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Research article

# Inverse Scattering Transform for the Fifth-Order Fractional KdV Equation with a Self-Consistent Source

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### Abstract

**In this paper, we constract and integrate the fifth-order Riesz fractional KdV (fRfKdV) equation with a self-consistent source in the class of rapidly decreasing functions via the inverse scattering transform(IST) method. We show that fifth-order Riesz fractional KdV equation with self-consistent sources is also an important theoretical model, since it is a completely integrable system. For one soliton cases, explicit formulas for the solutions of the problem under consideration are derived as examples to simulate their spatial structures and analyze their structural properties by selecting different values of fractional orders. The results indicate that both the source and the fractional order significantly influence the velocity of soliton propagation, while no energy dissipation occurs through out the entire motion process. This behavior corresponds to a super-dispersive transport mechanism, which is an experimentally testable prediction of the present theory. Furthermore, the analysis reveals that the absolute value of wave velocity becomes larger as $\epsilon$ increases. This study contributes valuable insights into nonlinear dynamics and soliton behavior, improving the understanding of multifaceted wave interactions in nonlinear fractional equations.**

***Keywords:*** *Riesz fractional Korteweg-de Vries equation, self-consistent source, inverse scattering transform method, Sturm-Liouville operator.*

***AMS Subject Classification (2020):*** *Primary: 37K10 ; Secondary: 37K15; 35R11.*

## 1. Introduction

The fifth-order KdV equation, sometimes referred to as a Kawahara-type equation, arises as a model for plasma waves, capillary–gravity waves, and various other dispersive phenomena [1]-[8].

Our goal is to constract and integrate the fifth-order fractional KdV equation with a self-consistent source in the class of rapidly decreasing functions via the inverse scattering transform (IST) method.

The fractional equations often predict physically measurable quantities that follow power laws. For example, in anomalous diffusion, the mean squared displacement is related to time by a power law $t^{\alpha}$, $\alpha > 0$[9, 10, 11, 12]. Similarly, for integrable soliton equations, fractional generalizations predict anomalous dispersion where the speed of localized solitonic waves are related to their amplitude by a power law[13].

In 2022, Ablowitz, Been, and Carr established for the first time that the Riesz fractional KdV equation[14]

$$q_t + \gamma(L^A)q_x = 0$$

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where

$$\gamma(L^A) = -4L^A|4L^A|^{\varepsilon}, \qquad L^A = -\frac{1}{4}\partial_x^2 - q + \frac{1}{2}q_x \int\limits_x^{\infty} dy$$

admits an inverse scattering formulation within the class of rapidly decaying functions. Following the work of Ablowitz, Been, and Carr, several nonlinear evolution equations involving the Riesz fractional derivative were also integrated by means of the direct and inverse scattering problem methods of the scattering theory [15, 17, 18, 19, 20, 21, 22].

In this paper, we study the integration of fifth-order fractional KdV equation with a self-consistent source by the inverse scattering method. We show that fRfKdV equation with a self-consistent source is also an important theoretical model, since it is a completely integrable system. For the one soliton case, explicit formulas for the solutions of the problem under consideration are derived as examples to simulate their spatial structures and analyze their structural properties by selecting different values of fractional orders.

The concept of self-consistent sources in integrable nonlinear evolution equations was first introduced by V. K. Melnikov, who showed that such equations can retain their integrability while describing richer physical and mathematical structures[23]. Physically, the sources appear in solitary waves with non-constant velocity and lead to a variety of dynamics of physical models. They have important applications in plasma physics, hydrodynamics, solid-state physics, etc.[24, 25, 26, 27, 28]. Recently, numerous studies have been devoted to the investigation of nonlinear evolution equations with a sources, motivated by both physical and mathematical considerations in[29]-[47].

## 2. Statement of the Problem

In the present paper, we consider the following fifth-order Riesz fractional Korteweg-de Vries equation with a self-consistent source

$$\begin{cases} u_t + \Omega(L^*)u_x = 4\sum\limits_{m=1}^{N} \dfrac{\partial}{\partial x}\left(|\varphi_m|^2\right), \\ -\varphi_m'' + u(x,t)\varphi_m = \lambda_m \varphi_m, \quad m = 1, \dots, N \end{cases} \tag{2.1}$$

under initial condition

$$u(x,0) = u_0(x), \qquad x \in \mathbb{R}. \tag{2.2}$$

Here, the operator $\Omega(L^*)$:

