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Research article

Periodic Gibbs measures for the soft-core Widom–Rowlinson model and their extremality

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Abstract

We consider both hard-core and soft-core Widom–Rowlinson models with spin values $\{-1,0,1\}$ on a Cayley tree of order $k \geq 2$, focusing on the Gibbs measures of these models. The models depend on three parameters: the order k of the tree, the interaction strength θ (describing $J<0$ and $J>0$ cases), and the particle intensity λ. The hard-core Widom–Rowlinson model corresponds to the case $\theta = 0$.

For $k = 2$ and $J > 0$, we prove the existence of two and four non-translation-invariant (non-TI) periodic Gibbs measures within the corresponding regions of the parameters θ and λ. Since the explicit forms of the four solutions to the system of functional equations are obtained, the conditions for the extremality and non-extremality of these periodic Gibbs measures are established. In the case $k = 3$ and $J > 0$, under the condition $\theta > 2$, explicit expressions for two critical values $\lambda_{\mathrm{cr},i}(\theta)$ $(i = 1, 2)$ are derived. Using these critical values, we show that there exist exactly two or exactly four non-translation-invariant periodic Gibbs measures in the corresponding regions of the parameter space.

Keywords: *Cayley tree, Widom–Rowlinson model, Gibbs-measure, translation-invariant Gibbs measure, periodic Gibbs measure, extremality, non-extremality, phase transition.*

AMS Subject Classification (2020): *Primary:82B20; Secondary: 60K35, 82B26, 37B10.*

1. Introduction

The Widom–Rowlinson model was originally introduced in Euclidean space as a model of point particles carrying positive or negative charges [20]. In its hard-core version, particles with opposite signs are prohibited from being located within a prescribed distance of each other. Under this hard-core constraint on a graph, particles of opposite signs cannot occupy adjacent vertices. By relaxing the hard-core constraint, one obtains the soft-core variant. The analysis of this model becomes more intricate due to the presence of an additional parameter that controls the ferromagnetic or antiferromagnetic interaction between particles of opposite signs.

Let us now focus on models defined on trees. For earlier studies of such models, we refer to [1, 2, 3]. This approach relies on a one-to-one correspondence between boundary laws and splitting Gibbs measures. In the Widom–Rowlinson model, boundary laws are represented by two-dimensional positive vectors that satisfy a nonlinear recurrence relation on the tree. Each boundary law determines a corresponding transition matrix for a Markov chain indexed by the tree [7, 17]. The problem of establishing the extremality of Gibbs measures is considerably more delicate and is usually studied separately [5, 11, 23, 24].

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In [21], Gibbs measures for the hard- and soft-core Widom–Rowlinson models on a Cayley tree were investigated in detail. In particular, a system of functional equations associated with the model was derived, and necessary and sufficient conditions for the existence and uniqueness of translation-invariant Gibbs measures were established. Moreover, the existence of multiple Gibbs measures for certain parameter values was proved, and the occurrence of phase transitions was demonstrated.

This paper is a direct continuation of the studies initiated in [21], where periodic Gibbs measures for the soft-core Widom–Rowlinson model on a Cayley tree were investigated. Positive solutions of the system of functional equations associated with the model are analyzed. Based on these equations, the existence of splitting periodic Gibbs measures is studied. Furthermore, conditions for the extremality and non-extremality of the obtained periodic Gibbs measures are established.The structure of the paper is organized as follows. In Section 2, we introduce the basic definitions and formulate the problem. Section 3 is devoted to translation-invariant splitting Gibbs measures (TISGMs), where some known results from [21] are recalled. Section 4 deals with periodic Gibbs measures; in particular, for a fixed invariant set, we identify non-translation-invariant periodic Gibbs measures for $k = 2$ and $k = 3$. Finally, Sections 5 and 6 are devoted to establishing conditions for the extremality and non-extremality of the measures obtained in the case $k = 2$.

2. Preliminaries

A Cayley tree $\Im^k$ of order $k \geq 1$ is an infinite, connected, acyclic regular graph in which each vertex has degree $k+1$. Let $\Im^k = (V, L, i)$, where V is the set of vertices $\Im^k$, L is the set of its edges, and i is the incidence function that assigns, to every edge $l \in L$, its endpoints $x, y \in V$. If $i(l) = \{x, y\}$, then x and y are called the *nearest neighbor vertices* and are denoted by $l = \langle x, y\rangle$. The set of nearest neighbor pairs $\langle x, x_1\rangle, \langle x_1, x_2\rangle, \dots, \langle x_{d-1}, y\rangle$ is called a *path* from x to y. The distance $d(x, y)$ between vertices x and y on a Cayley tree is the number of edges of the shortest path connecting the vertices x and y, i.e., $d(x, y) = \min\{d \mid \exists x = x_0, x_1, \dots, x_{d-1}, x_d = y \in V$ such that $\langle x_0, x_1\rangle, \dots, \langle x_{d-1}, x_d\rangle\}$.

For fixed $x^0 \in V$, we set

$$W_n = \{x \in V \mid d(x, x^0) = n\}, \quad V_n = \{x \in V \mid d(x, x^0) \leq n\}, \quad L_n = \{l = \langle x, y\rangle \in L \mid x, y \in V_n\}.$$

A point y is called a direct successor of a point x if $x \prec y$ and $d(x, y) = 1$. Note that in $\Im^k$, every vertex $x \neq x^0$ has k direct successors, and the vertex x^0 has $k+1$ successors. The set of direct successors of a vertex x is denoted by $S(x)$, i.e., if $x \in W_n$, then

$$S(x) = \{y_i \in W_{n+1} \mid d(x, y_i) = 1, i = 1, 2, \dots, k\}.$$

Let $\Phi = \{-1, 0, 1\}$ and $\sigma \in \Omega = \Phi^V$ be a configuration, i.e., $\sigma = \{\sigma(x) \in \Phi : x \in V\}$. In other words, in this model, to every vertex x, there is assigned one of the values $\sigma(x) \in \Phi = \{-1, 0, 1\}$. The set of all configurations on V is denoted by Ω. Similarly, one can define configurations in V_n (W_n, resp.), and the set of all configurations in V_n (W_n, resp.) is denoted by Ω_{V_n} (Ω_{W_n}, resp.).

Consider the set Φ as the set of vertices of some graph G. The activity set for the graph G is the function $\lambda : G \to \mathbb{R}_+$ from the set G to the set of positive real numbers. The value of λ_i of the function λ at a vertex $i \in \{-1, 0, 1\}$ is called its *activity*.

It is known from [21] that the Hamiltonian for the Soft-Core Widom–Rowlinson (SCWR) model of step n, i.e. on the configuration set Ω_{V_n}, is given by

$$H_n^{sc}(\sigma) = J \sum_{\{x,y\} \in L_n} \mathbf{1}(\sigma(x)\sigma(y) = -1) - \frac{\ln \lambda}{\beta} \sum_{x \in V_n} \sigma^2(x),$$

where $\beta = \frac{1}{T}$ and $T > 0$ is the temperature. The parameter $J \in \mathbb{R}$ can be seen as repulsion or attraction between particles of different charges depending on the sign of J and $\lambda > 0$ as an activity.

The associated finite volume Gibbs measure on Ω_{V_n} with external fields on the boundary, $\{h^x = (h_{-1,x}, h_{0,x}, h_{1,x}) \in \mathbb{R}^3 : x \in \Im^k\}$ is defined by

$$\mu_{n,\beta}(\sigma) = \frac{1}{Z_n} \exp\left(-\beta H_n^{sc}(\sigma) + \sum_{x \in W_n} h_{\sigma(x),x} \right).$$

The sequence $\mu_{n,\beta}$, $n \geq 1$ is said to be compatible if for all $n \geq 2$ and all $\sigma^{n-1} \in \Omega_{V_{n-1}}$ as well as $\omega^n \in \Omega_{W_n}$

$$\sum_{\omega^n \in \Omega_{W_n}} \mu_{n,\beta}(\sigma^{n-1} \vee \omega^n) = \mu_{n-1,\beta}(\sigma^{n-1})$$

holds. Here

$$\sigma^{n-1} \vee \omega^n = \begin{cases} \sigma^{n-1}, & \text{if } x \in V_{n-1}, \\ \omega^n, & \text{if } x \in W_n. \end{cases}$$

For a sequence of compatible finite volume Gibbs measures $\{\mu_{n,\beta}\}_{n=1}^{\infty}$, by Kolmogorov's extension theorem (see [25]), there exists a unique measure μ defined on the whole tree with $\mu(\sigma|_{V_n} = \sigma^n) = \mu_{n,\beta}(\sigma^n)$. Following [15], we call μ a splitting Gibbs measure. On a Cayley tree, the absence of cycles allows for a recursive decomposition of the system via a system of functional equations (boundary laws). Consequently, splitting Gibbs measures (SGMs) are closely related to limiting Gibbs measures (LGMs), often coinciding with physical LGMs under appropriate boundary conditions. In contrast, on the $\mathbb{Z}^d$ lattice, the presence of loops prevents any splitting property, meaning the concept of an SGM is absent. In this setting, DLR states or generic LGMs serve as the fundamental objects obtained via the thermodynamic limit. Thus, while the concept of SGMs closely coincides with that of LGMs on trees, these two notions are fundamentally distinct on lattices.

