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Research article

The Cauchy problem for a nonhomogeneous time-fractional beam vibration equation involving the Gerasimov–Caputo operator

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Abstract

In this paper, a nonhomogeneous initial value problem posed in a half-plane for a fourth-order differential equation involving a time-fractional derivative is investigated. The problem is divided into three parts. First, the solution of the corresponding problem with homogeneous initial conditions is obtained. Subsequently, using the obtained fundamental solution, an explicit expression for the solution of the nonhomogeneous initial value problem is derived.

Keywords: *Cauchy problem; beam vibration equation; Schrödinger equation; Mittag–Leffler function; Wright function; Gerasimov–Caputo fractional derivative; Riemann–Liouville fractional integral.*

AMS Subject Classification (2020): *Primary: 35R11; Secondary: 26A33; 33E12; 35A08; 74K10.*

1. Introduction

The study of fractional differential equations has experienced significant growth in recent years due to its broad range of theoretical and practical applications. Unlike classical differential equations based on integer-order derivatives, fractional models naturally incorporate memory-dependent behavior and spatial or temporal nonlocality, allowing for a more realistic description of many complex phenomena [1, 2, 3]. Consequently, differential equations involving fractional derivatives are now widely used in various fields, including materials science, viscoelasticity, heat and mass diffusion, mechanics, and engineering applications [4, 5].

In mechanical systems, particularly in problems related to the vibrations of rods, beams, and plates, the analysis of oscillatory processes plays an important role in structural mechanics, stability analysis, and engineering design. Such vibration phenomena are often modeled by fourth-order partial differential equations [6, 7]. Among these classes of equations, beam vibration equations occupy a special place because of their significant theoretical interest and wide range of practical applications.

It is well known that the vibration of a dynamic beam is governed by the following equation:

$$\rho\frac{\partial^2 u(x,t)}{\partial t^2} + EI\frac{\partial^4 u(x,t)}{\partial x^4} = f(x,t), \tag{1.1}$$

where $u(x,t)$ denotes the transverse displacement of the beam at position x and time t, ρ is the linear mass density representing the mass per unit length, and EI is the flexural rigidity of the beam, with E being Young's modulus and I the second moment of area of the cross-section. The function $f(x,t)$ represents the external force acting on the beam.

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Introducing appropriate notation, equation (1.1) can be rewritten in the following form:

$$\frac{\partial^2 u\,(x,t)}{\partial t^2} + a^2 \frac{\partial^4 u\,(x,t)}{\partial x^4} = F\,(x,t)\,.$$

Various initial, mixed, and inverse problems for equations of this type, as well as for equations obtained from them through suitable transformations, have been investigated in numerous studies [8, 9, 10, 11, 12, 13]. In addition, the existence and qualitative properties of solutions to the Cauchy problem and initial–boundary value problems in the case $F(x,t) = 0$ were analyzed in [14, 15].

Furthermore, increasing attention has recently been devoted to beam vibration equations involving fractional-order operators, such as

$${}^{C}D_{0t}^{\alpha}V(x,t) + \frac{\partial^4 V(x,t)}{\partial x^4} = g(x,t), \tag{1.2}$$

where $1 < \alpha < 2$, and ${}^{C}D_{0t}^{\alpha}$ denotes the Gerasimov–Caputo fractional derivative, defined by

$${}^{C}D_{0t}^{\alpha}f(x,t) = \frac{1}{\Gamma(2-\alpha)}\int_0^t (t-\tau)^{1-\alpha} f''(x,\tau)\,d\tau, \qquad 1 < \alpha < 2$$

When $\alpha = 2$, equation (1.2) reduces to the classical beam vibration equation.

Significant contributions to this rapidly developing research area can be found in [13, 16, 17] and the references therein. In [18, 19], various inverse problems for the fractional beam vibration equation were investigated based on the the considered direct problem. In [20], the authors investigated an initial–boundary value problem for a multidimensional fractional beam equation and established its solution by applying the method of separation of variables. The authors of [21] developed a fractional Euler–Bernoulli beam theory and successfully explained the experimentally observed stiffness enhancement in microbeams through comparisons with experimental data. Fractional beam vibration models were also employed in [5, 22] to describe nonlocal effects, memory properties, and material damage processes in a mathematically rigorous framework. In [23], several initial and boundary value problems for the beam vibration equation were solved by exploiting its connections with Brownian motion and the heat equation. Moreover, the fractional beam vibration equation was analyzed, revealing deep relationships among stochastic processes, anomalous diffusion, and fractional dynamics. Finally, in [27], a half-line initial value problem for the homogeneous fractional beam vibration equation was studied.

In this paper, we investigate the problem of finding a solution $V(x,t) \in M$ of the fractional differential equation (1.2) in the domain

$$\Omega = \{(x,t) : x \in \mathbb{R},\ t > 0\},$$

subject to the initial conditions

$$V(x,0) = \varphi(x), \qquad V_t(x,0) = \psi(x), \tag{1.3}$$

where $1 < \alpha < 2$, and $\varphi(x)$, $\psi(x)$, and $g(x,t)$ are given sufficiently smooth functions. Here, M denotes the class of functions that are α-times differentiable with respect to t and four times differentiable with respect to x in Ω, and are continuously differentiable once with respect to t and twice with respect to x in a neighborhood of the boundary of $\overline{\Omega} = (x,t) : x \in \mathbb{R}, ; t \geq 0$.

In addition, the following decay conditions at infinity are imposed:

$$\lim_{|x|\to\infty} V(x,t) = 0, \qquad \lim_{|x|\to\infty} V_{xx}(x,t) = 0. \tag{1.4}$$

To facilitate the subsequent analysis, we first recall several properties of the operators and special functions that will be used throughout the paper.

