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Research article

Initial and Initial-Boundary Value Problems for Multi-Parameter Integral-Differential Equations

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Abstract

This paper investigates initial and initial–boundary value problems for a class of multi-parameter integral-differential equations. In particular, we formulate the Cauchy problem for an ordinary differential equation involving a multi-parameter integral-differential operator. The problem is equivalently transformed into a Volterra integral equation of the second kind. By applying the method of successive approximations, we derive an explicit representation of the solution. As an application of the newly established results, we study direct and inverse source problems for a partial differential equation that incorporates the same multi-parameter integral-differential operator in the time variable. An example is presented to illustrate the theoretical findings.

Keywords: *Multi-parameter integral-differential operator; Cauchy problem; Inverse source problem; Mittag-Leffler function; sub-diffusion equation.*

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1. Introduction

Integral–differential operators are an essential mathematical tool for modeling complex phenomena across a wide spectrum of applied sciences, including physics, biology, economics, nanotechnology, and continuum mechanics. Their ability to simultaneously incorporate local dynamics and nonlocal interactions has led to successful applications in viscoelastic processes [1], nanotechnology [2], fractional dynamics [3], continuum mechanics [4], economic and financial modeling [5], as well as biological systems [6]. This broad applicability has stimulated growing interest in the development and analysis of new classes of integral–differential operators.

Comprehensive theoretical foundations of integral–differential operators, together with their connections to ordinary and partial differential equations and related applications, are presented in the monographs [7], [8]. In many modeling frameworks, integral–differential operators naturally arise in the description of complex processes characterized by memory effects and nonlocal behavior, particularly in fractal or heterogeneous media.

Motivated by these features, numerous generalized operator constructions have been proposed in recent years. In particular, various types of integral–differential operators have been introduced and systematically investigated in [8], [9], [10], [11], [12], [13], [22], [14], and [16], where their analytical properties and applications to ordinary and partial differential equations were explored. These studies demonstrate that integral–differential operators provide a flexible analytical framework for capturing long-range interactions and hereditary effects in mathematical models.

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Despite these advances, many existing operator families remain limited in their capacity to represent multiple interacting parameters that commonly appear in realistic systems. This observation motivates the development of more general operator structures that offer greater parametric freedom while preserving analytical tractability.

In this paper, we introduce a new multi-parameter integral–differential operator that extends several known operators as special cases. The proposed construction allows the incorporation of multiple structural characteristics within a unified framework. We investigate its fundamental properties and derive new operational relations, thereby establishing a foundation for future applications to generalized differential equations and related mathematical models.

2. Useful results

Theorem 2.1.[11] *If* $\alpha > 0,\ p > 0,$ *and* $f(x) \in L_1(a,b)$, then

$$ {}_K D_{a+}^{\alpha,p}\, {}_K I_{a+}^{\alpha,p} f(x) = f(x). \tag{2.1}$$

Here,

$$ {}_K I_{a+}^{\alpha,p} f(x) = \frac{p^{1-\alpha}}{\Gamma(\alpha)} \int_a^x \frac{t^{p-1} f(t)}{(x^p - t^p)^{1-\alpha}} dt, $$

and

$$ {}_K D_{a+}^{\alpha,p} f(x) = \frac{p^{\alpha-n+1}}{\Gamma(n-\alpha)} \left(x^{1-p} \frac{d}{dx} \right)^n \int_a^x \frac{t^{p-1} f(t)}{(x^p - t^p)^{\alpha-n+1}} dt. $$

Theorem 2.2.[11] *If* $n-1 < \alpha < n, n = [a]+1, n \in N, p > 0,\ {}_K I_{a+}^{n-\alpha,p} f(x) \in AC_{\delta_p}^n[a,b]$ *then*

$$ {}_K I_{a+}^{\alpha,p}\, {}_K D_{a+}^{\alpha,p} f(x) = f(x) - \sum_{j=1}^{n} \frac{\left(\frac{x^p - a^p}{p}\right)^{\alpha-j}}{\Gamma(\alpha-j+1)} \left[\lim_{x\to a+} \left(x^{1-p} \frac{d}{dx} \right)^{n-j} {}_K I_{a+}^{n-\alpha,p} f(x) \right], \tag{2.2}$$

$$ {}_K I_{b-}^{\alpha,p}\, {}_K D_{b-}^{\alpha,p} f(x) = f(x) - \sum_{j=1}^{n} \frac{(-1)^{n-j}\left(\frac{b^p - x^p}{p}\right)^{\alpha-j}}{\Gamma(\alpha-j+1)} \left[\lim_{x\to b-} \left(x^{1-p} \frac{d}{dx} \right)^{n-j} {}_K I_{b-}^{n-\alpha,p} f(x) \right]. \tag{2.3}$$

Here,

$$ {}_K I_{b-}^{\alpha,p} f(x) = \frac{p^{1-\alpha}}{\Gamma(\alpha)} \int_x^b \frac{t^{p-1} f(t)}{(t^p - x^p)^{1-\alpha}} dt, $$

and

$$ {}_K D_{b-}^{\alpha,p} f(x) = \frac{p^{\alpha-n+1}}{\Gamma(n-\alpha)} \left(-x^{1-p} \frac{d}{dx} \right)^n \int_x^b \frac{t^{p-1} f(t)}{(t^p - x^p)^{\alpha-n+1}} dt. $$

Theorem 2.3.[21] *If* $f(t) \in C^1[0.T]$ *obtains the maximum (minimum) at the point* $t_0 \in (0,T)$ *and* $\alpha \in (0,1)$, *then the following inequalities hold:*

$$ {}_K^C D_{0+}^{\alpha,p} f(t_0) \geq (\leqslant) \frac{p^\alpha {t_0}^{-\alpha p}}{\Gamma(1-\alpha)} \left[f(t_0) - f(0) \right] \geq 0 \,(\leqslant 0). \tag{2.4}$$

Here

$$ {}_{K}^{C}D_{0+}^{\alpha,p} f(x) = \frac{p^{\alpha}}{\Gamma(1-\alpha)} \int_{0}^{x} \frac{t^{p-1}}{(x^{p}-t^{p})^{\alpha}} \left(t^{1-p} \frac{d}{dt}\right) f(t)\, dt. $$

3. Analytical approach: ODE

In this section, we will use an analytical approach to obtain an explicit solution for the Cauchy problem.

3.1. New Multi-Parameter Integral-Differential operators.

Let us define the multi-parameter integral-differential operators, then study some properties of them.

Definition 3.1. Let $f(x) \in L_1(a,b)$, ${}_{K}I_{a+}^{(1-\mu)(n-\beta),p} f(x) \in AC_{\delta_p}^{n}[a,b]$, $n-1<\alpha\le n$, $n-1<\beta\le n$. The operators

$$ \mathbb{M}_{a+}^{(\alpha,\beta)\mu} f(x) = {}_{K}I_{a+}^{\mu(n-\alpha),p} \left(x^{1-p}\frac{d}{dx}\right)^{n} {}_{K}I_{a+}^{(1-\mu)(n-\beta),p} f(x), $$

$$ \mathbb{M}_{b-}^{(\alpha,\beta)\mu} f(x) = {}_{K}I_{b-}^{\mu(n-\alpha),p} \left(-x^{1-p}\frac{d}{dx}\right)^{n} {}_{K}I_{b-}^{(1-\mu)(n-\beta),p} f(x) $$

are called the left and right-sided multi-parameter integral-differential operators. Here $p \in \mathbb{R}^{+}$, α and β are orders of the operators and μ $(0\le\mu\le1)$ is type of the operators. Also,

$$ AC_{\delta_p}^{n}[a,b] = \left\{ f:[a,b]\to R \ and\ \delta_p^{n-1} f(x) \in AC[a,b]\ ,\ \delta_p = x^{1-p}\frac{d}{dx} \right\}, $$

$$ AC[a,b] = \left\{ f:[a,b]\to\mathbb{R}:\ f(x) = c + \int_{a}^{x} \varphi(t)dt,\ \varphi(t)\in L_1(a,b) \right\}, $$

$$ L_p(a,b) = \left\{ \|f\|_p = \left(\int_a^b |f(t)|^p dt\right)^{\frac{1}{p}};\ \|f\|_{\infty} = esssup_{a\le x\le b}|f(x)|;1\le p\le+\infty;\ -\infty\le a,b\le+\infty \right\}, $$

and $AC[a,b]$ is a class of absolutely continuous functions and $L_p(a,b)$ is a class of Lebesgue measurable functions.