The finite volume Gibbs measure for the Hard-Core Widom–Rowlinson model we get for $J > 0$ by $\lim_{\beta\to\infty} \mu_{n,\beta}^{sc} = \mu_n^{hc}$.

The following theorem gives conditions to make the finite volume Gibbs measures compatible.

Theorem 2.1. *[21] Let $0 < \beta < \infty$, $J \in \mathbb{R}$ and $\lambda > 0$. The sequence of probability measures $\{\mu_{n,\beta}^{sc}\}_{n=1}^{\infty}$ is compatible if and only if for any $x \in V \setminus \{x^0\}$ the following system of equations holds:*

$$\begin{cases} \tilde{h}_{+,x} = \ln\lambda + \sum_{y\in S(x)} f(\tilde{h}_{+,y}, \tilde{h}_{-,y}, \theta), \\ \tilde{h}_{-,x} = \ln\lambda + \sum_{y\in S(x)} f(\tilde{h}_{-,y}, \tilde{h}_{+,y}, \theta), \end{cases} \tag{2.1}$$

where $\theta = \exp(-J\beta)$,

$$f(x, y, \theta) = \ln \frac{1 + e^x + \theta e^y}{1 + e^x + e^y}$$

and $\tilde{h}_{\pm,y} = \ln\lambda + h_{\pm1,y} - h_{0,y}$.

3. Translation-invariant Gibbs measures

We seek translation-invariant solutions to the system (2.1). In the case of a TISGM, the external field vectors h^x are independent of x; that is, $h^x = h = (h_1, h_2, h_3) \in \mathbb{R}^3$ for all $x \in \Im^k$. The measure corresponding to such a solution is a TISGM. By introducing two new variables, $z_1 = e^{\tilde{h}_+}$ and $z_2 = e^{\tilde{h}_-}$, system (2.1) can be reduced to the following system of equations:

$$\begin{cases} z_1 = \lambda \left(\dfrac{1 + z_1 + \theta z_2}{1 + z_1 + z_2} \right)^k, \\ z_2 = \lambda \left(\dfrac{1 + \theta z_1 + z_2}{1 + z_1 + z_2} \right)^k. \end{cases} \tag{3.1}$$

The following results are known from [21].

- For the SCWR model with $J < 0$, there exist a critical value $\theta_{\rm cr}(k)$ and two critical values $\lambda_{{\rm cr},i}(k,\theta)$ $(i = 1, 2)$ such that for the number N_1 of TISGMs, the following assertions hold:

$$N_1 = \begin{cases} 1, & if \quad \theta \le \theta_{\rm cr} \; or \; \theta > \theta_{\rm cr} \; and \; \lambda \in (0, \lambda_{{\rm cr},2}) \cup (\lambda_{{\rm cr},1}, +\infty), \\ 2, & if \quad \theta > \theta_{\rm cr} \; and \; \lambda \in \{\lambda_{{\rm cr},2}, \lambda_{{\rm cr},1}\}, \\ 3, & if \quad \theta > \theta_{\rm cr} \; and \; \lambda \in (\lambda_{{\rm cr},2}, \lambda_{{\rm cr},1}), \end{cases}$$

where $\theta_{cr} = 2\left(\frac{k+1}{k-1}\right)^2 - 1$ and $\lambda_{cr,i} := \lambda_{cr,i}(k,\theta) = \frac{2^k z_i}{(1+\theta)^{k+1}} \left(\frac{1+\theta+2z_i}{2(1+z_i)}\right)^k$, where z_1 and z_2 are the solutions of the equation $2z^2 + [4 - (\theta - 1)(k - 1)]z + \theta + 1 = 0$.

- For the SCWR model with $J > 0$ and $k = 2$, there exists a critical value $\lambda_{\rm cr}(2) := \frac{1}{1-3\theta}\left(\frac{3}{2}\right)^2$ such that for the number N_2 of TISGMs, the following assertions hold:

$$N_2 = \begin{cases} 1, & if \quad \theta \ge \frac{1}{3} \; or \; \theta < \frac{1}{3} \; and \; \lambda \le \lambda_{\rm cr}(2), \\ 3, & if \quad \theta < \frac{1}{3} \; and \; \lambda > \lambda_{\rm cr}(2). \end{cases}$$

- For the SCWR model with $J > 0$ and $k = 3$, there exists a critical value $\lambda_{\rm cr}(3) := \frac{1}{2-4\theta}\left(\frac{4}{3}\right)^3$ such that for the number N_3 of TISGMs, the following assertions hold:

$$N_3 = \begin{cases} 1, & if \quad \theta \ge \frac{1}{2} \; or \; \theta < \frac{1}{2} \; and \; \lambda \le \lambda_{\rm cr}(3), \\ 3, & if \quad \theta < \frac{1}{2} \; and \; \lambda > \lambda_{\rm cr}(3). \end{cases}$$

- For the SCWR model with $J > 0$ and $k \ge 4$, the following assertions hold:
  1) If $\theta \ge \theta'_{\rm cr}(k)$, or if $\theta < \theta'_{\rm cr}(k)$ and $\lambda < \lambda'_{\rm cr}(k)$, then there exists a unique TISGM;
  2) If $\theta'_{\rm cr}(k) > \theta \ge \theta_{\rm cr}(k)$, or if $\theta < \theta_{\rm cr}(k)$ and $\lambda'_{\rm cr}(k) < \lambda \le \lambda_{\rm cr}(k)$, then there exists at least one TISGM;
  3) For $\theta < \theta_{\rm cr}(k)$ and $\lambda > \lambda_{\rm cr}(k)$, there exist at least three TISGMs,

  where

$$\theta'_{\rm cr}(k) := \frac{k-1}{k}, \quad \lambda'_{\rm cr}(k) = \frac{1}{k-1-k\theta}, \quad \text{and} \quad \lambda_{\rm cr}(k) = \frac{1}{k-1-\theta(k+1)}\left(\frac{k+1}{k}\right)^k.$$

Remark 3.1. For $\theta = 0$, the system of equations (3.1) reduces to the three-state hard-core model studied in [16, 18], which corresponds to the case where the activity graph G is a hinge graph.

4. Periodic Gibbs Measures in the Case $J > 0$

In this section, we study periodic Gibbs measures for the SCWR model.

It is well known that there exists a one-to-one correspondence between the vertex set V of a Cayley tree of order $k \geq 1$ and the group G_k, which is the free product of $k+1$ cyclic groups of order two with generators $a_1, a_2, \ldots, a_{k+1}$ (see [10]). Each element $x \in G_k$ (except for the identity element e) can be uniquely represented as a reduced word $x = a_{i_1} a_{i_2} \ldots a_{i_n}$, where $i_m \in \{1, \ldots, k+1\}$ for all $1 \leq m \leq n$ and $i_m \neq i_{m+1}$ for all $1 \leq m \leq n-1$. The number n of generators in this representation is called the length of the word x and is denoted by $|x|$.

Let $\widehat{G}_k$ be a normal subgroup of G_k of finite index $r \geq 1$, and let $G_k/\widehat{G}_k = \{H_1, \ldots, H_r\}$ denote the corresponding quotient group.

Definition 4.1. A collection of quantities $z = \{z_x, x \in G_k\}$ is said to be $\widehat{G}_k$-periodic if $z_{yx} = z_x$ for $\forall x \in G_k, y \in \widehat{G}_k$.

Definition 4.2. A measure μ is called $\widehat{G}_k$-periodic if it corresponds to a $\widehat{G}_k$-periodic collection of quantities z.