2. Preliminary functions and operators

Definition 2.1. Let $\alpha > 0$ and let $f(t)$ be an integrable function. The Riemann–Liouville fractional integral of order α is defined by

$$I_{0t}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau.$$

Definition 2.2. For $\alpha > 0$, $n-1 \le \alpha < n$ and $f \in C^n(0,t)$, the Riemann-Liouville fractional derivative is defined as follows:

$${}^{RL}D_{0t}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-\tau)^{n-\alpha-1} f(\tau)\, d\tau$$

Definition 2.3. For $\alpha > 0$, $n-1 < \alpha \le n$ and $f \in C^n(0,t)$, the Gerasimov-Caputo fractional derivative is defined as follows:

$${}^{C}D_{0t}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau)\, d\tau$$

Lemma 2.1. *If* $\alpha \ge 0$ *and* f *is a continuous function, then*

$${}^{C}D_{0,t}^{\alpha} I_{0,t}^{\alpha} f(t) = f(t) \tag{2.1}$$

$${}^{RL}D_{0,t}^{\alpha} I_{0,t}^{\alpha} f(t) = f(t).$$

Lemma 2.2. *Let* $\alpha > 0$, $n \notin \mathbb{N}$, $n = [\alpha] + 1$ *and* $f \in C^m[a,b]$. *Then* ${}^{C}D_{at}^{\alpha} f \in C[a,b]$ *and* ${}^{C}D_{at}^{\alpha} f(a) = 0$ *holds.*

Lemma 2.3. *Let* $0 < \alpha < 1$, $f \in A^1[a,b]$ *and* $f(0) = 0$. *Then*

$$I_{0t}^{\alpha}\, {}^{C}D_{0t}^{\alpha} f(t) = f(t), \tag{2.2}$$

$$I_{0t}^{\alpha}\, {}^{RL}D_{0t}^{\alpha} f(t) = f(t), \tag{2.3}$$

$${}^{RL}D_{0,t}^{\alpha} f(t) = {}^{C}D_{0,t}^{\alpha} f(t). \tag{2.4}$$

Lemma 2.4. *Let* $a < b$ *and* $n \in \mathbb{N}$, *and assume that* $f \in C^n[a,b]$. *Moreover, let* $\alpha, \beta > 0$ *be such that there exists* $l \in \mathbb{N}$, $l \le n$, *and* $\alpha, \alpha+\beta \in [l-1, l]$. *Then the following property holds:*

$${}^{C}D_{at}^{\alpha}\, {}^{C}D_{at}^{\beta} f(t) = {}^{C}D_{at}^{\alpha+\beta} f(t) \tag{2.5}$$

The proofs of the above lemmas can be found in [24].

Lemma 2.5. *([27]) Let* $\alpha \in (1,2)$, $f \in C^2(a,b) \cap C^1[a,b]$, *and* $f'(a) = 0$. *Then the following equality holds:*

$${}^{C}D_{at}^{\alpha} f(t) = {}^{C}D_{at}^{\alpha/2}\, {}^{C}D_{at}^{\alpha/2} f(t). \tag{2.6}$$

Lemma 2.6. *([29]) Let* $u(t) \in C[a,b]$, $t \in (a,b)$, *and* $\alpha \in (0,1)$. *Then*

$${}^{RL}D_{0t}^{\alpha} f^2(t) \ge 2f(t)\, {}^{RL}D_{0t}^{\alpha} f(t), \quad f(t) \in C[0,\infty). \tag{2.7}$$

Now we present the special functions that will be used in this work and the relations between them.

Definition 2.4. The one-parameter Mittag–Leffler function is defined as follows:

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \alpha \in \mathbb{C}, \operatorname{Re}\alpha > 0, z \in \mathbb{C}. \tag{2.8}$$

The two-parameter Mittag–Leffler function is defined as follows:

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \alpha, \beta \in \mathbb{C}, \operatorname{Re}\alpha > 0, \operatorname{Re}\beta > 0, z \in \mathbb{C} \tag{2.9}$$

For the Mittag–Leffler function, the following estimate holds:

Lemma 2.7. *[1] If $\alpha < 2$, β is an arbitrary real number, and μ satisfies the condition $\pi\alpha/2 < \mu < \min \pi, \pi\alpha$, and C is a real constant, then*

$$|E_{\alpha,\beta}(z)| \leq \frac{C}{1+|z|},$$
$$\mu \leq |\arg(z)| \leq \pi, \quad |z| \geq 0.$$

Definition 2.5. The Wright function is defined as follows:

$$W_{\lambda,\mu}(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!\Gamma(\lambda n + \mu)}, \lambda > -1, \mu \in \mathbb{C}, z \in \mathbb{C} \tag{2.10}$$

Definition 2.6. Also, an auxiliary function related to the Wright function was introduced in [3] as follows:

$$M_\nu(z) = W_{-\nu,1-\nu}(-z), 0 < \nu < 1$$

or

$$M_\nu(z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\Gamma(-\nu n + 1 - \nu)} = \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{(-z)^n}{n!} \Gamma(\nu n + \nu) \sin(\pi\nu(n+1)) \tag{2.11}$$

Definition 2.7. In [30], the following Wright-type function is introduced:

$$e_{\alpha,\beta}^{\mu,\delta}(x) = \sum_{n=0}^{+\infty} \frac{x^n}{\Gamma(\alpha n + \mu)\Gamma(\delta - \beta n)}. \tag{2.12}$$

