ and the zeros $\xi_1, \ldots, \xi_N$ alone.

Thus the divisor

$$D = \sum_{j=1}^{N} (\xi_j, \mu_j), \qquad \mu_j = \sigma_j\sqrt{R_{2N}(\xi_j)},$$

is part of the spectral data only after an admissible choice of signs $\sigma_1, \ldots, \sigma_N$ has been fixed.

Proof For each j, the relation

$$\mu_j^2 = R_{2N}(\xi_j)$$

follows from Proposition 2.1. Therefore, over the point $\lambda = \xi_j$ there are exactly two points on the hyperelliptic curve:

$$\left(\xi_j, \sqrt{R_{2N}(\xi_j)}\right), \qquad \left(\xi_j, -\sqrt{R_{2N}(\xi_j)}\right).$$

Hence the sign σ_j may take either value $+1$ or -1, and it is not fixed by the values of $\lambda_1, \ldots, \lambda_{2N}$ and $\xi_1, \ldots, \xi_N$ alone.

Therefore, in order to define the divisor on the curve uniquely, one must choose for each ξ_j one of the two points lying above it, i.e. one must specify the corresponding sign σ_j. The proof is complete.

The spectral data of equation (2.1) consist of the roots

$$\lambda_1, \lambda_2, \ldots, \lambda_{2N},$$

the spectral parameters

$$\xi_j, \qquad j = 1, \ldots, N,$$

and the corresponding sheet signs

$$\sigma_1, \ldots, \sigma_N.$$

These data determine the factorized forms of the polynomials $R_{2N}(\lambda)$ and $B_N(\lambda)$,

$$\begin{aligned} R_{2N}(\lambda) &= (1 + tg\frac{\theta_{N-1}}{2} tg\frac{\theta_0}{2})^2 \Pi_{j=1}^{2N}(\lambda - \lambda_j), \\ B_N(\lambda) &= tg\frac{\theta_0}{2} \Pi_{j=1}^{N}(\lambda - \xi_j), \end{aligned} \tag{2.24}$$

and together with the interpolation conditions

$$K_N(\xi_j) = \sigma_j \sqrt{R_{2N}(\xi_j)}, \qquad j = 1, \ldots, N, \tag{2.25}$$

they determine the divisor on the spectral curve and therefore provide the finite-gap spectral characterization of the system.

Using the representations (2.14)–(2.17) in (2.21), together with the polynomial identity (2.24), we derive the following identities for $\xi_k, \ k = \overline{1, N}$

$$\xi_1 \xi_2 \ldots . \xi_N = (-1)^{N+1} \frac{tg\frac{\theta_{N-1}}{2}}{tg\frac{\theta_0}{2}}. \tag{2.26}$$

We now present the main result concerning the trace formula for the discrete system (2.1), which is the key ingredient in the reconstruction of the solution

$$\{\theta_n\}_{n \in \mathbb{Z}}.$$

Let

$$a := tg\frac{\theta_0}{2}, \qquad b := tg\frac{\theta_{N-1}}{2}.$$

Then

$$\kappa = 1 - ab.$$

By Vieta's theorem applied to $B_N(\lambda)$,

$$\prod_{j=1}^{N} \xi_j = (-1)^{N+1} \frac{b}{a}.$$

Hence

$$ab = (-1)^{N+1} a^2 \prod_{j=1}^{N} \xi_j,$$

and therefore

$$tg^2\frac{\theta_0}{2} = a^2 = (-1)^{N+1} \frac{1 - \kappa}{\prod_{j=1}^{N} \xi_j}.$$

Define

$$\widetilde{B}_N(\lambda) := \prod_{j=1}^{N}(\lambda - \xi_j), \qquad \widetilde{B}'_N(\xi_j) = \prod_{\substack{m=1\\ m\neq j}}^{N}(\xi_j - \xi_m).$$

Since that the roots $\xi_1, \dots, \xi_N$ are simple and that

$$1 + (-1)^N \prod_{j=1}^{N} \xi_j \neq 0.$$

Choose the branch on the curve

$$\mu^2 = R_{2N}(\lambda)$$

by fixing

$$\nu_j := K_N(\xi_j) = \sigma_j \sqrt{R_{2N}(\xi_j)}, \qquad \sigma_j \in \{\pm 1\}.$$