$$\Omega(L^*) = 16(L^*)^2|4L^*|^{\varepsilon}, \qquad L^* = -\frac{1}{4}\partial_x^2 - u + \frac{1}{2}u_x \int\limits_x^{\infty} dy \tag{2.3}$$

is constructed from the anomalous dispersion relation

$$\Omega(k^2) = 16k^4|2k|^{2\epsilon}, \tag{2.4}$$

of the linearization part of Equation (2.1)

$$u_t + |-\partial_x^2|^{\varepsilon} u_{xxxxx} = 0,$$

replacing the spectral variable $k^2$ by the operator $L^*$ in (2.4) yields precisely the definition of $\Omega(L^*)$ introduced in (2.3). Here $|-\partial_x^2|^{\epsilon}$, $\epsilon \in (0,1)$ is the Riesz fractional derivative [48] defined as

$$\left(-\partial_x^2\right)^{\epsilon} u(x,t) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} \hat{u}(k,t)|k|^{2\epsilon} e^{ikx}\, dk,$$

$$\hat{u}(k,t) = \int_{-\infty}^{\infty} u(x,t)e^{-ikx}dx.$$

The explicit form of the fractional power $|4L^*|^{\varepsilon}$, $\varepsilon \in (0,1)$ will be given in section 5.

In (2.1), the functions $\varphi_m = \varphi_m(x,t)$, $m = 1, ..., N$, are eigenfunctions of the time-dependent operator

$$\mathcal{L}(t) := -\frac{d^2}{dx^2} + u(x,t), \quad x \in \mathbb{R}, \ t > 0,$$

associated with the eigenvalues $\lambda_m(t) = -\nu_m^2(t)$, $m = 1, ..., N$, which satisfy the following normalization condition

$$\int_{-\infty}^{\infty} |\varphi_m^2(x,t)|\, dx = \beta_m(t), \qquad m = 1, ..., N$$

where $\beta_m(t)$ are given positive continuous functions.

The real-valued initial function $u_0(x)$ is assumed to satisfy:

1. $$\int_{-\infty}^{\infty} (1+|x|)|u_0(x)|\, dx < \infty;$$

2. The operator
$$\mathcal{L}(0) := -\frac{d^2}{dx^2} + u_0(x), \quad x \in \mathbb{R},$$
has exactly $N$ simple negative eigenvalues $\lambda_j(0) < 0$, $j = 1, \ldots, N$.

In the problem (2.1)-(2.2), the functions $u(x,t)$ and $\varphi_m(x,t)$, $m = 1, ..., N$, are unknown.

The main aim of this work is to derive representations for the solutions

$$\{u(x,t),\ \varphi_1(x,t),\ \varphi_2(x,t), ..., \varphi_N(x,t)\}$$

of the problem (2.1)-(2.2) with the use of the inverse scattering method for the operator $\mathcal{L}(t)$.

**Note.** In (2.3), if we set $\epsilon = 0$, then we obtain the classical fifth-order KdV equation with a self-consistent source:

$$u_t + u_{xxxxx} + 10uu_{xxx} + 20u_x u_{xx} + 30u^2 u_x = G(x,t),$$

where $G(x,t) = 4\sum_{m=1}^{N} \frac{\partial}{\partial x}\left(|\varphi_m|^2\right)$.

## 3. Preliminaries

In this section, we give the basic facts about the scattering theory of the Sturm-Liouville operator $\mathcal{L}(t)$ [49, 50].

We consider the Sturm-Liouville scattering problem

$$\mathcal{L}(t)y \equiv -y'' + u(x,t)y = k^2 y, \qquad |x| < \infty, \ \forall t \geq 0 \tag{3.1}$$

where $k^2$ is the spectral parameter and $u(x,t)$ is real function satisfying condition

$$\int_{-\infty}^{\infty} (1+|x|)|u(x,t)|\, dx < \infty, \ \forall t \geq 0. \tag{3.2}$$

We denote by $f = f(x,k,t)$ and $g = g(x,k,t)$ the Jost solutions of (3.1) with the asymptotics

$$f(x,k,t) = e^{ikx} + o(1), \quad x \to \infty; \qquad g(x,k,t) = e^{-ikx} + o(1), \quad x \to -\infty, \quad (\operatorname{Im} k = 0). \tag{3.3}$$

Under condition (3.2), such solutions exist and are uniquely determined by asymptotics (3.3). The solutions $f(x,k,t)$ and $g(x,k,t)$ satisfy the representations

$$f(x,k,t) = e^{ikx} + \int_x^\infty K^+(x,s;t)e^{iks}\,ds, \qquad g(x,k,t) = e^{-ikx} + \int_{-\infty}^x K^-(x,s;t)e^{-iks}\,ds, \tag{3.4}$$

where kernels $K^+(x,s;t)$ and $K^-(x,s;t)$ are real functions related to the potential $u(x,t)$ by the relations

$$u(x,t) = \mp 2\frac{d}{dx}K^\pm(x,x;t).$$

With respect to the variable $k$, the Jost solutions $f(x,k,t)$ and $g(x,k,t)$ continue analytically into the upper half-plane $\operatorname{Im} k > 0$.