Between (2.10) and(2.12), the following relation holds:

$$W_{-\beta,\delta}(x) = e_{1,\beta}^{1,\delta}(x).$$

Lemma 2.8. *For (2.10), the following properties hold [28]:*

$$\frac{d^n}{dt^n}\left(t^{\mu-1}W_{\lambda,\mu}\left(-ct^\lambda\right)\right) = t^{\mu-n-1}W_{\lambda,\mu-n}\left(-ct^\lambda\right), \tag{2.13}$$

$${}^C D_{0t}^\alpha\left(t^{\mu-1}W_{\lambda,\mu}\left(-ct^\lambda\right)\right) = t^{\mu-\alpha-1}W_{\lambda,\mu-\alpha}\left(-ct^\lambda\right), \tag{2.14}$$

$$\frac{d}{dt}W_{\lambda,\mu}(t) = W_{\lambda,\mu+\lambda}(t) :: \tag{2.15}$$

$$|W_{\lambda,\mu}(t)| \leq C \exp\left(-\nu|t|^{\frac{1}{1-\delta}}\right), \quad |t| \to \infty$$

where $C = C(\lambda, \mu, \nu)$, $\lambda \in (0,1)$, $\varepsilon \in \mathbb{R}$, and $\nu < (1-\lambda)^{\frac{\lambda}{1-\lambda}} \cos\frac{\pi - |\arg t|}{1-\lambda}$; $\frac{1+\lambda}{2}\pi < |\arg t| \leq \pi$.

$$\int_0^{+\infty} W_{\lambda,\mu}(\delta t)\ dx = -\frac{1}{\lambda\Gamma(\lambda+\varepsilon)}, \quad \frac{1+\lambda}{2}\pi < |\arg\lambda| \leq \pi. \tag{2.16}$$

Furthermore, between (2.8) and (2.11), the following relation holds:

$$\mathcal{F}\left[\frac{1}{2}M_{\nu}\left(|x|\right)\right]=E_{2\nu}\left(-s^{2}\right) \tag{2.17}$$

where $\mathcal{F}$ denotes the Fourier transform and is defined as

$$\mathcal{F}\left(f\left(x\right)\right)=\int\limits_{-\infty}^{\infty}e^{-isx}f\left(x\right)dx.$$

Detailed properties of (2.8), (2.9), and (2.10), as well as the proof of , are given in [17]. For (2.12), the following lemma is provided in [30].

Lemma 2.9. *If $\pi \gg |\arg z| > \pi(\alpha+\beta)/2+\varepsilon,\ \varepsilon>0,\ k=0,1,2,\ldots,$ then as $z\to\infty$ the following limit relations hold:*

$$\lim_{|z|\to\infty}e^{\mu,\delta}_{\alpha,\beta}(z)=0,$$

$$\lim_{|z|\to\infty},ze^{\mu,\delta}_{\alpha,\beta}(z)=-\frac{1}{\Gamma(\mu-\alpha)\Gamma(\delta+\beta)},$$

$$\lim_{|z|\to\infty},z^{2}e^{\mu,-\beta-k}_{\alpha,\beta}(z)=\frac{1}{\Gamma(-k-\alpha)\Gamma(\delta+2\beta)},$$

$$\lim_{|z|\to\infty},z^{2}e^{\mu,-\beta-k}_{\alpha,\beta}(z)=\frac{1}{\Gamma(\mu-2\alpha)\Gamma(-k+\beta)}.$$

In [3], formulas (F.33) and (F.34) for $M_{\nu}(x)$ are generalized to $W_{\lambda,\mu}\left(x\right)$ as follows:

$$\int\limits_{0}^{\infty}x^{\delta}W_{\lambda,\mu}\left(-x\right)dx=\frac{\Gamma\left(\delta+1\right)}{2\pi i}\int\limits_{Ha}\frac{e^{\sigma}}{\sigma^{\mu-\lambda(\delta+1)}}=\frac{\Gamma\left(\delta+1\right)}{\Gamma\left(\mu-\lambda\left(\delta+1\right)\right)}.$$

$$\mathcal{F}\left[\frac{1}{2}W_{\lambda,\mu}\left(|x|\right)\right]=E_{-2\lambda,\mu-\lambda}\left(-\kappa^{2}\right) \tag{2.18}$$

3. Main results

Since equation (1.2) is linear, we represent the solution of the problem (1.2)–(1.4) in the following form

$$V(x,t)=V_1(x,t)+V_2(x,t)+V_3(x,t),\quad x\in\mathbb{R},\ t>0$$

where $V_1(x,t)$ and $V_2(x,t)$ are respectively solutions of the equation

$$^{C}D^{\alpha}_{0t}V\left(x,t\right)+V_{xxxx}\left(x,t\right)=0 \tag{3.1}$$

satisfying the conditions

$$V_1(x,0)=\varphi(x),V_{1t}(x,0)=0,x\in\mathbb{R}. \tag{3.2}$$

$$V_2(x,0)=0,V_{2t}(x,0)=\psi(x),x\in\mathbb{R}. \tag{3.3}$$

and $V_3(x,t)$ is the solution of equation (1.3) satisfying the conditions

$$V_3(x,0)=0, V_{3t}(x,0)=0, x\in\mathbb{R}. \tag{3.4}$$

If the solution $V_1(x,t)$ of equation (3.1) satisfying the initial conditions (3.2) is known, then the result of the following lemma is used to construct $V_2(x,t)$.

Lemma 3.1. *If $V_1(x,t)$ is a solution of the problem (3.1)-(3.2), then*

$$V_2(x,t)=\int_0^t V_1(x,s)\,ds \tag{3.5}$$

is a solution of equation (3.1) satisfying the following initial conditions:

$$V_2(x,0)=0, V_{2t}(x,0)=\varphi(x).$$

Proof. First, differentiating (3.5) with respect to t yields

$$V_{2t}(x,t)=V_1(x,t),$$

then applying the $(\alpha-1)$-order Gerasimov–Caputo derivative gives

$${}^C D_{0t}^{\alpha-1}\left(\frac{\partial V_2(x,t)}{\partial t}\right)={}^C D_{0t}^{\alpha-1}\left(V_1(x,t)\right) \tag{3.6}$$

Using Lemma 2.4 and relation (2.4), and taking condition (3.2) into account, we can rewrite (3.6) as

$${}^C D_{0t}^{\alpha}V_2(x,t)=\int_0^t {}^C D_{0t}^{\alpha}V_1(x,s)\,ds. \tag{3.7}$$