Then

$$K_N(\lambda) = \kappa\, \widetilde{B}_N(\lambda) + \sum_{j=1}^{N} \nu_j \frac{\widetilde{B}_N(\lambda)}{(\lambda - \xi_j)\widetilde{B}'_N(\xi_j)}.$$

Evaluating at $\lambda = 0$ and using $K_N(0) = -\kappa$, we obtain

$$\kappa = \frac{(-1)^N \left(\prod_{j=1}^{N} \xi_j\right) \sum_{j=1}^{N} \frac{\sigma_j \sqrt{R_{2N}(\xi_j)}}{\xi_j \prod_{m\neq j}(\xi_j - \xi_m)}}{1 + (-1)^N \prod_{j=1}^{N} \xi_j}.$$

Substituting this into the previous formula yields

$$A_0 = (-1)^N \left(\prod_{j=1}^{N} \xi_j \right),$$

$$\sum_{j=1}^{N} \frac{\sigma_j \sqrt{R_{2N}(\xi_j)}}{\xi_j \prod_{m\neq j}(\xi_j - \xi_m)} = \left(1 + \mathrm{tg}\,\frac{\theta_{N-1}}{2}\,\mathrm{tg}\,\frac{\theta_0}{2}\right) \left(\sum_{j=1}^{N} \frac{\sigma_j \sqrt{\widetilde{R}_{2N}(\xi_j)}}{\xi_j \prod_{m\neq j}(\xi_j - \xi_m)} \right)$$

$$D_0 = \sum_{j=1}^{N} \frac{\sigma_j \sqrt{\widetilde{R}_{2N}(\xi_j)}}{\xi_j \prod_{m\neq j}(\xi_j - \xi_m)}$$

Substituting this into the previous formula yields

$$\cos\theta_0 = \frac{1 - tg^2 \frac{\theta_0}{2}}{1 + tg^2 \frac{\theta_0}{2}}, \tag{2.27}$$

where

$$tg^2 \frac{\theta_0}{2} = \frac{-1 - A_0 + A_0 D_0}{A_0(1 + A_0 + A_0 D_0)} \tag{2.28}$$

3. Trace formula

To reconstruct the solution θ_n, $n \in \mathbb{Z}$, we shift the index in (2.1) by an arbitrary integer k and consider the translated system

$$y_{n+k+1}(z) = L_{n+k}(z) y_{n+k}(z), \tag{3.1}$$

where

$$L_{n+k}(z) = zP_{n+k} + \frac{1}{z} Q_{n+k}.$$

Let $\varphi_n(z,k)$ and $\phi_n(z,k)$ denote the fundamental solutions of the system (3.1) satisfying the initial conditions

$$\varphi_0(z,k) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \phi_0(z,k) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Lemma 3.1. *The fundamental solutions $\varphi_n(z,k)$ and $\phi_n(z,k)$ satisfy the relations*

$$\varphi_n(z,k) = s_{k,2}(z) c_{n+k}(z) - c_{k,2}(z) s_{n+k}(z), \tag{3.2}$$

and

$$\phi_n(z,k) = -s_{k,1}(z) c_{n+k}(z) + c_{k,1}(z) s_{n+k}(z). \tag{3.3}$$

Proof. Let

$$\Phi_k(z) = \big(c_k(z),\ s_k(z)\big).$$

Since the potential is N-periodic, i.e.,

$$\theta_{n+N} = \theta_n,$$

the solutions of (3.1) satisfy the monodromy relation

$$\Phi_{k+N}(z) = \Phi_k(z) M_N(z),$$

where the monodromy matrix is given by

$$M_N(z) = \begin{pmatrix} c_{N,1}(z) & s_{N,1}(z) \\ c_{N,2}(z) & s_{N,2}(z) \end{pmatrix}.$$

Consequently,

$$\begin{aligned} c_{N+k}(z) &= c_{N,1}(z) c_k(z) + c_{N,2}(z) s_k(z), \\ s_{N+k}(z) &= s_{N,1}(z) c_k(z) + s_{N,2}(z) s_k(z). \end{aligned}$$

As the solution Φ_{k+n} satisfy the system

$$\Phi_{k+n} = L_{n+k-1} \Phi_{k+n-1} = L_{n+k-1} L_{n+k-2} L_{n+k-3} \cdots L_k \Phi_k$$