We rewrite the system of equations (2.1) as follows:

$$\begin{cases} h_{+,x} = \ln \lambda + \sum\limits_{y \in S(x)} \ln \dfrac{1 + e^{h_{+,y}} + \theta e^{h_{-,y}}}{1 + e^{h_{+,y}} + e^{h_{-,y}}}, \\ h_{-,x} = \ln \lambda + \sum\limits_{y \in S(x)} \ln \dfrac{1 + \theta e^{h_{+,y}} + e^{h_{-,y}}}{1 + e^{h_{+,y}} + e^{h_{-,y}}}, \end{cases} \tag{4.1}$$

where $h_{+,x} = \ln z_{1,x}$ and $h_{-,x} = \ln z_{2,x}$. We consider periodic solutions of the system (4.1).

Let the function $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ with $h = (h_+, h_-) \mapsto F(h) = (F_1(h), F_2(h))$ be defined by

$$F_1(h) = \ln \frac{1 + e^{h_+} + \theta e^{h_-}}{1 + e^{h_+} + e^{h_-}}, \quad F_2(h) = \ln \frac{1 + \theta e^{h_+} + e^{h_-}}{1 + e^{h_+} + e^{h_-}}.$$

Then, we have the following assertion.

Proposition 4.1. *The function $F(h)$ is injective.*

Proof. We show that $F(h) = F(l)$ if and only if $h_+ = l_+$ and $h_- = l_-$.
Necessity. Let $F(h) = F(l)$. Then, $F_1(h) = F_1(l)$ and $F_2(h) = F_2(l)$, where $h = (h_+, h_-)$ and $l = (l_+, l_-)$. From these relations, we obtain the following system of equations:

$$\begin{cases} (z_1 - s_1)s_2 - (z_2 - s_2)(s_1 + 1) = 0, \\ (z_1 - s_1)(s_2 + 1) - (z_2 - s_2)s_1 = 0, \end{cases} \tag{4.2}$$

where $z_1 = e^{h_+}$, $z_2 = e^{h_-}$, $s_1 = e^{l_+}$ and $s_2 = e^{l_-}$. It is easy to see that the determinant of system (4.2) is nonvanishing:

$$\Delta = \begin{vmatrix} s_2 & -(s_1 + 1) \\ s_2 + 1 & -s_1 \end{vmatrix} = s_1 + s_2 + 1 > 0.$$

Therefore, system (4.2) has a unique solution $z_i = s_i$ for $i = 1, 2$.
Sufficiency is obvious. This completes the proof.

Let $G_k^{(2)}$ be the subgroup of G_k consisting of words of even length.

Theorem 4.1. *Let H_0 be a normal subgroup of a finite index in G_k. Then, for the SCWR model, any H_0-periodic splitting Gibbs measure is either translation-invariant or $G_k^{(2)}$-periodic.*

Proof. This theorem can be proved by using Proposition 4.1., similarly to the proof of Theorem 2 in [12]. This completes the proof.

It follows from Theorem 4.1. that if periodic Gibbs measures exist, then they are $G_k^{(2)}$-periodic.

In [21], the following system of functional equations for periodic Gibbs measures was studied:

$$\begin{cases} z_1 = \lambda\left(\dfrac{1+t_1+\theta t_2}{1+t_1+t_2}\right)^k, & z_2 = \lambda\left(\dfrac{1+t_2+\theta t_1}{1+t_1+t_2}\right)^k, \\ t_1 = \lambda\left(\dfrac{1+z_1+\theta z_2}{1+z_1+z_2}\right)^k, & t_2 = \lambda\left(\dfrac{1+z_2+\theta z_1}{1+z_1+z_2}\right)^k. \end{cases} \tag{4.3}$$

Consider the mapping $W : \mathbb{R}^4 \to \mathbb{R}^4$ defined by the following system of equations:

$$\begin{cases} z_1' = \lambda\left(\dfrac{1+t_1+\theta t_2}{1+t_1+t_2}\right)^k, \\ z_2' = \lambda\left(\dfrac{1+t_2+\theta t_1}{1+t_1+t_2}\right)^k, \\ t_1' = \lambda\left(\dfrac{1+z_1+\theta z_2}{1+z_1+z_2}\right)^k, \\ t_2' = \lambda\left(\dfrac{1+z_2+\theta z_1}{1+z_1+z_2}\right)^k. \end{cases} \tag{4.4}$$

Note that the system of equations (4.3) can be written as $z = W(z)$. Thus, to solve the system of equations (4.3), it is necessary to find the fixed points of the mapping W.

The following lemma holds.

Lemma 4.1. *The following sets are invariant with respect to the mapping W:*

$$I_1 = \{(z_1, z_2, t_1, t_2) \in \mathbb{R}^4 : z_1 = z_2 = t_1 = t_2\},$$

$$I_2 = \{(z_1, z_2, t_1, t_2) \in \mathbb{R}^4 : z_1 = z_2,\ t_1 = t_2\},$$

$$I_3 = \{(z_1, z_2, t_1, t_2) \in \mathbb{R}^4 : z_1 = t_1,\ z_2 = t_2\},$$

$$I_4 = \{(z_1, z_2, t_1, t_2) \in \mathbb{R}^4 : z_1 = t_2,\ z_2 = t_1\}.$$

The following theorem is known.

Theorem 4.2. *[21] For $k \geq 6$, $\theta < \frac{k^2-6k+1}{(k+1)^2}$ and $\lambda \in (\lambda^-(k,\theta), \lambda^+(k,\theta))$ on I_2 there are at least three $G_k^{(2)}$-periodic SGMs μ_0, μ_1, μ_2, where*

$$\lambda^{\pm}(k,\theta) = s^{\pm} \cdot \left(\frac{1+2s^{\pm}}{1+(1+\theta)s^{\pm}}\right)^k, \quad s^{\pm} = \frac{k-3-(k+1)\theta \pm \sqrt{(1-\theta)[k^2-6k+1-(k+1)^2\theta]}}{4(1+\theta)}.$$

For the invariant sets I_1, I_2, and I_3, the following theorem holds.

Theorem 4.3. *For the SCWR model, the following assertions hold:*

1. *Let $k \geq 2$. On the invariant sets I_1 and I_3, any $G_k^{(2)}$-periodic SGM is a TISGM.*

2. *Let $k \in \{2,3,4,5\}$ and $J < 0$. On the invariant set I_2, any $G_k^{(2)}$-periodic SGM coincides with a TISGM.*

Proof.

1. The assertions for the cases I_1 and I_3 are obvious.
2. Let us consider the case I_2. In this case, the system of equations (4.3) takes the form:

$$\begin{cases} z_1 = \lambda \left(\dfrac{1 + (1+\theta) z_2}{1 + 2z_2} \right)^k, \\ z_2 = \lambda \left(\dfrac{1 + (1+\theta) z_1}{1 + 2z_1} \right)^k. \end{cases} \tag{4.5}$$

Assume that $z_1 > 0$, $z_2 > 0$, $\lambda > 0$, and $\theta \in (0,1) \cup (1,\infty)$. We show that the system of equations (4.5) admits a solution of the form $z_1 = z_2$.

Introducing the notation $x = \sqrt[k]{z_1}$ and $y = \sqrt[k]{z_2}$, we rewrite the system of equations (4.5) as

$$\begin{cases} x = \sqrt[k]{\lambda} \left(\dfrac{1 + (1+\theta) y^k}{1 + 2y^k} \right), \\ y = \sqrt[k]{\lambda} \left(\dfrac{1 + (1+\theta) x^k}{1 + 2x^k} \right). \end{cases}$$

In this system of equations, dividing the first equation by the second and performing some algebraic transformations, we obtain

$$(x-y)\left(1 - 2xy \sum_{i=0}^{k-2} x^{k-2-i} y^i + (1+\theta) \sum_{i=0}^{k} x^{k-i} y^i + 2(1+\theta) x^k y^k \right) = 0. \tag{4.6}$$

It follows that $x = y$ or

$$1 - 2xy \sum_{i=0}^{k-2} x^{k-2-i} y^i + (1+\theta) \sum_{i=0}^{k} x^{k-i} y^i + 2(1+\theta) x^k y^k = 0. \tag{4.7}$$

In the first case, the system of equations (4.5) has a unique solution, and the measure corresponding to this solution coincides with the TISGM.

In the second case, we rewrite equation (4.7) with respect to θ as follows:

$$\theta(k) = \frac{2xy \sum_{i=0}^{k-2} x^{k-2-i} y^i - 1}{2x^k y^k + \sum_{i=0}^{k} x^{k-i} y^i} - 1$$

and we show that this expression for θ is always negative for $k \in \{2,3,4,5\}$.