Now differentiating both sides of (3.5) four times with respect to x, we obtain

$$V_{2xxxx}(x,t)=\int_0^t V_{1xxxx}(x,s)\,ds. \tag{3.8}$$

Adding (3.7) and (3.8), we obtain

$${}^C D_{0t}^{\alpha}V_2(x,t)+V_{2xxxx}(x,t)=\int_0^t {}^C D_{0t}^{\alpha}V_1(x,s)+V_{1xxxx}(x,s)\,ds \tag{3.9}$$

From (3.9), since the right-hand side of the equality is equal to zero, it follows that the left-hand side is also zero, and hence in (3.5) it follows that the function $V_2(x,t)$ also satisfies equation (3.1). Now we verify the initial conditions:

$$V_2(x,0)=\lim_{t\to 0}\int_0^t V_1(x,s)\,ds=0$$

and

$$V_{2t}(x,0)=V_1(x,0)=\varphi(x).$$

If we replace $\varphi(x)$ with $\psi(x)$, then we obtain the solution $V_2(x,t)$ of equation (3.1) satisfying the following initial conditions:

$$V_2(x,0)=0, V_{2t}(x,0)=\psi(x).$$

□

Lemma 3.2. *If $V_3(x,t)$ is a solution of problem (1.2)-(3.4), then*

$$V_3(x,t) = \int_0^t w(x,t-s,s)\,ds \tag{3.10}$$

is a solution of the equation

$${}^C D_{0t}^{\alpha} w(x,t,s) + w_{xxxx}(x,s) = 0$$

satisfying the initial conditions

$$w(x,0,s) = 0, \quad w_t(x,0,s) = \lim_{t\to 0} {}^C D_{0s}^{2-\alpha} g(x,s)$$

and is therefore a solution of the problem satisfying the initial conditions.

Proof. *Differentiating (3.10) with respect to t gives*

$$V_{3t}(x,t) = \int_0^t w_t(x,t-s,s)\,ds + w(x,0,t) - 0 = \int_0^t w_t(x,t-s,s)\,ds. \tag{3.11}$$

Now applying the $\alpha-1$-order derivative with respect to t to the obtained equality, we have

$$\begin{aligned}
{}^C D_{0t}^{\alpha-1} V_{3t}(x,t) &= {}^C D_{0t}^{\alpha} V_3(x,t) = {}^C D_{0t}^{\alpha-1} \int_0^t w_t(x,t-s,s)\,ds \\
&= I_{0t}^{2-\alpha} \frac{\partial}{\partial t} \int_0^t w_t(x,t-s,s)\,ds = I_{0t}^{2-\alpha} \left(\int_0^t w_{tt}(x,t-s,s)\,ds + w_t(x,0,t) \right) \\
&= \int_0^t {}^C D_{0t}^{\alpha} w(x,t-s,s)\,ds + I_{0t}^{2-\alpha} w_t(x,0,t) - \int_0^t w_{xxxx}(x,t-s,s)\,ds + f(x,t) \\
&= -\frac{\partial^4}{\partial x^4} \int_0^t w(x,t-s,s)\,ds + f(x,t) = -V_{3xxxx}(x,t) + f(x,t)
\end{aligned}$$

$${}^C D_{0t}^{\alpha} v(x,t) = -v_{xxxx}(x,t) + f(x,t)$$

Taking the limit $t \longrightarrow 0$ in (3.10) and (3.11), we obtain that the initial conditions (3.4) are satisfied.

Now we present the following theorem for solving problem (3.1)-(3.2).

Theorem 3.1. *[27] The function $V_1(x,t) \in M$ is a solution of problem (3.1)-(3.2) if and only if*

$$U(x,t) = V_1(x,t) + i I_{0t}^{\alpha/2} V_{1xx}(x,t) \tag{3.12}$$

is a solution of the equation

$${}^C D_{0t}^{\alpha/2} U(x,t) - i U_{xx}(x,t) = 0, (x,t) \in \Omega \tag{3.13}$$

satisfying the initial condition

$$U(x,0) = \varphi(x), x \in \mathbb{R} \tag{3.14}$$

where i is the imaginary unit.

Proof. *Necessity.* Assume that $V_1(x,t) \in M$ is a solution of problem (3.1)-(3.2). We show that the function $U(x,t)$ defined in (3.12) is a solution of problem (3.13)-(3.14). From (3.12), applying the Gerasimov–Caputo derivative of order $\alpha/2$ with respect to t yields

$$ {}^C D_{0t}^{\alpha/2} U(x,t) = {}^C D_{0t}^{\alpha/2} V_1(x,t) + i\, {}^C D_{0t}^{\alpha/2} I_{0t}^{\alpha/2} V_{1xx}(x,t) . $$

Using (2.1), we obtain

$$ {}^C D_{0t}^{\alpha/2} U(x,t) = {}^C D_{0t}^{\alpha/2} V_1(x,t) + iV_{1xx}(x,t) . \tag{3.15} $$

Taking the second-order derivative of (3.12) with respect to x and multiplying by i, we obtain

$$ iU_{xx}(x,t) = iV_{xx}(x,t) - I_{0t}^{\alpha/2} V_{xxxx}(x,t) \tag{3.16} $$

Subtracting (3.16) from (3.15), we arrive at

$$ {}^C D_{0t}^{\alpha/2} U(x,t) - iU_{xx}(x,t) = {}^C D_{0t}^{\alpha/2} V_1(x,t) + I_{0t}^{\alpha/2} V_{1xxxx}(x,t) \tag{3.17} $$

By conditions (3.2), Lemma 2.2, and relations (2.2),(2.6), we obtain

$$ {}^C D_{0t}^{\alpha/2} V_1(x,t) = I_{0t}^{\alpha/2}\, {}^C D_{0t}^{\alpha} V_1(x,t) \tag{3.18} $$

Taking into account (3.18), we can rewrite (3.17) as

$$ {}^C D_{0t}^{\alpha/2} U(x,t) - iU_{xx}(x,t) = I_{0t}^{\alpha/2} \left({}^C D_{0t}^{\alpha} V_1(x,t) + V_{1xxxx}(x,t) \right) . \tag{3.19} $$

Since $V_1(x,t)$ is a solution of equation (3.1), it follows from (3.19) that $U(x,t)$ is a solution of equation (3.13).