According to the definition of the monodromy matrix, we introduce the monodromy matrix associated with the shifted system (3.1) by

$$M_n^k(z) = \begin{pmatrix} \varphi_{n,1}(z,k) & \phi_{n,1}(z,k) \\ \varphi_{n,2}(z,k) & \phi_{n,2}(z,k) \end{pmatrix}, \tag{3.4}$$

which admits the representation

$$M_n^k(z) = L_{n+k-1}(z) L_{n+k-2}(z) \cdots L_k(z). \tag{3.5}$$

Consequently,

$$M_n^k(z) = \Phi_{k+n}(z) \Phi_k^{-1}(z),$$

from which relations (3.2) and (3.3) follow immediately. □

According to the shifted monodromy matrix, the discriminant of the shifted system is defined as

$$\Delta(z,k) = \varphi_{N,1}(z,k) + \phi_{N,2}(z,k). \tag{3.6}$$

Lemma 3.2. *The discriminant $\Delta(z,k)$ defined by (3.6) is independent of the shift parameter k.*

Proof. By the periodicity of the coefficients, the monodromy matrices satisfy the similarity relation

$$M_N^k = M_k M_N M_k^{-1}$$

and thus

$$\Delta(z,k) = \mathrm{tr}(M_k M_N M_k^{-1}) = \mathrm{tr}(M_N) = \Delta(z).$$

Hence, the roots of the equations

$$\Delta(z) = \pm 2$$

are independent of the shift parameter k. □

The inverse spectral problem for (2.1) consists in reconstructing the solution

$$\{\theta_n\}_{n\in\mathbb{Z}}$$

from the corresponding spectral data.

Corollary 3.1. *By formulas (2.27) and (2.28), the trace formula for the solution θ_k, $k \in \mathbb{Z}$, of equation (2.1) is given by*

$$\tan^2\frac{\theta_k}{2} = \frac{-1 - A_k + A_k D_k}{A_k(1 + A_k + A_k D_k)}, \tag{3.7}$$

where $\xi_{j,k}$, $j = 1,\dots,N$, are the roots of the equation

$$\varphi_{N,1}(z,k) = 0.$$

Lemma 3.3. *The spectral problem (2.1) can be uniquely reconstructed using its spectral data.*

Proof. Since the inverse spectral reconstruction relies on the spectral data at the kth step, we first derive recurrence relations for the polynomials $B_{N,k}(\lambda)$, $K_{N,k}(\lambda)$, and $S_{N,k}(\lambda)$.

According to the expansion (3.5) of the monodromy matrix M_n^k and periodicity of θ_n, it holds

$$M_N^k = L_{k-1} M_N^{k-1} L_{k-1}^{-1}, \tag{3.8}$$

and taking account (2.2) and (3.4) we have

$$\frac{z^N}{\gamma_N} M_N^k = \Psi_N^k \tag{3.9}$$

where

$$\Psi_N^k = \begin{pmatrix} T_{N,k} & B_{N,k} \\ S_{N,k} & U_{N,k} \end{pmatrix}.$$

According to the equalities (3.8) and (3.9), it follows that

$$\Psi_N^k = L_{k-1}\Psi_N^{k-1} L_{k-1}^{-1}, \tag{3.10}$$

$$L_{k-1} = \begin{pmatrix} \frac{z}{2}(1+\cos\theta_{k-1}) + \frac{1}{2z}(1-\cos\theta_{k-1}) & (\frac{z}{2} - \frac{1}{2z})\sin\theta_{k-1} \\ (\frac{z}{2} - \frac{1}{2z})\sin\theta_{k-1} & \frac{z}{2}(1-\cos\theta_{k-1}) + \frac{1}{2z}(1+\cos\theta_{k-1}) \end{pmatrix},$$

$$L_{k-1}^{-1} = \begin{pmatrix} \frac{z}{2}(1-\cos\theta_{k-1}) + \frac{1}{2z}(1+\cos\theta_{k-1}) & (\frac{1}{2z} - \frac{z}{2})\sin\theta_{k-1} \\ (\frac{1}{2z} - \frac{z}{2})\sin\theta_{k-1} & \frac{z}{2}(1+\cos\theta_{k-1}) + \frac{1}{2z}(1-\cos\theta_{k-1}) \end{pmatrix}.$$