Let us consider properties of the proposed operators, which will be useful in the next sections.

Lemma 3.1. *If* $p\in\mathbb{R}^{+}$, $n-1<\alpha,\beta\le n$, $\gamma=\beta+\mu(n-\beta)$, $\sigma=\beta+\mu(\alpha-\beta)$, $0\le\mu\le1$ *then*

$$ \mathbb{M}_{a+}^{(\alpha,\beta)\mu} f(x) = {}_{K}I_{a+}^{\gamma-\sigma,p}\, {}_{K}D_{a+}^{\gamma,p} f(x), $$

where $n-1<\gamma\le n$, $n-1<\sigma\le n$.

Proof.

$$ \mathbb{M}_{a+}^{(\alpha,\beta)\mu} f(x) = {}_{K}I_{a+}^{\mu(n-\alpha),p} \left(x^{1-p}\frac{d}{dx}\right)^{n} {}_{K}I_{a+}^{(1-\mu)(n-\beta),p} f(x) = $$

$$ {}_{K}I_{a+}^{\beta+\mu n-\mu\beta-\beta-\mu\alpha+\mu\beta,p} \left(x^{1-p}\frac{d}{dx}\right)^{n} {}_{K}I_{a+}^{n-\beta-\mu(n-\beta),p} f(x) = $$

$$ {}_{K}I_{a+}^{\beta+\mu(n-\beta)-(\beta+\mu(\alpha-\beta)),p} \left(x^{1-p}\frac{d}{dx}\right)^{n} {}_{K}I_{a+}^{n-\gamma,p} f(x) = {}_{K}I_{a+}^{\gamma-\sigma,p}\, {}_{K}D_{a+}^{\gamma,p} f(x). $$

Lemma 3.2. *If* $f(x) \in L_1(a,b)$, $p \in \mathbb{R}^+$, $0 < \alpha, \beta < 1$, $\gamma = \beta + \mu(1-\beta)$, $\sigma = \beta + \mu(\alpha - \beta)$, $0 \le \mu \le 1$, then

$$\mathbb{M}_{a+}^{(\alpha,\beta)\mu} f(x) = {}_K D_{a+}^{\sigma,p} \left(f(x) - \frac{{}_K I_{a+}^{1-\gamma,p} f(a)}{\Gamma(\gamma)} \left(\frac{x^p - t^p}{p} \right)^{\gamma-1} \right),$$

where ${}_K I_{a+}^{1-\gamma} f(a) = {}_K I_{a+}^{1-\gamma} f(x)\Big|_{x=a}$.

Proof. Using Lemma 3.1, we make the following evaluations:

$$\mathbb{M}_{a+}^{(\alpha,\beta)\mu} f(x) = {}_K I_{a+}^{\mu(1-\alpha),p} \left(x^{1-p} \frac{d}{dx} \right) {}_K I_{a+}^{(1-\mu)(1-\beta),p} f(x) = {}_K I_{a+}^{\gamma-\sigma,p} {}_K D_{a+}^{\gamma,p} f(x) =$$
$${}_K I_{a+}^{\gamma-\sigma,p} \left(x^{1-p} \frac{d}{dx} \right) {}_K I_{a+}^{1-\gamma,p} f(x) = \left(x^{1-p} \frac{d}{dx} \right) {}_K I_{a+}^{1,p} {}_K I_{a+}^{\gamma-\sigma,p} \left(x^{1-p} \frac{d}{dx} \right) {}_K I_{a+}^{1-\gamma,p} f(x) =$$
$$\left(x^{1-p} \frac{d}{dx} \right) {}_K I_{a+}^{\gamma-\sigma,p} {}_K I_{a+}^{1,p} \left(x^{1-p} \frac{d}{dx} \right) {}_K I_{a+}^{1-\gamma,p} f(x) =$$
$$\left(x^{1-p} \frac{d}{dx} \right) {}_K I_{a+}^{\gamma-\sigma,p} \left[{}_K I_{a+}^{1-\gamma,p} f(x) - {}_K I_{a+}^{1-\gamma,p} f(a) \right] =$$
$$\left(x^{1-p} \frac{d}{dx} \right) {}_K I_{a+}^{1-\sigma,p} f(x) - \left(x^{1-p} \frac{d}{dx} \right) {}_K I_{a+}^{\gamma-\sigma,p} \left[{}_K I_{a+}^{1-\gamma,p} f(a) \right] =$$
$$\left(x^{1-p} \frac{d}{dx} \right) {}_K I_{a+}^{1-\sigma,p} \left(f(x) - \frac{{}_K I_{a+}^{1-\gamma,p} f(a)}{\Gamma(1+\gamma-\sigma)} {}_K I_{a+}^{\sigma-1,p} \left(\frac{x^p - t^p}{p} \right)^{\gamma-\sigma} \right) =$$
$${}_K D_{a+}^{\sigma,p} \left(f(x) - \frac{{}_K I_{a+}^{1-\gamma,p} f(a)}{\Gamma(\gamma)} \left(\frac{x^p - t^p}{p} \right)^{\gamma-1} \right).$$

Lemma 3.3. *If* $0 < \gamma \le 1$, $0 < \sigma \le 1$, $0 \le \mu < 1$, $\gamma = \beta + \mu(1-\beta)$, $\sigma = \beta + \mu(\alpha-\beta)$, $f(x) \in L_1(a,b)$, ${}_K I_{a+}^{(1-\mu)(1-\beta),p} f(x) \in AC_{\delta_p}^n [a,b]$, *then*

$$\mathbb{M}_{a+}^{(\alpha,\beta)\mu} f(x) = {}_K^C D_{a+}^{\sigma,p} f(x) + \frac{f(a)}{\Gamma(1-\sigma)} \left(\frac{x^p - a^p}{p} \right)^{-\sigma}.$$

Proof. According to Definition 3.1, we have

$$\mathbb{M}_{a+}^{(\alpha,\beta)\mu} f(x) = {}_K I_{a+}^{\mu(1-\alpha),p} \left(x^{1-p} \frac{d}{dx} \right) {}_K I_{a+}^{(1-\mu)(1-\beta),p} f(x) = {}_K I_{a+}^{\mu(1-\alpha),p} \varphi(x),$$

where

$$\varphi(x) = \left(x^{1-p} \frac{d}{dx} \right) {}_K I_{a+}^{(1-\mu)(1-\beta),p} f(x).$$