Indeed,

$$\theta(2) = -\frac{2x^2y^2 + xy + 1 + (x-y)^2}{2x^2y^2 + x^2 + xy + y^2} < 0, \qquad \theta(3) = -\frac{2x^3y^3 + (x+y)(x-y)^2 + 1}{2x^3y^3 + x^3 + x^2y + xy^2 + y^3} < 0,$$

$$\theta(4) = -\frac{x^4y^4 + (x-y)^2(x^2+xy+y^2) + (x^2y^2 - \frac{1}{2})^2 + \frac{3}{4}}{2x^4y^4 + x^4 + x^3y + x^2y^2 + xy^3 + y^4} < 0,$$

$$\theta(5) = -\frac{x^5y^5 + (x-y)^2(x^3 + xy^2 + y^3)}{2x^5y^5 + x^5 + x^4y + x^3y^2 + x^2y^3 + xy^4 + y^5} < 0.$$

Hence, equation (4.6) has solutions only of the form $x = y$ for $k = 2,3,4,5$. Therefore, on I_2 and for $k \in \{2,3,4,5\}$, any $G_k^{(2)}$-periodic SGM is a TISGM. This completes the proof of Theorem 4.3.

The case I_4. In this case, from the system of equations (4.3), we obtain the following system:

$$\begin{cases} z_1 = \lambda \left(\frac{1+z_2+\theta z_1}{1+z_1+z_2}\right)^k, \\ z_2 = \lambda \left(\frac{1+z_1+\theta z_2}{1+z_1+z_2}\right)^k. \end{cases} \tag{4.8}$$

Subtracting the second equation of (4.8) from the first and performing some algebraic transformations, we obtain

$$(z_1 - z_2)\left(1 - \frac{\lambda}{(1+z_1+z_2)^k}(\theta-1)\left[\sum_{i=0}^{k-1}(1+z_2+\theta z_1)^{k-1-i}(1+z_1+\theta z_2)^i\right]\right) = 0. \tag{4.9}$$

If $\theta \in (0,1)$, then equation (4.9) has a unique solution $z_1 = z_2$, which corresponds to a TISGM.
Let $\theta > 1$. Then, from equation (4.9), it follows that $z_1 = z_2$ or

$$(1+z_1+z_2)^k = \lambda(\theta-1)\left[\sum_{i=0}^{k-1}(1+z_2+\theta z_1)^{k-1-i}(1+z_1+\theta z_2)^i\right]. \tag{4.10}$$

Now we solve equation (4.10) in the case $k = 2$. Then, we have

$$(1+z_1+z_2)^2 = \lambda(\theta-1)\left[1-\theta+(1+\theta)(1+z_1+z_2)\right]. \tag{4.11}$$

Solving this quadratic equation with respect to $1+z_1+z_2$, we obtain

$$1+z_1+z_2 = \frac{(\theta-1)}{2}\cdot g_{\pm}(\theta,\lambda), \tag{4.12}$$

where $g_{\pm}(\theta,\lambda) = \lambda(\theta+1) \pm \sqrt{\lambda^2(\theta+1)^2 - 4\lambda} :> 0$.
Substituting this formula into the first equation of (4.8), we get

$$z_1 = \lambda\left(\frac{2z_1 + g_{\pm}(\theta,\lambda)}{g_{\pm}(\theta,\lambda)}\right)^2 \quad \text{or} \quad z_1^2 + \left(g_{\pm}(\theta,\lambda) - \frac{g_{\pm}^2(\theta,\lambda)}{4\lambda}\right)z_1 + \frac{g_{\pm}^2(\theta,\lambda)}{4} = 0. \tag{4.13}$$

Equation (4.13) has the following solutions:

$$z_1^{(i)} = \frac{1}{16\lambda}\cdot\left(g_+(\theta,\lambda) \pm \sqrt{g_+^2(\theta,\lambda) - 8\lambda g_+(\theta,\lambda)}\right)^2, \quad i = 1,2,$$

$$z_1^{(j)} = \frac{1}{16\lambda}\cdot\left(g_-(\theta,\lambda) \pm \sqrt{g_-^2(\theta,\lambda) - 8\lambda g_-(\theta,\lambda)}\right)^2, \quad j = 3,4.$$

To determine the existence domain of the above solutions, we solve the inequality:

$$[g_{\pm}(\theta,\lambda) - 8\lambda]\cdot g_{\pm}(\theta,\lambda) \geq 0 \quad \text{or} \quad g_{\pm}(\theta,\lambda) - 8\lambda \geq 0.$$

The solutions $z_1^{(i)}$ $(i = 1,2)$ determined above are positive on M_1, M_2, and M_3, while the solutions $z_1^{(j)}$ $(j = 3,4)$ are positive on M_2, where

$$M_1 = \left\{(\theta,\lambda): \quad \theta \in (3,7), \quad \lambda \in \left(\frac{1}{4(\theta-3)}, \infty\right)\right\},$$

$$M_2 = \left\{(\theta,\lambda): \quad \theta \in [7,\infty), \quad \lambda \in \left[\frac{4}{(\theta+1)^2}, \frac{1}{4(\theta-3)}\right]\right\},$$

$$M_3 = \left\{(\theta,\lambda): \quad \theta \in [7,\infty), \quad \lambda \in \left(\frac{1}{4(\theta-3)}, \infty\right)\right\}.$$

Substituting the obtained solutions $z_1^{(i)}$ and $z_1^{(j)}$ into (4.12), we find the corresponding values of z_2. Thus, equation (4.11) has the following solutions:

a) On the set M_1, there exist two solutions: $(z_1^{(1)}, z_1^{(2)})$ and $(z_1^{(2)}, z_1^{(1)})$.
b) On the set M_2, there exist four solutions: $(z_1^{(1)}, z_1^{(2)})$, $(z_1^{(2)}, z_1^{(1)})$, $(z_1^{(3)}, z_1^{(4)})$, and $(z_1^{(4)}, z_1^{(3)})$.
c) On the set M_3, there exist two solutions: $(z_1^{(1)}, z_1^{(2)})$ and $(z_1^{(2)}, z_1^{(1)})$.

Let μ_1, μ_2, μ_3, and μ_4 be the measures corresponding to the solutions $(z_1^{(1)}, z_1^{(2)})$, $(z_1^{(2)}, z_1^{(1)})$, $(z_1^{(3)}, z_1^{(4)})$, and $(z_1^{(4)}, z_1^{(3)})$, respectively.

The following theorem holds.

Theorem 4.4. *Let $k = 2$. For the SCWR model on I_4, the following assertions hold:*

1. *Let $0 < \theta < 1$ and $\lambda > 0$. Then any $G_k^{(2)}$-periodic Gibbs measure coincides with the TISGM.*
2. *If $\theta > 1$, then for the number $N_{per}(2)$ of $G_k^{(2)}$-periodic non-TI Gibbs measures, the following assertions hold*

$$N_{per}(2) = \begin{cases} \emptyset & \textit{if} \quad \theta \in (1,3] \textit{ and } \lambda \in (0, \ \infty), \\ 2 \ (\mu_1, \mu_2), & \textit{if} \quad \theta \in (3,\infty) \textit{ and } \lambda \in \left(\frac{1}{4(\theta-3)}, \ \infty\right), \\ 4 \ (\mu_1, \mu_2, \mu_3, \mu_4), & \textit{if} \quad \theta \in [7,\infty) \textit{ and } \lambda \in \left[\frac{4}{(\theta+1)^2}, \ \frac{1}{4(\theta-3)}\right]. \end{cases}$$

Case $k = 3$. In this case, setting $u = \sqrt[3]{z_1} > 0$, $v = \sqrt[3]{z_2} > 0$, and $a = \sqrt[3]{\lambda}$, we rewrite the system of equations (4.8) in the following form:

$$\begin{cases} u = a\frac{1+v^3+\theta u^3}{1+u^3+v^3}, \\ v = a\frac{1+u^3+\theta v^3}{1+u^3+v^3}. \end{cases} \tag{4.14}$$

Let us denote the sum and the product of u and v by

$$s = u + v, \quad p = uv. \tag{4.15}$$

Assuming $u \neq v$, we divide the first equation of (4.14) by the second to obtain

$$1 + (u+v)(u^2+v^2) - \theta uv(u+v) = 0. \tag{4.16}$$

By using the algebraic identities $u^2 + v^2 = s^2 - 2p$ and $u^3 + v^3 = s(s^2 - 3p)$, from (4.16) we obtain the following expression for p:

$$p = \frac{1+s^3}{(\theta+2)s}. \tag{4.17}$$

Adding the first equation of system (4.14) to the second, we have

$$u + v = a \cdot \frac{2 + (1+\theta)(u^3+v^3)}{1+u^3+v^3}.$$

Using (4.15), this equality can be written as

$$s = a \cdot \frac{2 + (1+\theta)s(s^2-3p)}{1+s(s^2-3p)}. \tag{4.18}$$

Substituting the expression for p from (4.17) into (4.18), we obtain

$$s = a \cdot \frac{(1+\theta)s^3 - 1}{s^3+1}. \tag{4.19}$$

Let s be a solution to equation (4.19). Then, to find the corresponding values of u and v according to (4.15), we solve the system

$$u+v=s, \quad uv=\frac{1+s^3}{(\theta+2)s}.$$

This system can be reduced to the quadratic equation

$$u^2-su+\frac{1+s^3}{(\theta+2)s}=0.$$

The discriminant of this equation is non-negative if and only if $(\theta-2)s^3-4\geq 0$. From this inequality, it follows that θ and s must satisfy $\theta>\theta_c(3):=2, \quad s^3\geq\frac{4}{\theta-2}$.