From $V_1(x,t) \in M$ and relation (3.12), at $t=0$ we obtain

$$ U(x,0) = V(x,0) \tag{3.20} $$

which implies the initial condition (3.14).

2. *Sufficiency.* Let the function $U(x,t)$ defined by (3.12) be a solution of problem (3.13)-(3.14). We now show that $V_1 \in M$ is a solution of problem (3.1)-(3.2).

First, we verify the initial condition (3.2). From (3.20), it follows that $V_1(x,0) = \varphi(x)$. From (3.13) and (3.15), we obtain ${}^C D_{0t}^{\alpha/2} V_1(x,t) = i\frac{\partial^2}{\partial x^2}(U - V_1)$, and applying the $(1-\alpha/2)$-order Gerasimov–Caputo derivative, we get

$$ {}^C D_{0t}^{1-\alpha/2}\, {}^C D_{0t}^{\alpha/2} V_1(x,t) = i\, {}^C D_{0t}^{1-\alpha/2} (U_{xx} - V_{1xx}) , $$

From (2.5), we obtain $V_{1t}(x,t) = i\, {}^C D_{0t}^{1-\alpha/2}(U_{xx} - V_{1xx})$, and letting $t \to 0$, we get $V_{1t}(x,t) = 0$. Hence, from (3.19), it follows that $V_1(x,t)$ is a solution of problem (3.1)-(3.2). Thus, Theorem 2 is proved. □

From this theorem, the following corollary follows.

Remark 3.1. Assume that $U \in M$ is a solution of problem (3.13)–(3.14), where $\varphi(x)$ is a real-valued function. Then the function $V_1(x,t) = \operatorname{Re} U(x,t)$ is a solution of problem (3.1)–(3.2). Here, $\operatorname{Re} U(x,t)$ denotes the real part of the function $U(x,t)$.

3.1. Uniqueness of the solution

Theorem 3.2. *The solution of problem (1.2)–(1.4) is unique.* ***Proof.*** *It is sufficient to show that the problem*

$$ {}^C D_{0t}^{\alpha} V(x,t) + V_{xxxx}(x,t) = 0, \quad V(x,0) = 0, \quad V_t = 0 $$

admits only the trivial solution $V(x,t) \equiv 0$. The uniqueness of the solution to the present problem is an immediate consequence of Theorem 1 together with the uniqueness result established for problem (3.13)–(3.14). Therefore, it is enough to prove uniqueness for (3.13),(3.14).

Assume that $U_1(x,t)$ and $U_2(x,t)$ are two solutions of problem (3.13)-(3.14), and define

$$U(x,t) = U_1(x,t) - U_2(x,t).$$

Then $U(x,t)$ satisfies equation (3.13) and the homogeneous initial condition

$$U(x,0) = 0, \quad x \in \mathbb{R}, \ t > 0. \tag{3.21}$$

Now we apply the Fourier transform with respect to the variable x:

$$W(s,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-isx} U(x,t)\, dx \tag{3.22}$$

and obtain from (3.13) the following equation:

$${}^{C}D_{0t}^{\alpha/2} W(s,t) + is^2 W(s,t) = 0.$$

Let $W(s,t) = a(s,t) + ib(s,t)$. Substituting this into (3.22), we obtain

$${}^{C}D_{0t}^{\alpha/2} \left(a(s,t) + ib(s,t)\right) + is^2 \left(a(s,t) + ib(s,t)\right) = 0,$$

$${}^{C}D_{0t}^{\alpha/2} a(s,t) - s^2 b(s,t) + i\left({}^{C}D_{0t}^{\alpha/2} b(s,t) + s^2 a(s,t)\right) = 0.$$

Separating real and imaginary parts, we obtain the system

$${}^{C}D_{0t}^{\alpha/2} a(s,t) - s^2 b(s,t) = 0 \; {}^{C}D_{0t}^{\alpha/2} b(s,t) + s^2 a(s,t) = 0$$

Multiplying the first equation by $a(s,t)$ and the second by $b(s,t)$ and adding, we obtain

$$a(s,t)\, {}^{C}D_{0t}^{\alpha/2} a(s,t) + b(s,t)\, {}^{C}D_{0t}^{\alpha/2} b(s,t) = 0. \tag{3.23}$$

From condition (3.21) we obtain

$$W(s,0) = a(s,0) + ib(s,0) = 0, \quad a(s,0) = 0, \quad b(s,0) = 0.$$

Taking into account (2.4) and (2.7), relation (3.23) can be written as

$${}^{RL}D_{0t}^{\alpha/2} a^2(s,t) + {}^{RL}D_{0t}^{\alpha/2} b^2(s,t) \le 0.$$

Applying a fractional derivative of order $\alpha/2$ to both sides and using relation (2.3), we obtain

$$a^2(s,t) + b^2(s,t) \le 0. \tag{3.24}$$

From (3.24) it follows that $a(s,t) = 0$ and $b(s,t) = 0$. Hence $W(s,t) \equiv 0$. Since $U(x,t) \in C_{x,t}^{2,1}\left(\overline{\Omega}\right)$, it follows that $U(x,t) \equiv 0$ for $t > 0$. □

To solve problems (3.1)-(3.2), we use Remark 1. Then, we arrive at the following problem for the one-dimensional time-fractional Schrödinger equation with respect to the initial condition (3.13):

$${}^{C}D_{0t}^{\alpha/2} U(x,t) - iU_{xx}(x,t) = 0.$$

The solution of this problem was obtained in [25] in the following form:

$$U(x,t) = \int_{-\infty}^{\infty} \varphi(\xi)\, g_\alpha(x-\xi,t)\, d\xi,$$

where

$$g_\alpha(x-\xi,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(x-\xi)} E_{\frac{\alpha}{2},1}\left(-ik^2 t^{\frac{\alpha}{2}}\right) dk.$$