We will simplify this item using Definition 3.1:

$$\varphi(x)=\left(x^{1-p}\frac{d}{dx}\right){}_K I_{a+}^{(1-\mu)(1-\beta),p}f\left(x\right)=\left(x^{1-p}\frac{d}{dx}\right){}_K I_{a+}^{1-\gamma,p}f\left(x\right)=$$

$$\left(x^{1-p}\frac{d}{dx}\right)\frac{p^{\gamma}}{\Gamma\left(1-\gamma\right)}\int\limits_a^x\left(x^p-t^p\right)^{-\gamma}t^{p-1}f(t)dt=$$

$$\left(x^{1-p}\frac{d}{dx}\right)\frac{-p^{\gamma-1}}{\Gamma\left(1-\gamma\right)}\int\limits_a^x\left(x^p-t^p\right)^{-\gamma}f(t)d\left(x^p-t^p\right)=$$

$$\left(x^{1-p}\frac{d}{dx}\right)\frac{-p^{\gamma-1}}{\Gamma\left(1-\gamma\right)}\left[\frac{\left(x^p-t^p\right)^{1-\gamma}}{1-\gamma}f(t)\Big|_a^x-\int\limits_a^x\frac{\left(x^p-t^p\right)^{1-\gamma}}{1-\gamma}f'(t)dt\right]=$$

$$\left(x^{1-p}\frac{d}{dx}\right)\frac{p^{\gamma-1}}{\Gamma\left(1-\gamma\right)}\left[\frac{\left(x^p-a^p\right)^{1-\gamma}}{1-\gamma}f(a)+\int\limits_a^x\frac{\left(x^p-t^p\right)^{1-\gamma}}{1-\gamma}f'(t)dt\right]=$$

$$\frac{f(a)}{\Gamma\left(1-\gamma\right)}\left(\frac{x^p-a^p}{p}\right)^{-\gamma}+\frac{1}{\Gamma\left(1-\gamma\right)}\int\limits_a^x\left(\frac{x^p-a^p}{p}\right)^{-\gamma}f'(t)dt\ =$$

$$\frac{f(a)}{\Gamma\left(1-\gamma\right)}\left(\frac{x^p-a^p}{p}\right)^{-\gamma}+\frac{1}{\Gamma\left(1-\gamma\right)}\int\limits_a^x\left(\frac{x^p-a^p}{p}\right)^{-\gamma}t^{p-1}\left(t^{1-p}\frac{d}{dt}\right)f(t)dt=$$

$${}_K I_{a+}^{1-\gamma,p}\left(x^{1-p}\frac{d}{dx}\right)f(x)+\frac{f(a)}{\Gamma\left(1-\gamma\right)}\left(\frac{x^p-a^p}{p}\right)^{-\gamma}.$$

Hence,

$$\mathbb{M}_{a+}^{(\alpha,\beta)\mu}f(x)=$$

$${}_K I_{a+}^{\mu(1-\alpha),p}\left[{}_K I_{a+}^{1-\gamma,p}\left(x^{1-p}\frac{d}{dx}\right)f(x)+\frac{f(a)}{\Gamma\left(1-\gamma\right)}\left(\frac{x^p-a^p}{p}\right)^{-\gamma}\right]=$$

$${}_K I_{a+}^{\mu-\mu\alpha+1-\gamma,p}\left(x^{1-p}\frac{d}{dx}\right)f(x)+\frac{f(a)}{\Gamma\left(1-\gamma\right)}{}_K I_{a+}^{\mu(1-\alpha),p}\left(\frac{x^p-a^p}{p}\right)^{-\gamma}=$$

$${}_K I_{a+}^{\mu-\mu\alpha+1-\gamma,p}\left(x^{1-p}\frac{d}{dx}\right)f(x)+\frac{f(a)}{\Gamma\left(1+\mu-\mu\alpha-\gamma\right)}\left(\frac{x^p-a^p}{p}\right)^{\mu-\mu\alpha-\gamma}=$$

$${}_K I_{a+}^{1-\sigma,p}\left(x^{1-p}\frac{d}{dx}\right)f(x)+\frac{f(a)}{\Gamma\left(1-\sigma\right)}\left(\frac{x^p-a^p}{p}\right)^{-\sigma}={}_K^C D_{a+}^{\sigma,p}f(x)+\frac{f(a)\left(\frac{x^p-a^p}{p}\right)^{-\sigma}}{\Gamma\left(1-\sigma\right)}.$$

Lemma 3.4. *If* ${}_K I_{a+}^{n-\gamma,p}f(x)\in AC_{\delta_p}^n\left[a,b\right],n-1<\alpha,\beta<n$ *and* $0\le\mu\le1$ *then*

$${}_K I_{a+}^{\sigma,p}\,\mathbb{M}_{a+}^{(\alpha,\beta)\mu}f(x)=f(x)-\sum_{j=1}^{n}\frac{C_{n-j}}{\Gamma(\gamma-j+1)}\left(\frac{x^p-a^p}{p}\right)^{\gamma-j},$$

where $\gamma=\beta+\mu\left(n-\beta\right)$, $\sigma=\beta+\mu\left(\alpha-\beta\right)$, $C_{n-j}=\lim\limits_{x\to a+}\left(x^{1-p}\frac{d}{dx}\right)^{n-j}I_{a+}^{n-\gamma,p}f(x)$.

Proof. Applying Lemma 3.1, we obtain

$$ {}_K I_{a+}^{\sigma,p}\, \mathbb{M}_{a+}^{(\alpha,\beta)\mu} f(x) = {}_K I_{a+}^{\sigma,p}\, {}_K I_{a+}^{\gamma-\sigma,p}\, {}_K D_{a+}^{\gamma,p} f(x) = {}_K I_{a+}^{\gamma,p}\, {}_K D_{a+}^{\gamma,p} f(x). \tag{3.1} $$

Since ${}_K I_{a+}^{n-\gamma,p} f(x) \in AC_{\delta_p}^{n}[a,b]$, and according to (2.2),(3.1), one can easily get

$$ {}_K I_{a+}^{\gamma,p}\, {}_K D_{a+}^{\gamma,p} f(x) = f(x) - \sum_{j=1}^{n} \frac{1}{\Gamma(\gamma-j+1)} \left(\frac{x^p}{p}\right)^{\gamma-j} \left[\lim_{x\to a+} \left(x^{1-p}\frac{d}{dx}\right)^{n-j} I_{a+}^{n-\gamma,p} f(x) \right] = $$

$$ = f(x) - \sum_{j=1}^{n} \frac{C_{n-j}}{\Gamma(\gamma-j+1)} \left(\frac{x^p}{p}\right)^{\gamma-j}. $$

3.2. A Cauchy problem.

Let us consider the equation

$$ \mathbb{M}_{0+}^{(\alpha,\beta)\mu} u(t) - \lambda u(t) = f(t)\,,\ t \in (0,b), \tag{3.2} $$

with the following initial conditions:

$$ \lim_{x\to 0+} \left(t^{1-p}\frac{d}{dt}\right)^{n-k} {}_K I_{0+}^{n-\beta-\mu(n-\beta),p} u(t) = A_k\,,\ t \in [0,b],\ k = \overline{1,n}\,. \tag{3.3} $$

Here $n-1 < \alpha,\beta < n,\ 0 \le \mu \le 1\,,\ n \in \mathbb{N},\ p > 0,\ A_j\,, \lambda, p \in \mathbb{R}$ and $f(t)$ is a given function.

Lemma 3.5. The Cauchy problem (3.2)-(3.3) is equivalent to the following integral equation:

$$ u(t) - \lambda\, {}_K I_{0+}^{\sigma,p} u(t) = {}_K I_{0+}^{\sigma,p} f(t) + \sum_{j=1}^{n} \frac{A_j}{\Gamma(\gamma-j+1)} \left(\frac{t^p}{p}\right)^{\gamma-j}. \tag{3.4} $$

Here $\gamma = \beta + \mu\,(n-\beta)\,,\ \sigma = \beta + \mu\,(\alpha-\beta)\,.$

Proof. First, we rewrite (3.2) using Lemma 3.1, then we apply the ${}_K I_{0+}^{\sigma,p}$ integral operator to both sides of the obtained equality. In a final step, we will use Lemma 3.4. Let us show these processes.

$$ {}_K I_{0+}^{\sigma,p}\, {}_K I_{0+}^{\gamma-\sigma,p}\, {}_K D_{0+}^{\gamma,p} u(t) - \lambda {}_K I_{0+}^{\sigma,p} u(t) = {}_K I_{0+}^{\sigma,p} f(t) \to $$

$$ {}_K I_{0+}^{\gamma,p}\, {}_K D_{0+}^{\gamma,p} u(t) - \lambda {}_K I_{0+}^{\sigma,p} u(t) = {}_K I_{0+}^{\sigma,p} f(t) \to $$

$$ u(t) - \sum_{j=1}^{n} \frac{1}{\Gamma(\gamma-j+1)} \left(\frac{t^p}{p}\right)^{\gamma-j} \left[\lim_{x\to 0+} \left(t^{1-p}\frac{d}{dt}\right)^{n-j} I_{0+}^{n-\gamma,p} u(t) \right] = $$

$$ \lambda {}_K I_{0+}^{\sigma,p} u(t) + {}_K I_{0+}^{\sigma,p} f(t) $$

or

$$ u(t) - \lambda {}_K I_{0+}^{\sigma,p} u(t) = {}_K I_{0+}^{\sigma,p} f(t) + \sum_{j=1}^{n} \frac{\left(\frac{t^p}{p}\right)^{\gamma-j}}{\Gamma(\gamma-j+1)} \left[\lim_{x\to 0+} \left(t^{1-p}\frac{d}{dt}\right)^{n-j} I_{0+}^{n-\gamma,p} u(t) \right]. $$

Considering initial conditions (3.3), we obtain the integral equation (3.4).