Solving equation (4.19) for a, we obtain

$$a=s\cdot\frac{s^3+1}{(1+\theta)s^3-1}=:\eta(s). \tag{4.20}$$

The derivative of the function $\eta(s)$ is given by

$$\eta'(s)=\frac{(\theta+1)s^6-2(\theta+3)s^3-1}{((1+\theta)s^3-1)^2}.$$

The equation $\eta'(s)=0$ has a unique positive solution

$$s_1=\sqrt[3]{\frac{\theta+3+\sqrt{\theta^2+7\theta+10}}{\theta+1}},$$

and the function $\eta(s)$ attains its minimum at the point s_1.

Hence, we define the critical value as

$$a_{cr}^{(1)}=\eta(s_1)=\sqrt[3]{\frac{\theta+3+\sqrt{\theta^2+7\theta+10}}{\theta+1}}\cdot\frac{\sqrt{\theta^2+7\theta+10}-\theta+1}{3(\theta+1)},$$

$$a_{cr}^{(2)}=\max_{\sqrt[3]{\frac{4}{\theta-2}}\leq s\leq s_1}\eta(s)=\eta\left(\sqrt[3]{\frac{4}{\theta-2}}\right)=\frac{2}{3}\sqrt[3]{\frac{1}{2\theta-4}}.$$

If $a<a_{cr}^{(1)}$, equation (4.20) has no solutions; if $a=a_{cr}^{(1)}$ or $a\geq a_{cr}^{(2)}$, equation (4.20) has a unique solution; and if $a_{cr}^{(1)}<a<a_{cr}^{(2)}$, equation (4.20) has two solutions.

Let us denote

$$\lambda_{cr}^{(1)}=\left[a_{cr}^{(1)}\right]^3=\frac{\theta+3+\sqrt{\theta^2+7\theta+10}}{\theta+1}\cdot\left(\frac{\sqrt{\theta^2+7\theta+10}-\theta+1}{3(\theta+1)}\right)^3,$$

$$\lambda_{cr}^{(2)}=\left[a_{cr}^{(2)}\right]^3=\frac{1}{2\theta-4}\cdot\left(\frac{2}{3}\right)^3.$$

The following theorem holds for the system of equations (4.14):

Theorem 4.5. *Let $k=3$ and $\lambda>0$. For the SCWR model on I_4, the following assertions hold:*

1. *Let $0<\theta<1$. Then any $G_k^{(2)}$-periodic Gibbs measure coincides with the TISGM.*
2. *If $\theta>1$, then for the number $N_{per}(3)$ of $G_k^{(2)}$-periodic non-TI Gibbs measures, the following assertions hold*

$$N_{per}(3)=\begin{cases}\emptyset, & \text{if} \quad 1<\theta\leq 2 \text{ or } \theta>2 \text{ and } \lambda<\lambda_{cr}^{(1)},\\ 2, & \text{if} \quad \theta>2 \text{ and } \lambda=\lambda_{cr}^{(1)} \text{ or } \theta>2 \text{ and } \lambda>\lambda_{cr}^{(2)},\\ 4, & \text{if} \quad \theta>2 \text{ and } \lambda_{cr}^{(1)}<\lambda\leq\lambda_{cr}^{(2)}.\end{cases}$$

5. Extremality and Non-Extremality of Periodic Splitting Gibbs Measures

To investigate the non-extremality of the periodic Gibbs measures, we employ the method proposed in [24]. For this purpose, we consider a Markov chain with the state space $\{-1, 0, +1\}$ and the transition probability matrix $P_\mu = (P_{\sigma(x)\sigma(y)})$, which is determined by the found Gibbs measure μ. Here, $P_{\sigma(x)\sigma(y)}$ denotes the probability of transitioning from state $\sigma(x)$ to state $\sigma(y)$:

$$P_{\sigma(x),\sigma(y)} = \frac{\exp\left\{-J\beta\left(\mathbf{1}(\sigma(x)\cdot\sigma(y)=-1) + \frac{\ln\lambda}{\beta}\cdot\sigma^2(y)\right) + h_{\sigma(y),y}\right\}}{\sum\limits_{\sigma(y)\in\{-1,0,1\}} \exp\left\{-J\beta\left(\mathbf{1}(\sigma(x)\cdot\sigma(y)=-1) + \frac{\ln\lambda}{\beta}\cdot\sigma^2(y)\right) + h_{\sigma(y),y}\right\}}.$$

From this, it follows that:

$$P_{-1,-1} = \frac{z_1}{z_1+\theta z_2+1}, \qquad P_{-1,0} = \frac{1}{z_1+\theta z_2+1}, \qquad P_{-1,1} = \frac{\theta z_2}{z_1+\theta z_2+1},$$

$$P_{0,-1} = \frac{z_1}{z_1+z_2+1}, \qquad P_{0,0} = \frac{1}{z_1+z_2+1}, \qquad P_{0,1} = \frac{z_2}{z_1+z_2+1},$$

$$P_{1,-1} = \frac{\theta z_1}{\theta z_1+z_2+1}, \qquad P_{1,0} = \frac{1}{\theta z_1+z_2+1}, \qquad P_{1,1} = \frac{z_2}{\theta z_1+z_2+1},$$

where $z_1 = \exp(h_{-1,y} - J\ln\lambda - h_{0,y})$ and $z_2 = \exp(h_{1,y} - J\ln\lambda - h_{0,y})$.

Thus, the transition probability matrix P_μ is expressed in the following form:

$$\mathbb{P}_\mu = \begin{pmatrix} \frac{z_1}{z_1+\theta z_2+1} & \frac{1}{z_1+\theta z_2+1} & \frac{\theta z_2}{z_1+\theta z_2+1} \\ \frac{z_1}{z_1+z_2+1} & \frac{1}{z_1+z_2+1} & \frac{z_2}{z_1+z_2+1} \\ \frac{\theta z_1}{\theta z_1+z_2+1} & \frac{1}{\theta z_1+z_2+1} & \frac{z_2}{\theta z_1+z_2+1} \end{pmatrix}. \tag{5.1}$$

However, in the case of periodic measures, the matrix $\mathbb{P}$ depends on $z = (z_1, z_2)$ and $t = (t_1, t_2)$ where $z \neq t$, and $\left(z_1^{(1)}, z_1^{(2)}\right)$, $\left(z_1^{(2)}, z_1^{(1)}\right)$, $\left(z_1^{(3)}, z_1^{(4)}\right)$, $\left(z_1^{(4)}, z_1^{(3)}\right)$ are the $G_k^{(2)}$ - periodic solutions (non-TI) of the system (4.8) for $k = 2$. More precisely, we have the relation $\mathbb{P} \equiv \mathbb{P}_{z,t} = \mathbf{P}_{\mu_i}$, where $\mathbf{P}_{\mu_i}$ is the transition probability matrix defined by the given $G_k^{(2)}$ - periodic Gibbs measure μ_i (see [23]). It should be noted that the matrix $\mathbf{P}_{\mu_i}$ is equal to the product of two transition probability matrices:

$$\mathbf{P}_{\mu_i} = \mathbb{P}_z\mathbb{P}_t = \begin{pmatrix} \frac{z_1}{z_1+\theta z_2+1} & \frac{1}{z_1+\theta z_2+1} & \frac{\theta z_2}{z_1+\theta z_2+1} \\ \frac{z_1}{z_1+z_2+1} & \frac{1}{z_1+z_2+1} & \frac{z_2}{z_1+z_2+1} \\ \frac{\theta z_1}{\theta z_1+z_2+1} & \frac{1}{\theta z_1+z_2+1} & \frac{z_2}{\theta z_1+z_2+1} \end{pmatrix} \cdot \begin{pmatrix} \frac{t_1}{t_1+\theta t_2+1} & \frac{1}{t_1+\theta t_2+1} & \frac{\theta t_2}{t_1+\theta t_2+1} \\ \frac{t_1}{t_1+t_2+1} & \frac{1}{t_1+t_2+1} & \frac{t_2}{t_1+t_2+1} \\ \frac{\theta t_1}{\theta t_1+t_2+1} & \frac{1}{\theta t_1+t_2+1} & \frac{t_2}{\theta t_1+t_2+1} \end{pmatrix}.$$