Consequently, the equality (3.10) gives the following relations

$$\begin{aligned}
T_{N,k} = ((\frac{z}{2}(1+\cos\theta_{k-1}) + \frac{1}{2z}(1-\cos\theta_{k-1}))T_{N,k-1} + S_{N,k-1}(\frac{z}{2} - \frac{1}{2z})\sin\theta_{k-1})(\frac{z}{2}(1-\cos\theta_{k-1}) + \\
+\frac{1}{2z}(1+\cos\theta_{k-1})) + ((\frac{z}{2}(1+\cos\theta_{k-1}) + \frac{1}{2z}(1-\cos\theta_{k-1}))B_{N,k-1} + \\
+U_{N,k-1}(\frac{z}{2} - \frac{1}{2z})\sin\theta_{k-1})(\frac{1}{2z} - \frac{z}{2})\sin\theta_{k-1}
\end{aligned}$$

$$\begin{aligned}
S_{N,k} = (T_{N,k-1}(\frac{z}{2} - \frac{1}{2z})\sin\theta_{k-1} + (\frac{z}{2}(1-\cos\theta_{k-1}) + \frac{1}{2z}(1+\cos\theta_{k-1}))S_{N,k-1})(\frac{z}{2}(1-\cos\theta_{k-1}) + \\
+\frac{1}{2z}(1+\cos\theta_{k-1})) + (B_{N,k-1}(\frac{z}{2} - \frac{1}{2z})\sin\theta_{k-1} + (\frac{z}{2}(1-\cos\theta_{k-1}) + \\
+\frac{1}{2z}(1+\cos\theta_{k-1}))U_{N,k-1})(\frac{1}{2z} - \frac{z}{2})\sin\theta_{k-1}
\end{aligned}$$

$$\begin{aligned}
B_{N,k} = ((\frac{z}{2}(1+\cos\theta_{k-1}) + \frac{1}{2z}(1-\cos\theta_{k-1}))T_{N,k-1} + S_{N,k-1}(\frac{z}{2} - \frac{1}{2z})\sin\theta_{k-1})(\frac{1}{2z} - \frac{z}{2})\sin\theta_{k-1} + \\
((+\frac{z}{2}(1+\cos\theta_{k-1}) + \frac{1}{2z}(1-\cos\theta_{k-1}))B_{N,k-1} + U_{N,k-1}(\frac{z}{2} - \frac{1}{2z})\sin\theta_{k-1})(\frac{z}{2}(1+\cos\theta_{k-1}) + \\
+\frac{1}{2z}(1-\cos\theta_{k-1}))
\end{aligned} \tag{3.11}$$

$$\begin{aligned}
U_{N,k} = (T_{N,k-1}(\frac{z}{2} - \frac{1}{2z})\sin\theta_{k-1} + (\frac{z}{2}(1-\cos\theta_{k-1}) + \frac{1}{2z}(1+\cos\theta_{k-1}))S_{N,k-1})(\frac{1}{2z} - \frac{z}{2})\sin\theta_{k-1} + \\
(B_{N,k-1}(\frac{z}{2} - \frac{1}{2z})\sin\theta_{k-1} + (\frac{z}{2}(1-\cos\theta_{k-1}) + \frac{1}{2z}(1+\cos\theta_{k-1}))U_{N,k-1})(\frac{z}{2}(1-\cos\theta_{k-1}) \\
+\frac{1}{2z}(1-\cos\theta_{k-1}))
\end{aligned}$$

Employing the equality (2.21) we obtain

$$S_{N,0}(\lambda) = \frac{R_{2N}(\lambda) - K_{N,0}^2(\lambda)}{4B_{N,0}(\lambda)}. \tag{3.12}$$

We have thus obtained the following algorithm for solving the inverse problem for the (2.1).

We have the system of spectral parameters $\left\{\xi_j, \sigma_j\right\}_{j=1}^N$ and the numbers $\lambda_1,\ \lambda_2, ..., \lambda_{2N}$.

1. Applying (2.27), we deduce θ_0.
2. Using relations (2.14)-(2.17), and (2.22), one constructs the polynomials $R_{2N}(\lambda)$, $B_{N,0}(\lambda)$, and $K_{N,0}(\lambda)$.
3. The polynomial $S_{N,0}(\lambda)$ is then determined by the formula (3.12).
4. From equality (3.11) we obtain $B_{N,k}(\lambda)$, whose roots $\xi_{j,k}$, $j = 1, \ldots, N$, are subsequently computed.
5. Applying the trace formula (3.7), we determine the value of (θ_k).