Lemma 3.6. If $p > 0$ and ${}_K I_{0+}^{\sigma,p} f(t)$ is summable, then the integral equation (3.4) has a unique solution.

Proof. Let us rewrite (3.4) as the second kind Volterra integral equation:

$$u(t) = \lambda \int_0^t K(t,s)u(s)ds + g(t), \tag{3.5}$$

where

$$g(t) = {}_K I_{0+}^{\sigma,p} f(t) + \sum_{j=1}^{n} \frac{A_j}{\Gamma(\gamma - j + 1)} \left(\frac{t^p}{p}\right)^{\gamma - j}, \quad K(t,s) = \frac{\left(\frac{t^p - s^p}{p}\right)^{\sigma - 1}}{\Gamma(\sigma) s^{1-p}}.$$

We know that $n - 1 < \sigma < n$. If $p > 1$ the kernel of (3.5) integral equation is a continuous on $[0, b]$ and it has the estimate

$$|K(t,s)| = \left| \frac{1}{\Gamma(\sigma)} \left(\frac{t^p - s^p}{p}\right)^{\sigma - 1} s^{p-1} \right| \leq \frac{M_1}{\Gamma(\sigma)}, \quad M_1 = \max_{0 \leqslant s \leqslant t \leqslant b} \left(\frac{b^p - s^p}{p}\right)^{\sigma - 1} s^{p-1}.$$

If $p > 0$, the kernel of (3.5) integral equation has a weak singularity and can be estimated as follows:

$$|K(t,s)| = \left| \frac{\left(\frac{t^p - s^p}{p}\right)^{\sigma - 1}}{\Gamma(\sigma) s^{1-p}} \right| \leq \frac{\left(\frac{b^p - s^p}{p}\right)^{\sigma - 1}}{\Gamma(\sigma) s^{1-p}} = \frac{M}{s^{1-p}}, \quad M = \max_{0 \leqslant s \leqslant t \leqslant b} \frac{1}{\Gamma(\sigma)} \left(\frac{b^p - s^p}{p}\right)^{\sigma - 1}.$$

If ${}_K I_{0+}^{\sigma,p} f(t)$ is summable, then $g(t)$ will be also summable. Therefore, according to the general theory of the Volterra integral equations [18], there exists a unique solution of (3.5).

Now we find the solution of (3.4) integral equation by using the method of successive approximation.

$$u_0(t) = \sum_{j=1}^{n} \frac{A_j}{\Gamma(\gamma - j + 1)} \left(\frac{t^p}{p}\right)^{\gamma - j},$$

$$u_k(x) = u_0(t) + \lambda {}_K I_{0+}^{\sigma,p} u_{k-1}(t) + {}_K I_{0+}^{\sigma,p} f(t),$$

$$u_1(x) = u_0(t) + \lambda {}_K I_{0+}^{\sigma,p} u_0(t) + {}_K I_{0+}^{\sigma,p} f(t) = \sum_{k=1}^{2} \lambda^{(k-1)} {}_K I_{0+}^{(k-1)\sigma,p} u_0(t) + \sum_{k=1}^{1} {}_K I^k{}_{0+}^{\sigma,p} f(t),$$

$$u_2(x) = u_0(t) + \lambda {}_K I_{0+}^{\sigma,p} u_0(t) + \lambda^2 {}_K I_{0+}^{2\sigma,p} u_0(t) + \lambda {}_K I_{0+}^{2\sigma,p} f(t) + {}_K I_{0+}^{\sigma,p} f(t) =$$
$$\sum_{k=1}^{3} \lambda^{(k-1)} {}_K I_{0+}^{(k-1)\sigma,p} u_0(t) + \sum_{k=1}^{2} \lambda^{k-1} {}_K I^k{}_{0+}^{\sigma,p} f(t),$$

$$u_3(x) = \sum_{k=1}^{4} \lambda^{(k-1)} {}_K I_{0+}^{(k-1)\sigma,p} u_0(t) + \sum_{k=1}^{3} \lambda^{k-1} {}_K I^k{}_{0+}^{\sigma,p} f(t).$$

If we continue this process h times, we get

$$u_h(x) = \sum_{k=1}^{h+1} \lambda^{(k-1)} {}_K I_{0+}^{(k-1)\sigma,p} u_0(t) + \sum_{k=1}^{h} \lambda^{k-1} {}_K I^k{}_{0+}^{\sigma,p} f(t). \tag{3.6}$$

We calculate each term on the right side of the (3.6) equation. At first, we calculate the first term of the equation as follows:

$$\sum_{k=1}^{h+1}\lambda^{(k-1)}{}_{K}I_{0+}^{(k-1)\sigma,p}u_0(t)=\sum_{k=1}^{h+1}\lambda^{(k-1)}{}_{K}I_{0+}^{(k-1)\sigma,p}\left(\sum_{j=1}^{n}\frac{A_j}{\Gamma(\gamma-j+1)}\left(\frac{t^p}{p}\right)^{\gamma-j}\right)=$$
$$\sum_{j=1}^{n}\frac{A_j}{\Gamma(\gamma-j+1)}\sum_{k=1}^{h+1}\lambda^{(k-1)}{}_{K}I_{0+}^{(k-1)\sigma,p}\left(\frac{t^p}{p}\right)^{\gamma-j}=$$

$$\sum_{j=1}^{n}\frac{A_j}{\Gamma(\gamma-j+1)}\sum_{k=1}^{h+1}\frac{\lambda^{(k-1)}p^{1-(k-1)\sigma}}{\Gamma((k-1)\sigma)}\int_0^t\frac{s^{p-1}}{(t^p-s^p)^{1-(k-1)\sigma}}\left(\frac{s^p}{p}\right)^{\gamma-j}ds=$$
$$\sum_{j=1}^{n}\frac{A_j}{\Gamma(\gamma-j+1)}\sum_{k=1}^{h+1}\frac{\lambda^{(k-1)}p^{1-(k-1)\sigma}p^{j-\gamma}}{\Gamma((k-1)\sigma)}\int_0^t\frac{s^{p-1}s^{p(\gamma-j)}ds}{(t^p-s^p)^{1-(k-1)\sigma}}.$$