For real $k$, pairs of functions $\{f(x,k,t), f(x,-k,t)\}$ and $\{g(x,k,t), g(x,-k,t)\}$ are pairs of linearly independent solutions of equation (3.1), therefore

$$f(x,k,t) = b(k,t)g(x,k,t) + a(k,t)g(x,-k,t),$$

$$g(x,k,t) = -b(-k,t)f(x,k,t) + a(k,t)f(x,-k,t).$$

The functions

$$r^+(k,t) = -\frac{b(-k,t)}{a(k,t)}, \qquad r^-(k,t) = \frac{b(k,t)}{a(k,t)}$$

are called reflection coefficients (right and left, respectively). The coefficients $a = a(k,t)$ and $b = b(k,t)$ have the following properties (see [49]).

**A.** For real $k \neq 0$

$$a(-k,t) = \overline{a(k,t)}, \quad b(-k,t) = \overline{b(k,t)}, \quad |a(k,t)|^2 = 1 + |b(k,t)|^2;$$

$$a(k,t) = -\frac{1}{2ik}W\{f(x,k,t), g(x,k,t)\}, \qquad b(k,t) = \frac{1}{2ik}W\{f(x,k,t), g(x,-k,t)\};$$

$$a(k,t) = 1 + O\left(\frac{1}{k}\right), \qquad b(k,t) = O\left(\frac{1}{k}\right), \qquad k \to \pm\infty;$$

where

$$W\{f(x,k,t), g(x,k,t)\} \equiv f(x,k,t)g'(x,k,t) - f'(x,k,t)g(x,k,t);$$

**B.** With respect to the variable $k$, the function $a(k,t)$ extends analytically to the half-plane $\operatorname{Im} k > 0$ and there it has a finite number of zeros $k_n(t) = i\nu_n(t)$, $(\nu_n(t) > 0, \ \forall t \geq 0)$, $n = 1, 2, \ldots, N$, these zeros are simple, and $\lambda_n(t) = -\nu_n^2(t)$ is an eigenvalue of the operator $\mathcal{L}(t)$. In addition, the following relation holds

$$g(x, k_n(t), t) = \chi_n(t) f(x, k_n(t), t), \qquad n = 1, 2, \ldots, N. \tag{3.5}$$

The set

$$\{r^+(k,t), k \in \mathbb{R}; \nu_1(t), \nu_2(t), \ldots, \nu_N(t); \chi_1(t), \chi_2(t), \ldots, \chi_N(t)\} \tag{3.6}$$

is called the scattering data for problem (3.1). The direct scattering problem is to determine the scattering data from the potential $u(x,t)$, and the inverse problem is to reconstruct from the scattering data the potential

$u(x,t)$ of Eq.(3.1). The kernel $K^+(x,y)$ in representation (3.4) is a solution to the Gelfand-Levitan-Marchenko (GLM) integral equation

$$K^+(x,y;t) + F^+(x+y;t) + \int_x^\infty K^+(x,z;t)F^+(z+y;t)\,dz = 0, \qquad (y > x), \tag{3.7}$$

where

$$F^+(x,t) = \sum_{j=1}^N \alpha_j^+(t)e^{-\nu_j(t)x} + \frac{1}{2\pi}\int_{-\infty}^\infty r^+(k,t)e^{ikx}\,dk, \tag{3.8}$$

$$\alpha_j^+(t) = \frac{\chi_j(t)}{i\,\frac{da(z,t)}{dz}\big|_{z=i\nu_j(t)}}, \tag{3.9}$$

and $a(z,t)$ is the analytic continuation of the function $a(k,t)$ to the upper half-plane $\mathrm{Im}\,k > 0$.
Finally, the potential is reconstructed from the GLM kernel via