Since the Mittag–Leffler function $E\alpha/2,1\left(-ik^2 t^{\frac{\alpha}{2}}\right)$ is an even function, the above expression can be rewritten as

$$g_\alpha(x-\xi,t) = \frac{1}{\pi}\int_{0}^{\infty} \cos(k(x-\xi))\, E_{\frac{\alpha}{2},1}\left(-ik^2 t^{\frac{\alpha}{2}}\right) dk.$$

Using relation (2.17), we express the solution in terms of the Wright auxiliary function:

$$g_\alpha(x-\xi,t) = \frac{1}{2}\frac{1}{\sqrt{i\pi t^{\frac{\alpha}{4}}}} M_{\frac{\alpha}{4}}\left(\frac{|x-\xi|}{\sqrt{i t^{\frac{\alpha}{4}}}}\right)$$

From Result 1, the solution of problem (3.1)-(3.2) is given by

$$V_1(x,t) = \int_{-\infty}^{\infty} \varphi(\xi)\, G_{1\alpha}(x-\xi,t)\, d\xi, \tag{3.25}$$

where

$$G_{1\alpha}(x-\xi,t) = Re\left(g_{1\alpha}(x-\xi,t)\right) = \frac{1}{\pi}\int_{0}^{\infty} \cos(k(x-\xi))\, E_{\alpha,1}\left(-k^4 t^{\alpha}\right) dk,$$

or equivalently,

$$G_{1\alpha}(x-\xi,t) = \frac{1}{2t^{\frac{\alpha}{4}}}\sum_{n=0}^{\infty} \frac{\left(-\frac{|x-\xi|}{t^{\frac{\alpha}{4}}}\right)^n \cos\left(\frac{\pi}{4}n + \frac{\pi}{4}\right)}{n!\,\Gamma\left(-\frac{\alpha}{4}n + 1 - \frac{\alpha}{4}\right)}. \tag{3.26}$$

3.2. Existence of the solution.

We can write (3.25) in following form:

$$V_1(x,t) = \frac{1}{2}\int_{0}^{\infty} \left[\varphi\left(x-\eta t^{\alpha/4}\right) + \varphi\left(x+\eta t^{\alpha/4}\right)\right] \sum_{n=0}^{\infty} \frac{(-\eta)^n \cos\left(\frac{\pi}{4}n + \frac{\pi}{4}\right)}{n!\,\Gamma\left(-\frac{\alpha}{4}n + 1 - \frac{\alpha}{4}\right)} d\eta$$

If the function $\varphi(x)$ is bounded on $(-\infty,+\infty)$, then the improper integral in (3.25) converges uniformly in the domain $\overline{\Omega}$.

$$V_1(x,0) = \lim_{t\to 0}\frac{1}{2}\left[\int_{0}^{\infty} \left[\varphi\left(x-\eta t^{\alpha/4}\right) + \varphi\left(x+\eta t^{\alpha/4}\right)\right] \sum_{n=0}^{\infty} \frac{(-\eta)^n \cos\left(\frac{\pi}{4}n + \frac{\pi}{4}\right)}{n!\,\Gamma\left(-\frac{\alpha}{4}n + 1 - \frac{\alpha}{4}\right)} d\eta\right] =$$

$$= \varphi(x)\,\mathrm{Re}\left(\int_{0}^{\infty} M(-\zeta)\, d\zeta\right) = \varphi(x)$$

Now we show that $V_{1t}(x,0)=0$. For this purpose, we express the solution in terms of the Wright function and, using relation (2.13), take the derivative and integrating by parts 4 times, we obtain:

$$\frac{\partial}{\partial t}V_1(x,t)=V_t(x,0)\lim_{\xi\to x}t^{\alpha/2-1}\varphi''(\xi)\,\mathrm{Re}\left(W_{-\alpha/4,3\alpha/4}\left(-\frac{x-\xi}{\sqrt{i}}t^{-\alpha/4}\right)\right)$$

$$+\frac{t^{\alpha-1}}{2}\int\limits_0^\infty\left[\varphi^{(IV)}\left(x+\eta t^{\alpha/4}\right)+\varphi^{(IV)}\left(x-\eta t^{\alpha/4}\right)\right]\mathrm{Re}\left(\sqrt{i}W_{-\alpha/4,\alpha}\left(-\frac{\eta}{\sqrt{i}}\right)\right)d\eta$$

Taking into account the relations in Lemma 2.9:

$$V_{1t}(x,0)=\lim_{t\to0}\lim_{\xi\to x}t^{\alpha/2-1}\varphi''(\xi)\,\mathrm{Re}\left(W_{-\alpha/4,3\alpha/4}\left(-\frac{x-\xi}{\sqrt{i}}t^{-\alpha/4}\right)\right)$$

$$+\lim_{t\to0}\frac{t^{\alpha-1}}{2}\int\limits_0^\infty\left[\varphi^{(IV)}\left(x+\eta t^{\alpha/4}\right)+\varphi^{(IV)}\left(x-\eta t^{\alpha/4}\right)\right]\mathrm{Re}\left(\sqrt{i}W_{-\alpha/4,\alpha}\left(-\frac{\eta}{\sqrt{i}}\right)\right)d\eta=0$$

Theorem 3.3. *Let $\varphi^{(k)}(x)\in C(\mathbb{R})$ and $\left|\varphi^{(k)}(x)\right|<\infty$, $k=0,1,2,3,4$. Then the function defined by the following representation is a solution of problem (3.1)-(3.2).*

$$V_1(x,t)=\int\limits_{-\infty}^{\infty}\varphi(\xi)\,G_{1\alpha}(x-\xi,t)\,d\xi,$$

where

$$G_{1\alpha}(x-\xi,t)=\frac{t^{-\alpha/4}}{2}\sum_{n=0}^{+\infty}\frac{\left(-\frac{|x-\xi|}{t^{\alpha/4}}\right)^n\cos\left(\frac{n+1}{4}\pi\right)}{\Gamma\left(-\frac{\alpha}{4}n+1-\frac{\alpha}{4}\right)}.$$