We continue the calculation by changing the variable of the last integral, by the following remarks $\left|y=\frac{s^p}{t^p},\ s^{p-1}ds=p^{-1}t^p dy, s^p=t^p y\right|$ and obtain

$$\sum_{j=1}^{n}\frac{A_j}{\Gamma(\gamma-j+1)}\sum_{k=1}^{h+1}\frac{\lambda^{(k-1)}p^{1-(k-1)\sigma}p^{j-\gamma}}{\Gamma((k-1)\sigma)}\int_0^1\frac{(t^p y)^{(\gamma-j)}p^{-1}t^p dy}{(t^p-t^p y)^{1-(k-1)\sigma}}=$$
$$\sum_{j=1}^{n}\frac{A_j}{\Gamma(\gamma-j+1)}\sum_{k=1}^{h+1}\frac{\lambda^{(k-1)}}{\Gamma((k-1)\sigma)}\left(\frac{t^p}{p}\right)^{(k-1)\sigma+\gamma-j}\int_0^1 y^{\gamma-j+1-1}(1-y)^{(k-1)\sigma-1}dy=$$
$$\sum_{j=1}^{n}\frac{A_j}{\Gamma(\gamma-j+1)}\sum_{k=1}^{h+1}\frac{\lambda^{(k-1)}}{\Gamma((k-1)\sigma)}\frac{\Gamma(\gamma-j+1)\Gamma((k-1)\sigma)}{\Gamma((k-1)\sigma+\gamma-j+1)}\left(\frac{t^p}{p}\right)^{(k-1)\sigma+\gamma-j}=$$
$$\sum_{j=1}^{n}A_j\sum_{k=1}^{h+1}\frac{\lambda^{(k-1)}}{\Gamma((k-1)\sigma+\gamma-j+1)}\left(\frac{t^p}{p}\right)^{(k-1)\sigma+\gamma-j}.$$

Now we compute the second term of the equation (3.6).

$$\sum_{k=1}^{h}\lambda^{k-1}{}_{K}I_{0+}^{k\sigma,p}f(t)=\sum_{k=1}^{h}\frac{\lambda^{k-1}p^{1-k\sigma}}{\Gamma(k\sigma)}\int_0^t\frac{s^{p-1}f(s)}{(t^p-s^p)^{1-k\sigma}}ds=$$
$$\int_0^t\sum_{k=1}^{h}\frac{\lambda^{k-1}}{\Gamma(k\sigma)}\left(\frac{t^p-s^p}{p}\right)^{k\sigma-1}s^{p-1}f(s)ds.$$

If we rewrite each term on the right side of equation (3.6) with the appropriate form, we have

$$u_h(x)=\sum_{j=1}^{n}A_j\sum_{k=1}^{h+1}\frac{\lambda^{(k-1)}}{\Gamma((k-1)\sigma+\gamma-j+1)}\left(\frac{t^p}{p}\right)^{(k-1)\sigma+\gamma-j}+$$
$$\int_0^t\sum_{k=1}^{h}\frac{\lambda^{k-1}}{\Gamma(k\sigma)}\left(\frac{t^p-s^p}{p}\right)^{k\sigma-1}s^{p-1}f(s)ds. \tag{3.7}$$

If we compute $\lim_{h\to+\infty} u_h(t)$ by using (3.7) and after that we simplify the obtained function, then we get $u(t)$ which is the solution of the integral equation (3.4):

$$u(t) = \sum_{j=1}^{n} A_j\left(\frac{t^p}{p}\right)^{\beta+\mu(n-\beta)-j} E_{\beta+\mu(\alpha-\beta),\beta+\mu(n-\beta)+1-j}\left[\lambda\left(\frac{t^p}{p}\right)^{\beta+\mu(\alpha-\beta)}\right]+$$

$$\int_0^t s^{p-1}\left(\frac{t^p-s^p}{p}\right)^{\beta+\mu(\alpha-\beta)-1} E_{\beta+\mu(\alpha-\beta),\beta+\mu(\alpha-\beta)}\left[\lambda\left(\frac{t^p-s^p}{p}\right)^{\beta+\mu(\alpha-\beta)}\right] f(s)ds. \tag{3.8}$$

Theorem 3.1 *Let $n-1<\alpha,\beta<n,\ 0\le\mu\le1,\ n\in\mathbb{N},\ p>0$. If $f(t)\in C^1[0,b]$ and ${}_K I_{0+}^{\sigma,p} f(t)$ is a summable function, then there exists a unique solution for (3.2)-(3.3).*

Proof. According to Lemma 3.5, the problem (3.2) - (3.3) is equivalent to the integral equation (3.4). Then, based on Lemma 3.6, one can state that (3.8) is a unique solution to the problem (3.2) - (3.3).

3.3. Solvability of the Cauchy problem in different classes of functions.

In this subsection, we provide a statement on the solvability of the Cauchy problem (3.2)-(3.3) in different classes of functions. Namely, the following statement holds:

Theorem 3.3 *If $f(s)\in C[0,b]$, $n-1<\gamma,\sigma<n$, then a unique solution of the problem (3.2)-(3.3) exists and $\left(\frac{t^p}{p}\right)^{n-\gamma} u(t)\in C[0,b]$.*

Proof. The proof of this statement can be deduced easily if one considers the continuity of the Mittag-Leffler function.

Let us investigate problem (3.2)-(3.3) in other class of functions.

Definition 3.2[22] $L_1^p(0,b)$ class of functions is defined as

$$L_1^p(0,b)=\left\{u:(0,b)\to R,\ \ u(t)t^{p-1}\in L_1(0,b),\ \ u(t)t^{p-1}=g'(t),\ \ g(t)\in AC^1_{\delta_p}(0,b]\right\},$$

$$\|f(t)\|_{L_1^p(0,b)}=\int_0^b \left|t^{p-1}f(t)\right|dt\,,\quad (0<p<+\infty)\,.$$

If $p=1$ then $L_1^p(0,b)=L_1(0,b)$.

We know from [11], if $f(t)\in AC^n_{\delta_p}(a,b]$, then the following equalities are valid:

$$f(t)=\sum_{k=0}^{n-1} C_k\left(\frac{t^p}{p}\right)^k+\frac{1}{\Gamma(n)}\int_0^t\left(\frac{t^p-\tau^p}{p}\right)^{n-1}\varphi(\tau)d\tau,$$

$$C_k=\lim_{t\to0+}\frac{1}{k!}\left(t^{1-p}\frac{d}{dt}\right)^k f(t)\ ,\ \varphi(\tau)\in L_1^p(0,b).$$

Theorem 3.4 *If $n-1<\alpha,\beta<n$ and $f(t)\in L_1^p(0,b)$, then $u(t)\in L_1^p(0,b)$.*

Proof. The proof of this theorem can be easily obtained in an analytical way based on the definition of the function and function (3.8), which is the solution of the Cauchy problem (3.2)-(3.3).

Theorem 3.5 *If $n-1<\alpha,\beta<n$ and $f(t)\in C^1[0,b]$, then for $u(t)$ which the solution of Cauchy problem, the following ${}_K I_{0+}^{(1-\mu)(n-\beta)},\ u(t)\in AC^n_{\delta_p}(0,b]$ and $u(t)\in L_1^p(0,b)$ holds true.*

Proof. The proof of this theorem can be easily obtained analytically based on the definitions of the functions.

3.4. A priori estimates.

In this subsection, we present a priori estimates.

Lemma 3.7. *If* $0 < \alpha, \beta < 1$, $g(a) = 0$, $0 \le \mu \le 1$ *and* $g(t)$ *is an absolutely continuous function, then we have this inequality*

$$g(t)\,\mathbb{M}_{a+}^{(\alpha,\beta)\mu,p} g(t) \ge \frac{1}{2}\,\mathbb{M}_{a+}^{(\alpha,\beta)\mu,p} g^2(t)\,.$$

Proof. If we apply $g(a) = 0$ (condition of Lemma 3.7), from Lemma 3.3 we get

$$\mathbb{M}_{a+}^{(\alpha,\beta)\mu,p} g(t) = {}_{K}^{C}D_{a+}^{\sigma,p} g(t)\,. \tag{3.9}$$

If we show that

$$g(t)\mathbb{M}_{a+}^{(\alpha,\beta)\mu,p} g(t) - \frac{1}{2}\mathbb{M}_{a+}^{(\alpha,\beta)\mu,p} g^2(t) \ge 0\,, \tag{3.10}$$

is true, then it is enough to prove Lemma 3.7.