4. Evolution of spectral parameters

Theorem 4.1. *Let $\{\theta_n(t)\}_{n\in\mathbb{Z}}$ be a solution of (1.1)–(1.2). Then the periodic and antiperiodic spectra of the associated spectral problem (1.3) remain invariant under the time evolution. Furthermore, the spectral parameters $\xi_j(t)$, $j = 1,\dots,N$, satisfy the following evolution equations:*

$$\dot{\xi}_j(t) = \frac{\omega_0(1 + (-1)^{N+1}\prod_{j=1}^{N}\xi_j)tg^2\frac{\theta_0}{2})\sigma_j\sqrt{\widetilde{R}_{2N}}}{tg\frac{\theta_0}{2}\prod_{k\neq j}^{N}(\xi_j-\xi_k)} \tag{4.1}$$

Proof Using the expression (2.13) for the monodromy matrix M_N, he derivative of M_N with respect to t takes

$$\dot{M}_N = \frac{d(L_{N-1}L_{N-2}L_{N-3}\cdots L_0)}{dt}. \tag{4.2}$$

Using the Lax representation

$$\dot{L}_n = H_{n+1}L_n - L_nH_n.$$

for the derivative of L_j, $j=\overline{1,\,N}$ in equality (4.2) and after simplifications we have

$$\dot{M}_n = \frac{d(L_{n-1}\cdot L_{n-2}\cdot ...L_1\cdot L_0)}{dt} = \frac{dL_{n-1}}{dt}L_{n-2}\cdot ...L_1\cdot L_0 + L_{n-1}\frac{dL_{n-2}}{dt}\cdot ...L_1\cdot L_0 + \\ +... + L_{n-1}L_{n-2}\cdot ...L_1\cdot\frac{dL_0}{dt}$$

Due to the periodicity in N, we get

$$\dot{M}_N = (H_NL_{N-1} - L_{N-1}H_{N-1})\cdot L_{N-2}\cdot ...L_1\cdot L_0 + L_{N-1}\cdot(H_{N-1}L_{N-2} - L_{N-2}H_{N-2})\cdot ...L_1\cdot L_0 + \\ +L_{N-1}\cdot L_{N-2}...L_1\cdot(H_0L_0 - L_0H_0) = H_NL_{N-1}\cdot L_{N-2}\cdot ...L_1\cdot L_0 - L_{N-1}\cdot L_{N-2}\cdot ...L_1\cdot L_0\cdot H_0.$$

As a result, we have the equality

$$\dot{M}_N = H_NM_N - M_NH_0 \tag{4.3}$$

and we find that

$$\dot{s}_{N,1} = \omega_N(s_{N,2} - c_{N,1}). \tag{4.4}$$

Using (2.2), (2.26), (2.24) and (2.25) in the equality (4.4) we get (4.1).

Lemma 4.1. *$\lambda_k(t)$ is independent of the parameter t for all $k = \overline{1,\,2N}$.*

Proof Demonstrating that $\lambda_k(t)$ is independent of the parameter t, is equivalent to proving that the derivative of $\Delta(z)$ with respect to t is zero. Since the equalities (4.3) and

$$\Delta(z,t) = \mathrm{tr}(M_N)$$

hold, the derivative of $\Delta(z)$ with respect to t gives

$$\dot{\Delta}(z) = \mathrm{tr}\,\dot{M}_N = 0. \tag{4.5}$$

Relation (4.5) implies that the zeros of $\Delta^2(z) - 4 = 0$ remain constant in t.

Conclusion

In this paper, we investigated the periodic differential–difference sine–Gordon equation by means of the inverse spectral problem method. We established the spectral theory for the associated periodic discrete linear system and obtained explicit polynomial representations of its fundamental solutions. Based on these representations, we derived trace formulas that reconstruct the potential from the spectral parameters.

Furthermore, by introducing the shifted monodromy matrix, we developed a constructive algorithm for solving the inverse spectral problem and recovering the periodic solution from the spectral data. Finally, we proved the invariance of the periodic and antiperiodic spectra under the time evolution and derived a discrete analogue of the Dubrovin equations governing the evolution of the auxiliary spectral parameters.

The obtained results provide a complete inverse spectral characterization of the periodic differential–difference sine–Gordon equation and establish a finite-gap integration framework for this discrete integrable system.

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Author's contributions

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