$$u(x,t) = -2\frac{\partial}{\partial x}K^+(x,x;t). \tag{3.10}$$

It is worthy to remark that the functions

$$h_n(x,t) = \left(\frac{da(k,t)}{dk}\bigg|_{k=i\nu_n(t)}\right)^{-1}\frac{d}{dk}\left[g(x,k;t) - \chi_n(t)f(x,k;t)\right]\bigg|_{k=i\nu_n(t)}, \quad n = 1,2,...,N \tag{3.11}$$

are solutions of the equations (3.1). For $\mathrm{Im}k > 0$, using (3.3), we obtain the following asymptotics

$$h_n(x,t) \to e^{\nu_n(t)x}, \quad x \to +\infty, \tag{3.12}$$

$$h_n(x,t) \to -\chi_n(t)e^{-\nu_n(t)x}, \quad x \to -\infty. \tag{3.13}$$

From asymptotics (3.3), (3.12) and (3.13) we have

$$\begin{aligned} W\{h_n(x,t),\ f(x,k_n(t);t)\} &= -2\nu_n(t), \\ W\{h_n(x,t),\ g(x,k_n(t);t)\} &= -2\nu_n(t)\chi_n(t). \end{aligned} \tag{3.14}$$

## 4. The time dependence of the scattering data

Following the procedure in cf. Ref. [14] the time dependence of the scattering data is given by

$$\frac{dr^+(k,t)}{dt} = 32ik^5|4k^2|^\epsilon r^+(k,t), \tag{4.1}$$

$$\frac{d\chi_n(t)}{dt} = \left(32ik_n^5|4k_n^2|^\epsilon + 2\beta_n(t)\right)\chi_n(t), \tag{4.2}$$

$$\frac{d\nu_n(t)}{dt} = 0, \qquad n = 1,\dots,N. \tag{4.3}$$

**Remark 1.** Taking into account (3.9) and (4.2), it is easy to show that

$$\frac{d\alpha_n^+(t)}{dt} = \left(32ik_n^5|4k_n^2|^\epsilon + 2\beta_n(t)\right)\alpha_n^+(t).$$

**Remark 2.** The obtained results completely determine the time evolution of scattering data, which allows solving problem (2.1)–(2.2) by the following algorithm. Assume that $u_0(x)$ is given.

1. For given functions $u_0(x)$, find the scattering data

$$\{r^+(k,0), k \in \mathbb{R};\ \nu_1(0), \nu_2(0), \dots, \nu_N(0);\ \chi_1(0), \chi_2(0), \dots, \chi_N(0)\}$$

for $\mathcal{L}(0)$.

2. From (4.1)–(4.3), obtain the time evolution of the scattering data

$$\{r^+(k,t), k \in \mathbb{R};\ \nu_1(t), \nu_2(t), \dots, \nu_N(t);\ \chi_1(t), \chi_2(t), \dots, \chi_N(t)\}$$

for $\mathcal{L}(t)$.

3. With this scattering data, uniquely determine the function $F^+(x,t)$ from relation (3.8).

4. Substituting $F^+(x,t)$ in Gelfand-Levitan-Marchenko integral equations (3.7) and solving this system, obtain $K^+(x,y,t)$.

5. Further, from (3.10) find the solution $u(x,t)$.

## 5. Explicit Form of the fKdV equation with a self-consistent source

As shown in [51], the eigenfunctions of $L^*$ are any of the three functions: $\{\partial_x g^2, \partial_x f^2, \partial_x(gf)\}$ which can be represent generically as $\Psi^*$ with eigenvalue $\lambda = k^2$. Here $g$ and $f$ are Jost solutions of (3.1). Then, we have

$$\Omega(L^*)\Psi^* = \Omega(k^2)\Psi^*.$$

The set of eigenfunctions $\Psi^*$ is complete [52]. Hence, any function $h(x,t)$ that is sufficiently regular with respect to $x$ admits the expansion

$$h(x,t) = \int\limits_{\Gamma_\infty} \frac{\tau^2(k,t)}{4\pi i k} dk \int\limits_{-\infty}^{\infty} \mathcal{G}(x,y,k;t) h(y,t) dy,$$

where $\Gamma_\infty = \lim_{R\to\infty} \Gamma_R$ with $\Gamma_R$ the semicircular contour in the upper half plane evaluated from $k = -R$ to $k = R$. $\tau$ is the transmission coefficient defined by the relation

$$f(x,k;t)\tau(k,t) = g(x,-k;t) + r^+(k,t) g(x,k;t),$$

and

$$\mathcal{G}(x,y,k;t) = \partial_x\big(f^2(x,k;t) g^2(y,k;t) - g^2(x,k;t) f^2(y,k;t)\big). \tag{14}$$