Let us show that (3.10) is true, by using (3.9):

$$\begin{aligned}
g(t){}_{K}^{C}D_{a+}^{\sigma,p} g(t) - \frac{1}{2}{}_{K}^{C}D_{a+}^{\sigma,p} g^2(t) &= g(t)\frac{p^\sigma}{\Gamma(1-\sigma)}\int\limits_a^t \frac{s^{p-1}}{(t^p-s^p)^\sigma}\left(s^{1-p}\frac{d}{ds}\right)g(s)ds - \\
&\frac{1}{2}\frac{p^\sigma}{\Gamma(1-\sigma)}\int\limits_a^t \frac{s^{p-1}}{(t^p-s^p)^\sigma}\left(s^{1-p}\frac{d}{ds}\right)g^2(s)ds = \\
&g(t)\frac{p^\sigma}{\Gamma(1-\sigma)}\int\limits_a^t \frac{g'(s)}{(t^p-s^p)^\sigma}ds - \frac{1}{2}\frac{p^\sigma}{\Gamma(1-\sigma)}\int\limits_a^t \frac{2g(s)g'(s)}{(t^p-s^p)^\sigma}ds = \\
\frac{p^\sigma}{\Gamma(1-\sigma)}\int\limits_a^t \frac{g'(s)\left[g(t)-g(s)\right]}{(t^p-s^p)^\sigma}ds &= \frac{p^\sigma}{\Gamma(1-\sigma)}\int\limits_a^t \frac{g'(s)}{(t^p-s^p)^\sigma}ds\int\limits_s^t g'(\eta)\,d\eta = \\
&\frac{p^\sigma}{\Gamma(1-\sigma)}\int\limits_a^t g'(\eta)\,d\eta\int\limits_0^\eta \frac{g'(s)}{(t^p-s^p)^\sigma}ds = \\
&\frac{p^\sigma}{\Gamma(1-\sigma)}\int\limits_a^t (t^p-\eta^p)^\sigma\frac{g'(\eta)}{(t^p-\eta^p)^\sigma}d\eta\int\limits_a^\eta \frac{g'(s)}{(t^p-s^p)^\sigma}ds = \\
&\frac{p^\sigma}{2\Gamma(1-\sigma)}\int\limits_a^t (t^p-\eta^p)^\sigma\frac{d}{d\eta}\left(\int\limits_a^\eta \frac{g'(s)}{(t^p-s^p)^\sigma}ds\right)^2 d\eta = \\
&\frac{p^\sigma}{2\Gamma(1-\sigma)}\left[(t^p-\eta^p)^\sigma\left(\int\limits_a^\eta \frac{g'(s)}{(t^p-s^p)^\sigma}ds\right)^2\Bigg|_{\eta=a}^{\eta=t} - \right. \\
&\left.\int\limits_a^t (-p)\,\eta^{p-1}\sigma(t^p-\eta^p)^{\sigma-1}\left(\int\limits_a^\eta \frac{g'(s)}{(t^p-s^p)^\sigma}ds\right)^2 d\eta\right] =
\end{aligned}$$

$$\frac{p^{\sigma+1}\sigma}{2\Gamma\left(1-\sigma\right)}\int\limits_{a}^{t}\frac{\eta^{p-1}}{\left(t^{p}-\eta^{p}\right)^{1-\sigma}}\left(\int\limits_{a}^{\eta}\frac{g'\left(s\right)}{\left(t^{p}-s^{p}\right)^{\sigma}}ds\right)^{2}d\eta\geqslant 0.$$

Theorem 3.6. *If $f\left(t\right)\in C^{1}[0.T]$, $\alpha\in(0,1)$, $\beta\in(0,1)$, and function $f(t)$ gets its maximum (minimum) value at the point $t_0\in(0,T)$, then following inequalities are hold:*

$$\mathbb{M}_{0+}^{(\alpha,\beta)\mu}f(t_0)\geq(\leqslant)\frac{p^{\sigma}t_0{}^{-\sigma p}}{\Gamma\left(1-\sigma\right)}f\left(t_0\right). \tag{3.11}$$

Furthermore, if $f\left(t_0\right)\geq 0(\leqslant 0)$, then we obtain $\mathbb{M}_{0+}^{(\alpha,\beta)\mu}f(t_0)\geq 0(\leqslant 0)$.
Here $\sigma=\beta+\mu\left(\alpha-\beta\right)$ and $0<\sigma<1$, $0\leq\mu<1$.

Proof. Using the result of Lemma 3.3 and Theorem 2.3, if the $f(t)$ function gets its maximum value at the point $t_0\in(0,T)$, then we can obtain

$$\mathbb{M}_{0+}^{(\alpha,\beta)\mu}f(t_0)={}_{K}^{C}D_{0+}^{\sigma,p}f(t_0)+\frac{p^{\sigma}t_0{}^{-\sigma p}}{\Gamma\left(1-\sigma\right)}f(0)\geq\frac{p^{\sigma}t_0{}^{-\sigma p}}{\Gamma\left(1-\sigma\right)}\left[f\left(t_0\right)-f(0)\right]+\frac{p^{\sigma}t_0{}^{-\sigma p}}{\Gamma\left(1-\sigma\right)}f(0)=$$
$$\frac{p^{\sigma}t_0{}^{-\sigma p}}{\Gamma\left(1-\sigma\right)}f(t_0).$$

As a result, we can write the following inequality:

$$\mathbb{M}_{0+}^{(\alpha,\beta)\mu}f(t_0)\geq\frac{p^{\sigma}t_0{}^{-\sigma p}}{\Gamma\left(1-\sigma\right)}f(t_0). \tag{3.12}$$

Based on (3.12), it is easy to know that if $f\left(t_0\right)\geq 0$, then $\mathbb{M}_{0+}^{(\alpha,\beta)\mu}f(t_0)\geq 0$ is true.
The second part of Theorem 3.6 can be proved similarly.

4. Analytical approach: PDE

4.1. A direct problem.

Let us consider the following PDE

$$\mathbb{M}_{0+}^{(\alpha,\beta)\mu}u(t,x)-u_{xx}(t,x)=f(t,x) \tag{4.1}$$

in a domain $\Omega=\{(t,x):\ 0<t<T,\ 0<x<1\}$. Here $n-1<\alpha,\beta<n$, and $f(t,x)$ is given a real function. We are interested in finding $u(t,x)$ that a regular solution of the Eq. (4.1) in Ω, satisfying

- regularity conditions: $u_{xx}\in C(\Omega)$, $\mathbb{M}_{0+}^{(\alpha,\beta)\mu}u(t,x)\in C(\Omega)$, $\left(\frac{t^p}{p}\right)^{n-\gamma}u(t,x)\in C(\bar{\Omega})$;
- initial conditions:

$$\lim_{t\to 0+}\left(t^{1-p}\frac{d}{dt}\right)^{n-k}{}_{K}I_{0t}^{(1-\mu)(n-\beta),p}u(t,x)=A_k(x),\ \ 0\leq x\leq 1,\ \ k=\overline{1,n}; \tag{4.2}$$

- boundary conditions:

$$u(t,0)=u(t,1)=0\ ,\ \ 0<t\leq T. \tag{4.3}$$

Here $A_k(x)$ are given functions.
We search a solution of the problem (4.1),(4.2)-(4.3) as follows:

$$u(t,x) = \sum_{m=1}^{+\infty} v_m(t)\sin m\pi x \ . \tag{4.4}$$

Substituting (4.4) into (4.1) and (4.2) we will obtain the following Cauchy problem for $u_m(t)$:

$$\begin{cases} \mathbb{M}_{0+}^{(\alpha,\beta)\mu} v_m(t) + (m\pi)^2 v_m(t) = f_m(t), \\ \lim\limits_{t\to 0+}\left(t^{1-p}\dfrac{d}{dt}\right)^{n-k} {}_K I_{0+}^{(1-\mu)(n-\beta),p} v_m(t) = A_{k,m}, \end{cases} \tag{4.5}$$

where $A_{k,m} = 2\int\limits_0^1 A_k(x)\sin m\pi x dx$, $f_m(t) = 2\int\limits_0^1 f(t,x)\sin m\pi x dx$.
The solution of (4.5) can be represented as follows:

$$\begin{gathered} v_m(t) = \sum_{j=1}^{n} A_{j,m}\left(\frac{t^p}{p}\right)^{\gamma-j} E_{\sigma,\gamma+1-j}\left[-(m\pi)^2\left(\frac{t^p}{p}\right)^{\sigma}\right] + \\ + \int\limits_0^t s^{p-1}\left(\frac{t^p-s^p}{p}\right)^{\sigma-1} E_{\sigma,\sigma}\left[-(m\pi)^2\left(\frac{t^p-s^p}{p}\right)^{\sigma}\right] f_m(s)ds \ , \end{gathered} \tag{4.6}$$

where $\gamma = \beta + \mu(n-\beta)$, $\sigma = \beta + \mu(\alpha-\beta)$.