3.3. Justification of the solution

Using (2.14) and (2.13), we take the fractional derivative of order α with respect to t and the fourth-order derivative with respect to x of $V_1(x,t)$, respectively, and obtain

$${}^{C}D_{0t}^{\alpha}V_1(x,t)=\frac{t^{-5\alpha/4}}{\sqrt{i}}\int\limits_{-\infty}^{+\infty}\varphi(\xi)\,\mathrm{Re}\left(\frac{1}{\sqrt{i}}W_{-\alpha/4,1-5\alpha/4}\left(-\frac{|x-\xi|}{\sqrt{i}}t^{-\alpha/4}\right)\right)$$

$$\frac{\partial^4}{\partial x^4}V(x,t)=-\frac{t^{-5\alpha/4}}{2}\int\limits_{-\infty}^{+\infty}\varphi(\xi)\,\mathrm{Re}\left(\frac{1}{\sqrt{i}}W_{-\alpha/4,1-5\alpha/4}\left(-\frac{|x-\xi|}{\sqrt{i}}t^{-\alpha/4}\right)\right)d\xi$$

and we have

$${}^{C}D_{0t}^{\alpha}V_1(x,t)+\frac{\partial^4}{\partial x^4}V_1(x,t)=0$$

Lemma 3.3. *The function*

$$G_{1\alpha}(x-\xi,t)=\frac{t^{-\alpha/4}}{2}\sum_{n=0}^{+\infty}\frac{\left(-\frac{|x-\xi|}{t^{\alpha/4}}\right)^n\cos\left(\frac{n+1}{4}\pi\right)}{\Gamma\left(-\frac{\alpha}{4}n+1-\frac{\alpha}{4}\right)}.$$

possesses the following properties:

(a) $G_{1\alpha} \in C^{\infty}(\Omega)$;
(b) $G_{1\alpha}$ *is a solution of equation (3.1) in the domain* Ω;
(c) $\lim_{t\to 0} G_{1\alpha}(x-\xi,t) = 0$;
(d) $\int_{-\infty}^{+\infty} G_{1\alpha}(x-\xi,t)\,d\xi = 1$.

Using the solution $V_1(x,t)$ and applying Lemma 3.1, we construct the function $V_2(x,t)$ satisfying problem (3.1)-(3.3). From (3.5), we obtain

$$V_2(x,t) = \int_0^t V_1(x,s)\,ds = \int_0^t \int_{-\infty}^{+\infty} \psi(\xi)\,G_{1\alpha}(x-\xi,s)\,d\xi\,ds.$$

Interchanging the order of integration yields

$$V_2(x,t) = \int_{-\infty}^{+\infty} \psi(\xi) G_{2\alpha}(x-\xi,t)\,d\xi, \tag{3.27}$$

where

$$G_{2\alpha}(x-\xi,t) = \int_0^t G_{1\alpha}(x-\xi,s)\,ds = \frac{1}{\pi}\int_0^{\infty} \cos\big(k(x-\xi)\big)\left(\int_0^t E_{\alpha,1}\Big(-k^4 s^{\alpha}\Big)\,ds\right)dk.$$

According to formula (2.1.53) in [2], the above integral can be written as

$$G_{2\alpha}(x-\xi,t) = \frac{t}{\pi}\int_0^{\infty} E_{\alpha,2}\Big(-k^4 t^{\alpha}\Big)\cos\big(k(x-\xi)\big)\,dk.$$

Using relation (2.18) with $\lambda = -\alpha/4, \mu = 2$ and

$$\mathcal{F}^{-1}\left[E_{\alpha/2,2}\left(-\kappa^2\right)\right] \to W_{-\alpha/4,2-\alpha/4}(-x),$$

we obtain

$$G_{2\alpha}(x-\xi,t) = \frac{t^{1-\alpha/4}}{2}\sum_{n=0}^{\infty} \frac{\left(-\frac{|x-\xi|}{t^{\alpha/4}}\right)^n \cos\left(\frac{(n+1)\pi}{4}\right)}{n!\,\Gamma\left(-\frac{\alpha n}{4}+2-\frac{\alpha}{4}\right)}. \tag{3.28}$$

Theorem 3.4. *If* $\psi(x) \in C(\mathbb{R})$ *and* $|\psi(x)| < \infty$, *then the following function is a solution of problem (3.1)-(3.3):*

$$V_2(x,t) = \int_{-\infty}^{\infty} \psi(\xi)\,G_{2\alpha}(x-\xi,t)\,d\xi,$$

where

$$G_{2\alpha}(x-\xi,t) = \frac{t^{1-\alpha/4}}{2}\sum_{n=0}^{+\infty} \frac{\left(-\frac{|x-\xi|}{t^{\alpha/4}}\right)^n \cos\left(\frac{n+1}{4}\pi\right)}{\Gamma\left(-\frac{\alpha}{4}n+2-\frac{\alpha}{4}\right)}.$$

The justification of the solution of problem (3.1)–(3.3) is carried out similarly to that of $V_1(x,t)$.

Lemma 3.4.

$$G_{2\alpha}(x-\xi,t) = \frac{t^{1-\alpha/4}}{2}\sum_{n=0}^{+\infty} \frac{\left(-\frac{|x-\xi|}{t^{\alpha/4}}\right)^n \cos\left(\frac{n+1}{4}\pi\right)}{\Gamma\left(-\frac{\alpha}{4}n+2-\frac{\alpha}{4}\right)}$$

has the following properties:

a) $G_{2\alpha} \in C^{\infty}(\Omega)$,
b) $G_{2\alpha}$ *is a solution of equation (3.1) in the domain* Ω,
c) $\lim\limits_{t\to 0} G_{2\alpha}(x-\xi,t) = 0$,
d) $\frac{\partial}{\partial t} G_{2\alpha}(x-\xi,t) = G_{1\alpha}(x-\xi,t)$,
e) $\int\limits_{-\infty}^{+\infty} G_{2\alpha}(x-\xi,t)\,d\xi = 1$.