The operator $\Omega(L^*)$ acting on an arbitrary function $h(x,t)$ may be represented by the spectral expansion

$$\Omega(L^*) h(x,t) = \int\limits_{\Gamma_\infty} \Omega(k^2) \frac{\tau^2(k,t)}{4\pi i k} dk \int\limits_{-\infty}^{\infty} \mathcal{G}(x,y,k;t) h(y,t) dy. \tag{5.1}$$

Hence, equation (5.1) provides an explicit representation of fKdV with self-consistent source, i.e. equation (5.1) with $h(x,t) = u_x(x,t)$ and $\Omega(L^*) = 16(L^*)^2 |4L^*|^\varepsilon$ which may be written as

$$u_t + \int\limits_{\Gamma_\infty} |4k^2|^\epsilon \frac{\tau^2(k,t)}{4\pi i k} dk \int\limits_{-\infty}^{\infty} \mathcal{G}(x,y,k;t) \left(u_{yyyyy} + 10 u u_{yyy} + 20 u_y u_{yy} + 30 u^2 u_y\right) dy = G(x,t). \tag{5.2}$$

If we set $\epsilon = 0$ in (5.2) then we recover fifth-order KdV equation with a self-consistent source:

$$u_t + u_{xxxxx} + 10uu_{xxx} + 20u_x u_{xx} + 30u^2 u_x = G(x,t).$$

The expression (5.1) can be evaluated using contour integration to give a representation of $\Omega(L^*)$ in terms of integrals along the real line and a sum over discrete values along the imaginary axis:

$$\Omega(L^*)u_x(x,t) = \iint_{\mathbb{R}^2} \Omega\Big(k^2\Big) \frac{\tau^2(k,t)}{4\pi ik} \mathcal{G}_c(x,y,k;t)\, u_y(y,t)\, dy\, dk + \sum_{j=1}^{N} \Omega(-\nu_j^2) \int_{-\infty}^{\infty} \mathcal{G}_{d,j}(x,y,t) u_y(y,t) dy. \tag{5.3}$$

Here the continuous contribution $\mathcal{G}_c$ is defined by

$$\mathcal{G}_c(x,y,k;t) = \partial_x\big(f^2(x,k;t)g^2(y,k;t) - g^2(x,k;t)f^2(y,k;t)\big)$$

and the discrete contribution, which comes from the poles of $\tau(k,t)$ at $k_j(t) = i\nu_j(t)$, $j = 1,2,\ldots,N$, is given by

$$\mathcal{G}_{d,j}(x,y,t) = i\eta_j(t)\frac{\partial}{\partial x}\big[f^2(y,k;t)f(x,k;t)\partial_k g(x,k;t) - f^2(y,k;t)g(x,k;t)\partial_k f(x,k;t)\big]_{k=k_j(t)} -$$

$$-i\eta_j(t)\frac{\partial}{\partial x}\big[f^2(x,k;t)g(y,k;t)\partial_k f(y,k;t) - f^2(x,k;t)g(y,k;t)\partial_k f(y,k;t)\big]_{k=k_j(t)}, \tag{5.4}$$

where

$$\eta_j(t) = \frac{\chi_j(t)}{\nu_j(t)(\partial_k a(k_j(t)))^2}. \tag{5.5}$$

## 6. Example

We consider the problem (2.1)–(2.2) for the one soliton initial condition

$$u(x,0) = -\frac{2}{ch^2 x},$$

with $\beta_1(t) = 5t^4$. In this case, $G(x,t) = 4\frac{\partial}{\partial x}\left(\varphi_1^2(x,t)\right)$ and the solution has the form