In the following subsection, we consider the inverse source problem for sub-diffusion $(0 < \alpha, \beta < 1)$ equations.

4.2. Inverse source problem.

Let us assume that $0 < \alpha, \beta < 1$ and $f(t,x) = h(x)$. We are interested to find a pair of functions $\{u(t,x),\ h(x)\}$, satisfying Eq. (4.1) in Ω together with conditions (4.2),(4.3) and the following over-determination condition

$$u(\xi,x) = \varphi(x),\ 0 \le x \le 1 \tag{4.7}$$

for a fixed $\xi \in (0,T]$, where $\varphi(x)$ is a given function . Based on (4.4),(4.6), considering that $n = 1$ one can get

$$\begin{gathered} u(t,x) = \sum_{m=1}^{+\infty}\sin m\pi x\left[A_{1,m}\left(\frac{t^p}{p}\right)^{\gamma-1} E_{\sigma,\gamma}\left[-(m\pi)^2\left(\frac{t^p}{p}\right)^{\sigma}\right] + \right. \\ \left. + h_m\int\limits_0^t s^{p-1}\left(\frac{t^p-s^p}{p}\right)^{\sigma-1} E_{\sigma,\sigma}\left[-(m\pi)^2\left(\frac{t^p-s^p}{p}\right)^{\sigma}\right] ds\right]. \end{gathered} \tag{4.8}$$

Considering that

$$\int\limits_0^t s^{p-1}\left(\frac{t^p-s^p}{p}\right)^{\sigma-1} E_{\sigma,\sigma}\left[-(m\pi)^2\left(\frac{t^p-s^p}{p}\right)^{\sigma}\right] ds = \left(\frac{t^p}{p}\right)^{\sigma} E_{\sigma,\sigma+1}\left[-(m\pi)^2\left(\frac{t^p}{p}\right)^{\sigma}\right],$$

we deduce

$$\begin{gathered} u(t,x) = \sum_{m=1}^{+\infty}\sin m\pi x\left[A_{1,m}\left(\frac{t^p}{p}\right)^{\gamma-1} E_{\sigma,\gamma}\left[-(m\pi)^2\left(\frac{t^p}{p}\right)^{\sigma}\right] + \right. \\ \left. + h_m\left(\frac{t^p}{p}\right)^{\sigma} E_{\sigma,\sigma+1}\left[-(m\pi)^2\left(\frac{t^p}{p}\right)^{\sigma}\right]\right]. \end{gathered} \tag{4.9}$$

Using over-determination condition (4.7), we find h_m as follows:

$$h_m = \left(\frac{\xi^p}{p}\right)^{\mu(1-\alpha)-1} \frac{\varphi_m\left(\frac{\xi^p}{p}\right)^{1-\gamma} - A_{1,m}E_{\sigma,\sigma}\left[-(m\pi)^2\left(\frac{\xi^p}{p}\right)^{\sigma}\right]}{E_{\sigma,\sigma+1}\left[-(m\pi)^2\left(\frac{\xi^p}{p}\right)^{\sigma}\right]}. \tag{4.10}$$

It is known that if $0<\sigma<1$, then $E_{\sigma,\sigma+1}(\dots)$ has no zeros . If $0<\alpha,\beta<1$, then definitely $0<\sigma<1$.
If we substitute (4.10) into (4.9), we get

$$\begin{aligned} u(t,x) = \sum_{m=1}^{+\infty} \sin m\pi x \Bigg[A_{1,m}\left(\frac{t^p}{p}\right)^{\gamma-1} E_{\sigma,\gamma}\left[-(m\pi)^2\left(\frac{t^p}{p}\right)^{\sigma}\right] + \left(\frac{t^p}{p}\right)^{\sigma}\left(\frac{\xi^p}{p}\right)^{\mu(1-\alpha)-1} \times \\ \frac{\varphi_m\left(\frac{\xi^p}{p}\right)^{1-\gamma} - A_{1,m}E_{\sigma,\sigma}\left[-(m\pi)^2\left(\frac{\xi^p}{p}\right)^{\sigma}\right]}{E_{\sigma,\sigma+1}\left[-(m\pi)^2\left(\frac{\xi^p}{p}\right)^{\sigma}\right]} E_{\sigma,\sigma+1}\left[-(m\pi)^2\left(\frac{t^p}{p}\right)^{\sigma}\right]\Bigg]. \end{aligned} \tag{4.11}$$

To prove the uniform convergence of infinite series corresponding to $h(x)$, $u(t,x)$, $u_{xx}(t,x)$ and $\mathbb{M}_{0t}^{(\alpha,\beta)\mu}u(t,x)$, we will use the following estimates:

$$\begin{gathered} \left|E_{\alpha,\beta}(-z)\right| \le \frac{C}{1+|z|} \le \frac{C}{|z|}, \\ \sum_{m=1}^{+\infty}\left|A_{1,m}\right| < \sum_{m=1}^{+\infty}\frac{1}{m\pi}\left|A_{1,m}^{(1)}\right| < \sum_{m=1}^{+\infty}\frac{1}{(m\pi)^2} + \|A_1{}'(x)\|_{L_2}^2, \\ \sum_{m=1}^{+\infty}|\varphi_m| < \sum_{m=1}^{+\infty}\frac{1}{m\pi}\left|\varphi_m^{(1)}\right| < \sum_{m=1}^{+\infty}\frac{1}{(m\pi)^2} + \|\varphi'(x)\|_{L_2}^2 . \end{gathered} \tag{4.12}$$

Based on (4.10), considering (4.12) we get

$$|h(x)| \le \sum_{m=1}^{+\infty}|h_m| < \left(\frac{\xi^p}{p}\right)^{\mu(1-\alpha)-1}\left(C_1 + \|\varphi'(x)\|_{L_2}^2 + \|A_1{}'(x)\|_{L_2}^2\right) .$$

Similarly, based on (4.11), considering (4.12) one can get

$$\begin{gathered} |u(t,x)| \le \sum_{m=1}^{+\infty}|v_m(t)| \le \left(\frac{t^p}{p}\right)^{\gamma-1}\|A_1(x)\|_C + \\ \left(\frac{t^p}{p}\right)^{\sigma}\left(\left(\frac{\xi^p}{p}\right)^{\mu(1-\alpha)-1}\|A_1(x)\|_C + \left(\frac{\xi^p}{p}\right)^{\mu(1-\alpha)-\gamma}\|\varphi(x)\|_C\right), \end{gathered}$$

$$\begin{gathered} |u_{xx}(t,x)| \le \sum_{m=1}^{+\infty}(m\pi)^2|v_m(t)| < \left(\frac{t^p}{p}\right)^{\gamma-1}\left(C_2 + \|A_1{}'(x)\|_{L_2}^2\right) + \\ \left(\frac{t^p}{p}\right)^{\sigma}\left[\left(\frac{\xi^p}{p}\right)^{\mu(1-\alpha)-1}\left(C_3 + \|A_1{}'(x)\|_{L_2}^2\right) + \left(\frac{\xi^p}{p}\right)^{\mu(1-\alpha)-\gamma}\left(C_4 + \|\varphi'(x)\|_{L_2}^2\right)\right] . \end{gathered}$$

The convergence of an infinite series corresponding to $\mathbb{M}_{0+}^{(\alpha,\beta)\mu}u(t,x)$ can be obtained for Eq.(4.1).
Also, we must note that the uniqueness of the solution to the direct problem can be proved using the maximum principle that was proposed in [21]. The inverse source problem can be proved based on the completeness of the Fourier-sinus series in $L_2(0,1)$.