Using the solution (z_1, z_2, t_1, t_2) and the I_4 invariant, the transition matrix $\mathbf{P}_{\mu_i}$ can be expressed as follows:

$$\mathbf{P}_{\mu_i} = \begin{pmatrix} \frac{z_2}{C}\left(\frac{z_1}{A} + \frac{1}{B} + \frac{\theta^2 z_2}{C}\right) & \frac{1}{C}\left(\frac{z_1}{A} + \frac{1}{B} + \frac{\theta z_2}{C}\right) & \frac{z_1}{C}\left(\frac{\theta z_1}{A} + \frac{1}{B} + \frac{\theta z_2}{C}\right) \\ \frac{z_2}{B}\left(\frac{z_1}{A} + \frac{1}{B} + \frac{\theta z_2}{C}\right) & \frac{1}{B}\left(\frac{z_1}{A} + \frac{1}{B} + \frac{z_2}{C}\right) & \frac{z_1}{B}\left(\frac{\theta z_1}{A} + \frac{1}{B} + \frac{z_2}{C}\right) \\ \frac{z_2}{A}\left(\frac{\theta z_1}{A} + \frac{1}{B} + \frac{\theta z_2}{C}\right) & \frac{1}{A}\left(\frac{\theta z_1}{A} + \frac{1}{B} + \frac{z_2}{C}\right) & \frac{z_1}{A}\left(\frac{\theta^2 z_1}{A} + \frac{1}{B} + \frac{z_2}{C}\right) \end{pmatrix}, \tag{5.2}$$

where $A = \theta z_1 + z_2 + 1$, $B = z_1 + z_2 + 1$, $C = z_1 + \theta z_2 + 1$.

Consequently, the matrix $\mathbf{P}_{\mu_i}$ defines a Markov chain corresponding to a two-step transition on the Cayley tree of order k^2, which can be viewed as a chain on the tree's even-level vertices.

5.1. Non-Extremality Conditions for the Splitting Gibbs Measures

The Kesten-Stigum sufficient condition for the non-extremality of the Gibbs measure μ_i $(i = 1, 2, 3, 4)$, corresponding to the matrix $\mathbf{P}_{\mu_i}$ is given by $k^2 s_1^2 > 1$, where s_1 is the second largest eigenvalue of $\mathbf{P}_{\mu_i}$ in absolute value.

Since $\mathbf{P}_{\mu_i}$ is a stochastic matrix, its largest eigenvalue is $s_0 = 1$. The remaining two eigenvalues are given by:

$$s_1(z_1, z_2, \theta) = \frac{(\theta - 1)^2}{2(z_1 + z_2 + 1)^2 (z_1 + \theta z_2 + 1)^2 (\theta z_1 + z_2 + 1)^2} \left(E + H\sqrt{D} \right),$$

$$s_2(z_1, z_2, \theta) = \frac{(\theta - 1)^2}{2(z_1 + z_2 + 1)^2 (z_1 + \theta z_2 + 1)^2 (\theta z_1 + z_2 + 1)^2} \left(E - H\sqrt{D} \right),$$

where

$$E = ((\theta + 1)z_1 z_2 (z_1 + z_2))^2 + 2z_1^2 z_2^2 (z_1 + z_2 + 1)(\theta^2 + 4\theta + 2) + (z_1^2 + z_2^2)(2z_1 + 2z_2 + 4z_1 z_2 + 1)$$
$$+ 2z_1 z_2 (z_1^3 + z_2^3 + \theta z_1^2 + \theta z_2^2 + \theta z_1 + \theta z_2) + z_1^4 + z_2^4,$$

$$D = ((\theta + 1)z_1 z_2 (z_1 + z_2))^2 + (z_1 - z_2)^2 (z_1 + z_2 + 1)^2 + 2z_1 z_2 (z_1^2 + z_2^2)(z_1 + z_2 + 1)$$
$$- 2\theta z_1 z_2 \left((z_1 - z_2)^2 - 3z_1 z_2 (z_1 + z_2) + z_1^3 + z_2^3 \right),$$

$$H = (\theta + 1)z_1 z_2 (z_1 + z_2 + 2) + z_1^2 + z_2^2 + z_1 + z_2.$$

Lemma 5.1. *Let $k = 2$. Then, for any solution (z_1, z_2) to the system of equations (4.8), the following equality holds:*

$$s = \max\{|s_1(z_1, z_2, \theta)|, |s_2(z_1, z_2, \theta)|\} = s_1(z_1, z_2, \theta),$$

where the pair (z_1, z_2) takes the values $\left(z_1^{(1)}, z_1^{(2)}\right)$, $\left(z_1^{(2)}, z_1^{(1)}\right)$, $\left(z_1^{(3)}, z_1^{(4)}\right)$ and $\left(z_1^{(4)}, z_1^{(3)}\right)$.

The Kesten–Stigum condition for determining the non-extremality region of the Gibbs measures is given as follows:

$$s_1(z_1, z_2, \theta) > \frac{1}{2}. \tag{5.3}$$

By means of this inequality, the non-extremality regions of the measures μ_i are determined for the given parameter values. Since the function $s_1(z_1, z_2, \theta)$ is symmetric with respect to the variables z_1 and z_2, it is sufficient to verify the condition (5.3) only for the pairs of solutions $\left(z_1^{(1)}, z_1^{(2)}\right)$ and $\left(z_1^{(3)}, z_1^{(4)}\right)$. Due to the complexity of the inequality (5.3), solving it analytically is significantly difficult. Therefore, we employ numerical computations and graphical analysis.

1. We examine the inequality (5.3) for the solution $\left(z_1^{(1)}, z_1^{(2)}\right)$ defined in the domains M_1, M_2, and M_3. To this end, we consider the following function:

$$l_1(z_1^{(1)}, z_1^{(2)}, \theta, \lambda) = s_1(z_1^{(1)}, z_1^{(2)}, \theta) - \frac{1}{2}.$$

By comparing the function l_1 with $l = 0$ in the domains M_1, M_2, and M_3 using Python software, it is determined that there is no domain where these functions take positive values (see Fig. 1., a)–c)).

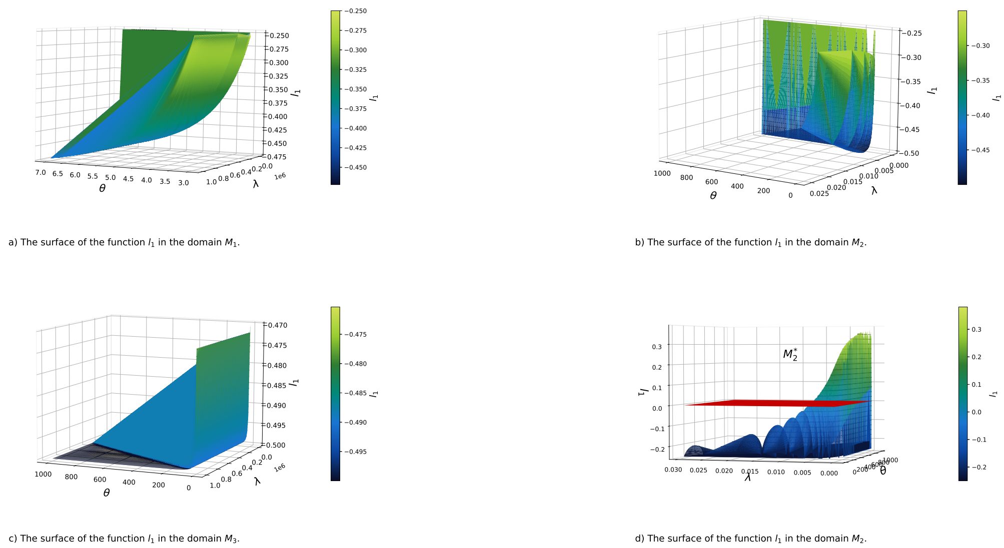
Remark 5.1. The measures μ_1 and μ_2 do not possess a non-extremality domain. Consequently, these measures may be extreme.