Now, in order to construct the solution $V_3(x,t)$, we use Lemma 3.2. The solution of the function $w(x,t,s)$ appearing there is also determined in the form of (3.27):

$$w(x,t,s) = \int\limits_{-\infty}^{+\infty} {}^{C}D_{0t}^{2-\alpha} g(x,s)\, G_{2\alpha}(x-\xi,t)\,d\xi$$

Hence, by means of (3.10), we obtain the solution $V_3(x,t)$:

$$V_3(x,t) = \int\limits_0^t w(x,t-s,s)\,ds = \int\limits_0^t \int\limits_{-\infty}^{+\infty} {}^{C}D_{0S}^{2-\alpha} g(\xi,s)\, G_{2\alpha}(x-\xi,t-s)\,d\xi ds \tag{3.29}$$

Taking into account the following fractional integration by parts formula from [28],

$$\int_{\eta}^{y} h(t) D_{\eta t}^{-\delta} g(t)\,dt = \int_{\eta}^{y} g(t) D_{yt}^{-\delta} h(t)\,dt, \quad \delta > 0;$$

expression (3.29) can be written in the form

$$V_3(x,t) = \int\limits_0^t \int\limits_{-\infty}^{+\infty} g(\xi,s)\, {}^{C}D_{t-,s}^{2-\alpha} G_{2\alpha}(x-\xi,t-s)\,d\xi ds. \tag{3.30}$$

where

$$G_{3\alpha}(x-\xi,t-s) == \frac{(t-s)^{\frac{3\alpha}{4}-1}}{2} \sum_{n=0}^{\infty} \frac{\left(-\frac{|x-\xi|}{\sqrt{i(t-s)^{\alpha/2}}}\right) \cos\left(\frac{n+1}{4}\pi\right)}{\Gamma\left(-\frac{\alpha}{4}n + \frac{3\alpha}{4}\right)} \tag{3.31}$$

Theorem 3.5. *If* $g(x,t) \in C\left(\overline{\Omega}\right)$ *and* $|g(x,t)| < \infty$, *then the function defined by (3.7) satisfies problem (1.2)-(3.4):*

$$V_3(x,t) = \int\limits_0^t \int\limits_{-\infty}^{+\infty} g(\xi,s)\, {}^{C}D_{t-,s}^{2-\alpha} G_{2\alpha}(x-\xi,t-s)\,d\xi ds.$$

where

$$G_{3\alpha}(x-\xi,t-s) = \frac{(t-s)^{3\alpha/4-1}}{2} \sum_{n=0}^{+\infty} \frac{\left(-\frac{|x-\xi|}{(t-s)^{\alpha/4}}\right)^n \cos\left(\frac{n+1}{4}\pi\right)}{\Gamma\left(-\frac{\alpha}{4}n + \frac{3\alpha}{4}\right)}.$$

The justification of the solution of problem (1.2)-(3.4) is carried out similarly to that of $V_1(x,t)$.

Lemma 3.5.

$$G_{3\alpha}(x-\xi,t-s) = \frac{t^{3\alpha/4-1}}{2} \sum_{n=0}^{+\infty} \frac{\left(-\frac{|x-\xi|}{t^{\alpha/4}}\right)^n \cos\left(\frac{n+1}{4}\pi\right)}{\Gamma\left(-\frac{\alpha}{4}n + \frac{3\alpha}{4}\right)}$$

has the following properties:

a) $G_{3\alpha} \in C^{\infty}(\Omega)$,
b) $G_{3\alpha}$ *is a solution of equation (1.2) in the domain* Ω,
c) $\lim\limits_{t \to 0} G_{3\alpha}(x-\xi, t) = 0$,
d) $\int\limits_{-\infty}^{+\infty} G_{3\alpha}(x-\xi, t)\, d\xi = 1$,
e) ${}^{C}D_{0t}^{2-\alpha} G_{2\alpha}(x-\xi, t) = G_{3\alpha}(x-\xi, t)$,
f) $G_{3\alpha}(x-\xi, t)$ *is the fundamental solution of equation (1.2).*

Using (3.25), (3.27), and (3.30), the solution of problem (1.2)-(1.5) can be written in the form

$$\begin{aligned} V(x,t) &= V_1(x,t) + V_2(x,t) + V_3(x,t) \\ &= \int\limits_{-\infty}^{+\infty} \left[\varphi(\xi)\, G_{1\alpha}(x-\xi, t) + \psi(\xi)\, G_{2\alpha}(x-\xi, t) + \int\limits_{0}^{t} g(\xi, s)\, G_{3\alpha}(x-\xi, t-s)\, ds \right] d\xi \end{aligned}$$

where $G_{1\alpha}(x-\xi, t)$, $G_{2\alpha}(x-\xi, t)$, and $G_{3\alpha}(x-\xi, t-s)$ are defined by (3.26), (3.28), and (3.31), respectively.

4. Conclusion

In this paper, the Cauchy problem for a fourth-order partial differential equation involving a time-fractional derivative was investigated in the half-plane. This equation can be regarded as a mathematical model of vibrations of a homogeneous beam. Despite the mathematical complexity of the model, the original Cauchy problem can be converted, by means of two consecutive order-reduction procedures, into an equivalent Cauchy problem for the homogeneous Schrödinger equation. This transformation enables the construction of an explicit solution formula for the time-fractional beam vibration equation through the solution of the resulting auxiliary problem.

The obtained results demonstrate the existence of a solution for arbitrary functions specified in the initial conditions. The proposed approach differs methodologically from previously known investigations and provides a theoretical foundation for further studies of fractional-order beam models and related higher-order evolution equations.

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