$$u(x,t) = -\frac{2}{ch^2\big(x - 4^{2+\varepsilon}t - t^5\big)}. \tag{6.1}$$

## 7. Graphical representation of soliton solution of the equation (2.1)

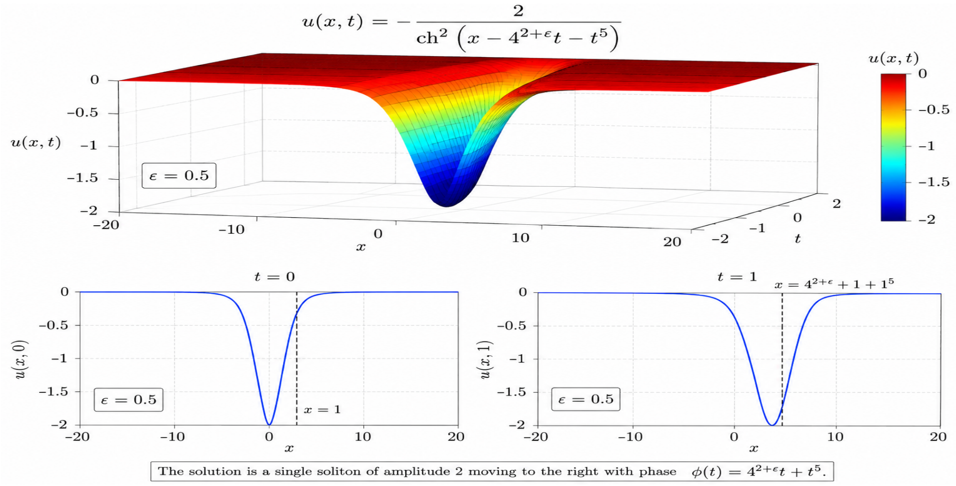


**Figure 1.** One soliton solution of the fRfKdV equation with a self-consistent source for $\varepsilon = \frac{1}{2}$, $\beta_1(t) = 5t^4$.

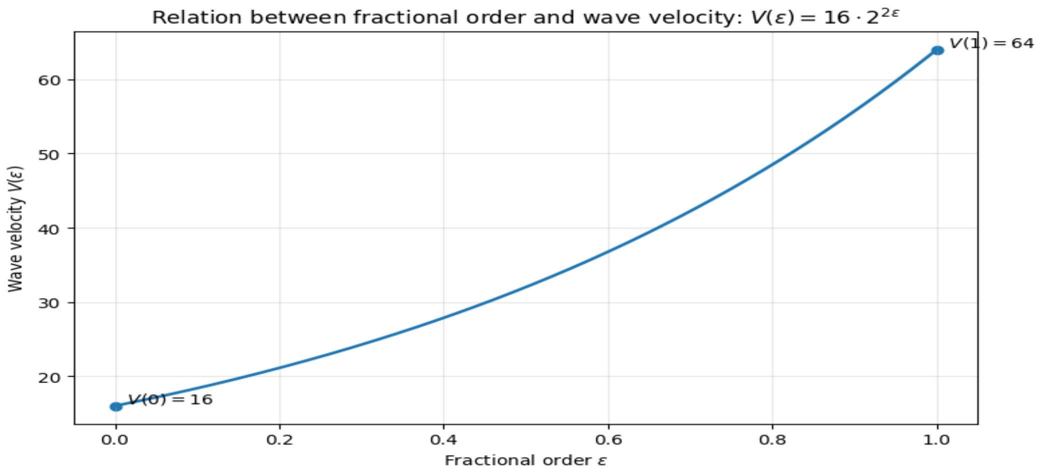


**Figure 2.** Relationship between the fractional derivative order $\varepsilon \in (0, 1)$ and the wave velocity $v(\varepsilon, t) = 4^{2+\varepsilon} + 5t^4$ for the one soliton case.

Figure 2 indicates that the soliton velocity consists of a fractional-order-dependent constant term and a time-dependent contribution. The analysis shows that as the fractional derivative order $\epsilon$ increases, the absolute value of the wave velocity also increases. From a physical point of view, this result implies that the fractional derivative enhances the effective propagation speed of the soliton. Moreover, the presence of the time-dependent term $5t^4$ leads to a non-uniform motion of the soliton, meaning that the soliton accelerates as time evolves. This behavior reflects the influence of memory effects and nonlocal properties introduced by the fractional derivative, as well as the contribution of the self-consistent source to the soliton dynamics.

## 8. Conclusion

The ITS method has been successfully used to obtain the soliton solutions of the fifth-order Riesz fractional KdV equation with a self-consistent source. We have determined the time evolution of the scattering data for the Sturm-Liouville problem associated with the solution of the fifth-order Riesz fractional KdV equation with a self-consistent source. Then, using the solution of the inverse scattering problem with respect to the time-dependent scattering data, we recover the desired solution of the fifth-order Riesz fractional KdV equation with a self-consistent source. For the one soliton case, explicit formulas for the solution of the problem under consideration are obtained. Finally, we showed that the one-soliton solutions of the fifth-order Riesz fractional KdV equation with a self-consistent source have power law relationships between the solitons amplitude and velocity. The self-consistent source further modifies the energy balance of the system by continuously interacting with the wave field, thereby accelerating the soliton dynamics.

theorem

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