2. Next, we examine the inequality (5.3) for the solution $\left(z_1^{(3)}, z_1^{(4)}\right)$ in the domain M_2.

For the solution $\left(z_1^{(3)}, z_1^{(4)}\right)$, the domain where the function

$$l_1(z_1^{(3)}, z_1^{(4)}, \theta, \lambda) = s_1(z_1^{(3)}, z_1^{(4)}, \theta) - \frac{1}{2}$$

takes positive values is determined, and this domain satisfies inequality (5.3) (see Fig.1., d)).



a) The surface of the function l_1 in the domain M_1.

b) The surface of the function l_1 in the domain M_2.

c) The surface of the function l_1 in the domain M_3.

d) The surface of the function l_1 in the domain M_2.

Fig. 1.

Theorem 5.1. *For the measures μ_3 and μ_4 defined in the domain M_2, there exist subdomain M_2^* ($M_2^* \subsetneq M_2$), respectively, such that in these domain the measures μ_3 and μ_4 are not extreme.*

5.2. Extremality of the Gibbs measures

To study the extremality, we employ the methods and necessary definitions from [22]. The deletion of any edge $\langle x^0, x^1\rangle = l \in L$ from the Cayley tree $\mathfrak{I}^k$ breaks the tree into two components $\mathfrak{I}^{k_0}_{x^0}$ and $\mathfrak{I}^{k_1}_{x^1}$, each of which is called a *semi-infinite tree* or a *Cayley semitree*.

Consider the finite complete subtree S containing all initial points of the semitree $\mathfrak{I}^{k_0}_{x^0}$. The boundary ∂S of the subtree S consists of the nearest neighbors of its vertices, which lie in $\mathfrak{I}^{k_0}_{x^0} \setminus S$. We identify the subtree S with its vertex set. We denote the set of all edges in S by A and ∂A by $E(A)$.

In [22], two key quantities κ and γ were introduced, which play an important role in the study of the extremality of translation-invariant Gibbs measures. These quantities are properties of the set $\{\mu_x\}$ of Gibbs measures, where x is a fixed boundary condition and S is an arbitrary initial complete finite subtree of $\mathfrak{I}^{k_0}_{x^0}$. Given an initial subtree $S \subset \mathfrak{I}^{k_0}_{x^0}$ and any vertex $x \in S$, we write S_x for the (maximal) subtree of S with initial point x. If x is not an initial point of S, then μ_x^S denotes the Gibbs measure in which the "ancestor" x has spin s and the configuration on the lower boundary of S_x (i.e., on $\partial S_x \setminus \{x\}$, the ancestor x) is determined by l.

We define the norm distance between two measures μ_1 and μ_2 on Ω by

$$\|\mu_1-\mu_2\|_x=\frac{1}{2}\sum_{i\in\{-1,0,1\}}|\mu_1(\sigma(x)=i)-\mu_2(\sigma(x)=i)|.$$

Let $\xi^{x,s}$ be the configuration obtained from ξ by setting the spin at x to s. Following [22], we define

$$\kappa=\kappa(\mu)=\sup_{x\in\Gamma^k}\max_{\xi,\xi':\xi\equiv\xi' \text{ on } S\setminus\{x\}}\left\|\mu_A^{\xi}-\mu_A^{\xi'}\right\|_x,$$

$$\gamma=\gamma(\mu)=\sup_{A\subset\Gamma^k}\max_{\substack{y\in\partial A\\ x\in A}}\left\|\mu_A^{\xi,y,s}-\mu_A^{\xi}\right\|_x,$$

where the maximum is over all boundary conditions ξ, all $y\in\partial A$, all neighbors $x\in A$ of each vertex y, and all spins $s\in\{-1,0,1\}$. A sufficient condition for the extremality of a Gibbs measure μ is $k\kappa(\mu)\gamma(\mu)<1$; for the $G_k^{(2)}$ -periodic measures under consideration, this condition reads $k^2\kappa(\mu)\gamma(\mu)<1$ (see [23]).

The value of κ is defined by the following formula:

$$\kappa=\frac{1}{2}\max_{i\in\{-1,0,1\}}\sum_j|P_{ij}-P_{ji}|.$$

It follows that $|P_{il}-P_{jl}|=0$ for $i=j$. For $i\neq j$, (5.2) gives $\sum_{l\in\{-1,0,1\}}|P_{il}-P_{jl}|$.

Let $i=-1$ and $j=0$, then

$$\sum_{l\in\{-1,0,1\}}|P_{-1l}-P_{0l}|=\begin{cases}\dfrac{2z_1z_2(\theta-1)^2\Big((\theta+1)z_1z_2(z_1+z_2+2)+z_1^2+z_2^2+z_1+z_2\Big)}{(\theta z_2+z_1+1)^2(\theta z_1+z_2+1)(z_1+z_2+1)^2}, & if \quad z_1>z_2,\\[2ex] \dfrac{2z_2^2(\theta-1)^2\Big((\theta+1)z_1^2(z_1+z_2+2)+(2z_1+1)(z_2+1)+\theta z_1\Big)}{(\theta z_2+z_1+1)^2(\theta z_1+z_2+1)(z_1+z_2+1)^2}, & if \quad z_1<z_2.\end{cases}$$

Let $i=-1$ and $j=1$, then

$$\sum_{l\in\{-1,0,1\}}|P_{-1l}-P_{1l}|=\begin{cases}\dfrac{2z_1(\theta-1)^2\Big((\theta+1)^2(z_1+z_2+1)z_1z_2^2+\theta(z_1+3z_2+2)z_1z_2+(z_1+z_2)(z_1+z_2+z_2^2)+(2z_1z_2+1)z_1\Big)}{(\theta z_2+z_1+1)^2(\theta z_1+z_2+1)^2(z_1+z_2+1)}, & if \quad z_1>z_2,\\[2ex] \dfrac{2z_2(\theta-1)^2\Big((\theta+1)^2(z_1+z_2+1)z_1^2z_2+\theta(3z_1+z_2+2)z_1z_2+(z_1+z_2)(z_1+z_2+z_1^2)+(2z_1z_2+1)z_2\Big)}{(\theta z_2+z_1+1)^2(\theta z_1+z_2+1)^2(z_1+z_2+1)}, & if \quad z_1<z_2.\end{cases}$$

Let $i=0$ and $j=1$, then

$$\sum_{l\in\{-1,0,1\}}|P_{0l}-P_{1l}|=\begin{cases}\dfrac{2z_1^2(\theta-1)^2\Big((\theta+1)z_2^2(z_1+z_2+2)+(2z_2+1)(z_1+1)+\theta z_2\Big)}{(\theta z_1+z_2+1)^2(z_1+z_2+1)^2(\theta z_2+z_1+1)}, & if \quad z_1>z_2,\\[2ex] \dfrac{2z_1z_2(\theta-1)^2\Big(\theta+1)z_1z_2(z_1+z_2+2)+z_1^2+z_2^2+z_1+z_2\Big)}{(\theta z_1+z_2+1)^2(z_1+z_2+1)^2(\theta z_2+z_1+1)}, & if \quad z_1<z_2.\end{cases}$$

As a result, we obtain:

$$\kappa=\begin{cases}\dfrac{z_1(\theta-1)^2\Big((\theta+1)^2(z_1+z_2+1)z_1z_2^2+\theta(z_1+3z_2+2)z_1z_2+(z_1+z_2)(z_1+z_2+z_2^2)+(2z_1z_2+1)z_1\Big)}{(\theta z_2+z_1+1)^2(\theta z_1+z_2+1)^2(z_1+z_2+1)}, & if \quad z_1>z_2,\\[2ex] \dfrac{z_2(\theta-1)^2\Big((\theta+1)^2(z_1+z_2+1)z_1^2z_2+\theta(3z_1+z_2+2)z_1z_2+(z_1+z_2)(z_1+z_2+z_1^2)+(2z_1z_2+1)z_2\Big)}{(\theta z_2+z_1+1)^2(\theta z_1+z_2+1)^2(z_1+z_2+1)} & if \quad z_1<z_2.\end{cases}$$