Example. Let $A_1(x) = \varphi(x) = \sin \pi x$. Then the solution of the inverse source problem will have the form

$$u(t,x) = \sin \pi x \left[\left(\frac{t^p}{p}\right) E_{\sigma,\gamma} \left[-\pi^2 \left(\frac{t^p}{p}\right)^{\sigma} \right] + \right.$$

$$\left. + \left(\frac{t^p}{p}\right)^{\sigma} \left(\frac{\xi^p}{p}\right)^{\mu(1-\alpha)-1} \frac{\left(\frac{\xi^p}{p}\right)^{1-\gamma} - E_{\sigma,\sigma}\left[-\pi^2\left(\frac{\xi^p}{p}\right)^{\sigma}\right]}{E_{\sigma,\sigma+1}\left[-\pi^2\left(\frac{\xi^p}{p}\right)^{\sigma}\right]} E_{\sigma,\sigma+1}\left[-\pi^2\left(\frac{t^p}{p}\right)^{\sigma}\right] \right],$$

where $\gamma = \beta + (1-\beta)$, $\sigma = \beta + \mu(\alpha - \beta)$, $0 \le \mu \le 1$, $p \in \mathbb{R}^+$, $\xi \in (0, T]$.
1) $\alpha = 0,7$; $\beta = 0,7$; $p = 2$; $\mu = 0,5$; $\xi = 1$
2) $\alpha = 0,9$; $\beta = 0,8$; $p = 2$; $\mu = 0,5$; $\xi = 1$
With these values, we will present numerical results in the corresponding graphs. Figures 1 illustrate the numerical results calculated for $u(t,x)$ with the 2 settings.

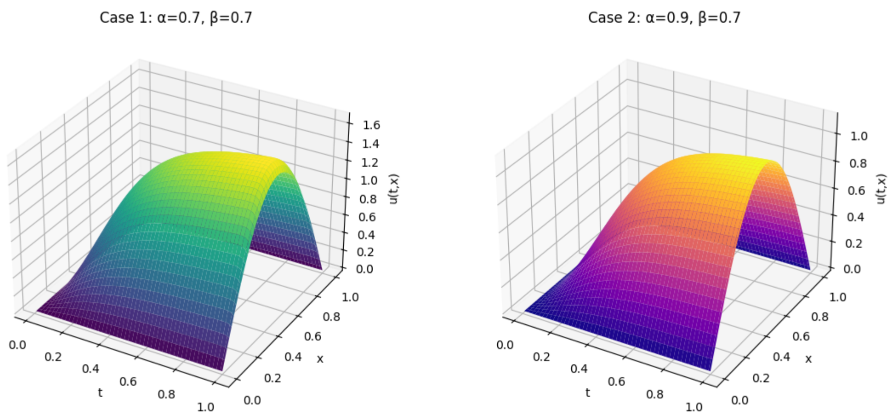

Figure 1. The numerical solutions of $u(x,t)$ when $T = 1$.

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Author's contributions

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References

[1] Mainardi, F.: Fractional Calculus and Waves in Linear Viscoelasticity: An Introduction to Mathematical Models. World Scientific. Singapore (2010).

[2] Baleano, D., Guvens, Z. B., Machado, J. T.: New Trends in Nanotechnology and Fractional Calculus Applications. Springer. Singapore (2010).

[3] Tarasov, V. E.: Fractional Dynamics: Applications of Fractional Calculus to Dynamics of Particles, Fields and Media. Springer. New York(2010).

[4] Atanackovic, T., Pilipovic, S., Stankovic, B., Zorica, D.: Fractional Calculus with Applications in Mechanics: Vibrations and Diffusion Processes. Wiley-ISTE. London (2014).

[5] Korbel, J., Luchko, Y.: *Modeling financial processes with a space-time fractional diffusion equation of varying order.* Fractional Calculus and Applied Analysis. **19** (19), 1414-1433 (2016). https://DOI: 10.1515/fca-2016-0073

[6] Copot, D., Tenreiro, M. J., Bates, J.: *The role of fractional calculus in modeling biological phenomena: A review.* Communications in Nonlinear Science and Numerical Simulation. **51**, 141-159 (2017). https://doi.org/10.1016/j.cnsns.2017.04.001

[7] Samko, S. G., Kilbas, A.A., Marichev, O.I.: Fractional Integrals and Derivatives: Theory and Applications. Gordon and Breach Science Publishers. Philadelphia (1993).

[8] Kilbas, A. A., Srivastava, H. M., Trujillo, J. J.: Theory and Applications of Fractional Differential Equations. Gordon and Breach Science Publishers. Amsterdam (2006).

[9] Baleanu, D., Fernandez, A.: *On Fractional Operators and Their Classifications.* Mathematics. **7** (9), 830 (2019). https://doi.org/10.3390/math7090830

[10] Katugampola, U. N.: *New approach to a generalized fractional integral.*Bulletin of Mathematical Analysis and Applications **6** (4), 1-15. (2014). https://doi.org/10.48550/arXiv.1106.0965

[11] Lupinska, B.: *Properties of the Katugampola fractional operators.* Tatra Mountains Mathematical Publications. **79** (2), 135-148 (2021). https://doi.org/10.2478/tmmp-2021-0024

[12] Olivera, D. S., Olivera, E. D.: *Hilfer-Katugampola fractional derivatives.* Computational and Applied Mathematics. **37** (1), 3672-3690 (2018). https://doi.org/10.1007/s40314-017-0536-8

[13] Abbas, S., Mouffak, B., Hamidi, N., Yong, Z.: *Implicit coupled Hilfer-Hadamard fractional differential systems under weak topologies.* Advances in Difference Equations. 328 (2018). https://doi.org/10.1186/s13662-018-1787-4

[14] Bulavatsky, M. B.: *Closed form of the solutions of some boundary-value problems for anomalous diffusion equation with Hilfer's generalized derivative.* Cybernetics and Systems Analysis. **50** (4), 570-577 (2014). https://doi.org/10.1007/s10559-014-9645-1

[15] Karimov, E., Ruzhansky, M., Toshtemirov, B.: *Solvability of the boundary-value problem for a mixed equation involving hyper-Bessel fractional differential operator and bi-ordinal Hilfer fractional derivative.* Mathematical Methods in the Applied Sciences. **46** (1), 54-70 (2023). https://doi.org/10.1002/mma.8491

[16] Luchko, Y.: *General fractional integrals and derivatives and their applications.*Physica D Nonlinear Phenomena. **455** (9), 133906 (2023). https://doi.org/10.1016/j.physd.2023.133906

[17] Hilfer, R.: Applications of Fractional Calculus in Physics. World Scientific. Germany (2000).

[18] Salohiddinov, M.: Integral equations. Yangiyul polygraph service. Tashkent (2007). (in Uzbek).

[19] Gorenflo, R., Kilbas, A. A., Mainardi, F., Rogosin, S.: Mittag-Leffler functions, related topics and applications. Springer. London (2014).

[20] Magnus, R.: Fundamental Mathematical Analysis. Springer. Reykjavik (2020).

[21] Cao, L., Kong, H., Zeng, S. D.: *Maximum principles for time-fractional Caputo- -Katugampola diffusion equations.* Journal of Nonlinear Sciences and Applications. **10** (4), 2257–22679 (2017). https://doi.org/10.22436/jnsa.010.04.75

[22] Kerbal, S., Khasanov, Sh.: *On Kattugampola-Prabhakar fractional integral-differential operators.* Gulf Journal of Mathematics. **20** (1), 190–221 (2025). https://doi.org/10.56947/gjom.v20i.2856

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