Next, we calculate the parameter γ for these solutions:

$$\gamma = \max\left\{\|\mu_A^{\eta^{y,-1}} - \mu_A^{\eta^{y,0}}\|_x, \|\mu_A^{\eta^{y,0}} - \mu_A^{\eta^{y,1}}\|_x, \|\mu_A^{\eta^{y,-1}} - \mu_y^{\eta^{y,1}}\|_x\right\},$$

where the total variation distances between the conditional measures are computed as follows:

$$\|\mu_A^{\eta^{y,-1}} - \mu_A^{\eta^{y,0}}\| = \frac{1}{2}\sum_{s\in\{-1,0,1\}} |\mu_A^{\eta^{y,-1}}(\sigma(x)=s) - \mu_A^{\eta^{y,0}}(\sigma(x)=s)| =$$

$$= \frac{1}{2}(|P_{-1,-1} - P_{0,-1}| + |P_{-1,0} - P_{0,0}| + |P_{-1,1} - P_{0,1}|) =$$

$$= \begin{cases} \dfrac{2z_1z_2(\theta-1)^2\Big((\theta+1)z_1z_2(z_1+z_2+2)+z_1^2+z_2^2+z_1+z_2\Big)}{(\theta z_2+z_1+1)^2(\theta z_1+z_2+1)(z_1+z_2+1)^2}, & if \quad z_1 > z_2, \\ \dfrac{2z_2^2(\theta-1)^2\Big((\theta+1)z_1^2(z_1+z_2+2)+(2z_1+1)(z_2+1)+\theta z_1\Big)}{(\theta z_2+z_1+1)^2(\theta z_1+z_2+1)(z_1+z_2+1)^2}, & if \quad z_1 < z_2, \end{cases}$$

$$\|\mu_A^{\eta^{y,-1}} - \mu_A^{\eta^{y,1}}\|_x = \frac{1}{2}\sum_{s\in\{-1,0,1\}} |P_{-1,s} - P_{1,s}| =$$

$$= \begin{cases} \dfrac{2z_1(\theta-1)^2\Big((\theta+1)^2(z_1+z_2+1)z_1z_2^2+\theta(z_1+3z_2+2)z_1z_2+(z_1+z_2)(z_1+z_2+z_2^2)+(2z_1z_2+1)z_1\Big)}{(\theta z_2+z_1+1)^2(\theta z_1+z_2+1)^2(z_1+z_2+1)}, & if \quad z_1 > z_2, \\ \dfrac{2z_2(\theta-1)^2\Big((\theta+1)^2(z_1+z_2+1)z_1^2z_2+\theta(3z_1+z_2+2)z_1z_2+(z_1+z_2)(z_1+z_2+z_1^2)+(2z_1z_2+1)z_2\Big)}{(\theta z_2+z_1+1)^2(\theta z_1+z_2+1)^2(z_1+z_2+1)}, & if \quad z_1 < z_2, \end{cases}$$

$$\|\mu_A^{\eta^{y,0}} - \mu_A^{\eta^{y,1}}\|_x = \frac{1}{2}\sum_{s\in\{-1,0,1\}} |P_{0,s} - P_{1,s}| = \begin{cases} \dfrac{2z_1^2(\theta-1)^2\Big((\theta+1)z_2^2(z_1+z_2+2)+(2z_2+1)(z_1+1)+\theta z_2\Big)}{(\theta z_1+z_2+1)^2(z_1+z_2+1)^2(\theta z_2+z_1+1)}, & if \quad z_1 > z_2, \\ \dfrac{2z_1z_2(\theta-1)^2\Big(\theta+1)z_1z_2(z_1+z_2+2)+z_1^2+z_2^2+z_1+z_2\Big)}{(\theta z_1+z_2+1)^2(z_1+z_2+1)^2(\theta z_2+z_1+1)}, & if \quad z_1 < z_2. \end{cases}$$

From the above, it follows that:

$$\gamma = \begin{cases} \dfrac{z_1(\theta-1)^2\Big((\theta+1)^2(z_1+z_2+1)z_1z_2^2+\theta(z_1+3z_2+2)z_1z_2+(z_1+z_2)(z_1+z_2+z_2^2)+(2z_1z_2+1)z_1\Big)}{(\theta z_2+z_1+1)^2(\theta z_1+z_2+1)^2(z_1+z_2+1)}, & if \quad z_1 > z_2, \\ \dfrac{z_2(\theta-1)^2\Big((\theta+1)^2(z_1+z_2+1)z_1^2z_2+\theta(3z_1+z_2+2)z_1z_2+(z_1+z_2)(z_1+z_2+z_1^2)+(2z_1z_2+1)z_2\Big)}{(\theta z_2+z_1+1)^2(\theta z_1+z_2+1)^2(z_1+z_2+1)} & if \quad z_1 < z_2. \end{cases}$$

It is easy to verify that the function γ is symmetric with respect to the variables z_1 and z_2. Therefore, without loss of generality, in what follows we present all analyses and graphical illustrations only for the case $z_1 > z_2$.

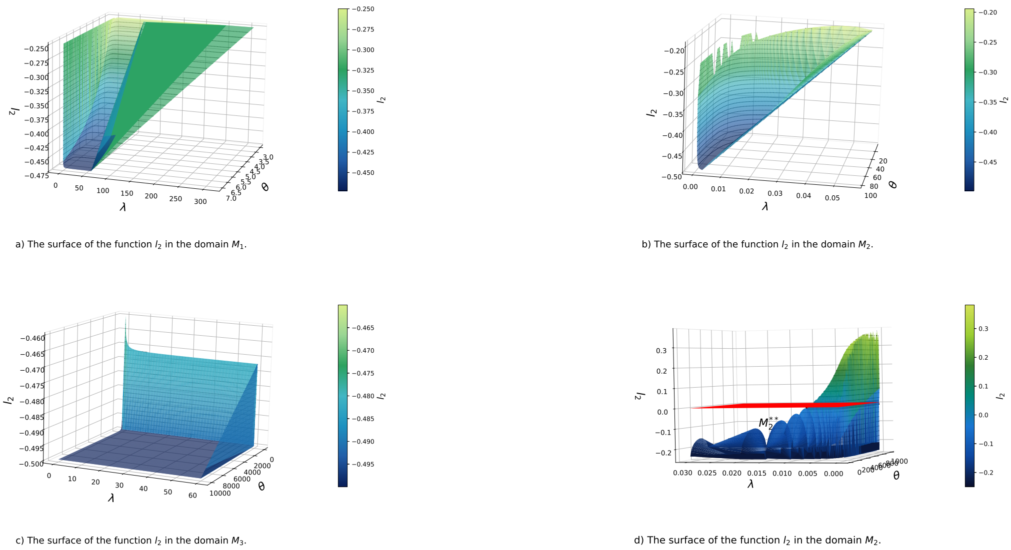
It is observed that for all measures corresponding to the identified solutions in the case $k = 2$, the relation $\kappa = \gamma$ holds. Thus, the condition $4\kappa\gamma < 1$ is equivalent to:

$$\gamma < 1/2. \tag{5.4}$$

This inequality determines the regions of extremality for the measures μ_i. Since the expression for the parameter γ and the solutions of system (4.8) are rather complicated, it is difficult to analyze this inequality analytically. Therefore, we use numerical calculations and graphical analysis to identify the regions of extremality.

1. We examine the inequality (5.4) for the solutions $\left(z_1^{(1)}, z_1^{(2)}\right)$ defined in the domains M_1, M_2, and M_3. To this end, we introduce the following function: $l_2 = \gamma(z_1^{(1)}, z_1^{(2)}, \theta) - \frac{1}{2}$. By numerically analyzing the values of l_2 in these domains via Python software, it is established that the function takes only negative values across all considered regions, i.e., $l_2 < 0$ (see Fig. 2., a)–c)).

2. Next, we investigate the inequality (5.4) for the solution $(z_1^{(3)}, z_1^{(4)})$ defined in the domain M_2. To this end, we introduce the following function: $l_2 = \gamma(z_1^{(3)}, z_1^{(4)}, \theta) - \frac{1}{2}$. By numerically analyzing this function via Python software, we establish the existence of a subregion $M_2^{**} \subset M_2$ such that $l_2 < 0$ (see Fig. 2., d)).



a) The surface of the function l_2 in the domain M_1.

b) The surface of the function l_2 in the domain M_2.

c) The surface of the function l_2 in the domain M_3.

d) The surface of the function l_2 in the domain M_2.

Fig. 2.

Theorem 5.2. *For the periodic Gibbs measures μ_i ($i = 1, 2, 3, 4$), the following assertions hold:*

1. *The measures μ_1 and μ_2 are extremal throughout the domains M_1, M_2, and M_3;*
2. *For the measures μ_3 and μ_4, there exists a proper subregion M_2^{**} ($M_2^{**} \subsetneq M_2$) in which these measures are extremal.*

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