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Research article

Hidden Potts-Observed SOS Models on a Two-Layer Structure

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Abstract

In this paper, we introduce a two-layer hidden Potts-observed SOS model on Cayley trees of arbitrary order. The model is constructed through a hidden Markov structure in which the observable SOS configuration is generated from an underlying latent Potts field by means of a local stochastic transformation. For the proposed model, we derive the compatibility conditions for Gibbs measures on Cayley trees of arbitrary order and investigate translation-invariant Gibbs states. In particular, explicit forms of translation-invariant Gibbs measures are obtained for Cayley trees of orders one and two, and conditions for their existence are established. Moreover, for these Gibbs measures, the hidden states are reconstructed and predicted from the observable layer. The obtained results provide a rigorous probabilistic framework for multi-state hidden spin systems and extend hidden Markov methods to Potts-SOS interactions on hierarchical graphs. The proposed approach may be useful in the study of spatial stochastic systems, graphical models, and related inference problems involving latent structures.

Keywords: *Cayley tree; Potts model; SOS model; hidden Markov structure; Gibbs measure; probabilistic inference.*

AMS Subject Classification (2020): *Primary: 82B20; Secondary: 60K35, 82B26, 60J05, 05C05, 68T05.*

1. Introduction

Models on hierarchical graphs have attracted considerable attention in statistical mechanics due to their ability to describe complex interaction mechanisms while remaining mathematically tractable. Among such graphs, the Cayley tree occupies a special position since many questions concerning Gibbs measures, phase transitions, extremality, and ground states can be studied rigorously. Classical foundations of the theory of Gibbs measures and phase transitions were established in [13, 12, 23, 9], while structural properties of Cayley trees and related graph-theoretical aspects were discussed in [11]. Comprehensive accounts of Gibbs measures on Cayley trees and their applications can be found in the monographs [20, 21, 1].

Among multi-state lattice systems, the Potts model and its various generalizations have been extensively investigated. Exact methods and thermodynamic properties of such models were developed in the classical work [3]. Subsequently, Potts models on Cayley trees were studied from different viewpoints, including phase transitions, competing interactions, periodic Gibbs measures, and ground-state structures [2, 5, 19, 4, 22]. Another important direction concerns Potts-SOS models, which combine discrete Potts interactions with solid-on-solid (SOS) type interactions. Gibbs measures, extremality properties, limiting Gibbs measures, and

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ground states for Potts-SOS models on Cayley trees were investigated in [14, 15, 17, 18], revealing a rich variety of equilibrium phenomena.

In parallel with these developments, hidden-variable models have become an important tool in machine learning, probabilistic inference, and graphical modeling. Hidden Markov structures provide a natural framework for describing situations in which the observed data arise from an underlying latent process. Recently, Gibbsian approaches to hidden Markov models have been proposed, establishing connections between statistical mechanics and modern inference methods [7, 1]. However, most existing studies have focused on binary-state systems, while multi-state hidden models remain much less understood. Recently, a hidden Markov framework for multi-state Potts systems was proposed in [16]. That work established the Gibbsian formalism for hidden Potts models on Cayley trees and demonstrated that hidden-state interactions can be successfully incorporated into the theory of Gibbs measures. These results motivate the study of more sophisticated hidden multi-layer systems involving different types of interactions at the hidden and observable levels.

Motivated by these developments, in the present paper we introduce a two-layer hidden Potts-observed SOS model on the Cayley tree. In the proposed framework, the hidden layer is represented by a three-state Potts configuration, whereas the observable layer is described by an SOS configuration generated through a local stochastic mechanism. This construction naturally combines ideas from hidden Markov models and Potts-SOS systems, providing a new class of Gibbsian graphical models with latent structure.

The main objective of this work is to establish a rigorous Gibbsian description of the proposed model. We derive compatibility conditions for Gibbs measures on Cayley trees of arbitrary order and investigate translation-invariant Gibbs measures. In particular, explicit forms of translation-invariant Gibbs measures are obtained for Cayley trees of orders one and two. Furthermore, we study the reconstruction and prediction of hidden states from the observable layer and identify parameter regions in which multiple Gibbs measures coexist.

The obtained results provide a bridge between the theory of Gibbs measures, Potts-SOS models, and hidden Markov structures. They also open new perspectives for the study of latent-variable systems on hierarchical graphs and their potential applications in probabilistic inference and machine learning.

1.1. The model

We now introduce the model under consideration. Let $\Gamma^k = (V, L)$ be a Cayley tree of order k, where V denotes the set of vertices and L denotes the set of edges. To each vertex $x \in V$ we assign a Potts-SOS spin $s(x)$ taking values in the finite set

$$\Phi = \{0, 1, 2\}.$$

A configuration of hidden spins is a mapping $s : V \to \Phi$, and the set of all such configurations is denoted by

$$\Omega = \Phi^V.$$

Similarly, $\sigma \in \Omega$ denotes the observable configuration.

The Hamiltonian $H(s, \sigma)$ of the hidden Markov model, which describes the energy associated with a pair of configurations (s, σ)–including both the Potts-SOS interactions and the dependence between the hidden Potts configuration $s \in \Omega$ and the observable SOS configuration $\sigma \in \Omega$–is defined as follows:

$$H(s, \sigma) = -J \sum_{\langle x,y \rangle} \big(\delta_{s(x)s(y)} - |\sigma(x) - \sigma(y)|\big) - \sum_{x \in V} p\big(s(x), \sigma(x)\big), \tag{1.1}$$

where $J \in \mathbb{R}$ is the interaction parameter, δ_{ij} is the Kronecker delta, and $p : \Phi \times \Phi \to \mathbb{R}$ is an external field depending on both hidden and observed states.

Remark 1.1. A more general formulation of (1.1) can be introduced by assigning separate coupling constants to the Potts and SOS interactions:

$$H(s,\sigma) = -J_1 \sum_{\langle x,y\rangle} \delta_{s(x)s(y)} + J_2 \sum_{\langle x,y\rangle} |\sigma(x)-\sigma(y)| - \sum_{x\in V} p\big(s(x),\sigma(x)\big), \tag{1.2}$$

where $J_1, J_2 \in \mathbb{R}$ are independent interaction parameters. Although the main results can be extended to this more general framework, the resulting compatibility equations become considerably more complicated. For this reason, in the present work we focus on the symmetric situation $J_1 = J_2 = J$, which already exhibits a rich and nontrivial behavior.

Remark 1.2. The hidden Potts interaction considered in (1.1) may also be replaced by more general interactions. In particular, one can study hidden models generated by Potts-SOS type Hamiltonians. Various aspects of Potts-SOS models on Cayley trees, including Gibbs measures, extremality questions, and ground-state structures, have been investigated in [14, 15, 18]. The development of hidden Markov counterparts of such models remains an interesting direction for future research.

2. The compatibility of measures

Define a finite-dimensional distribution of a probability measure μ_n in the $\Omega_n \times \Omega_n$ as

$$\mu_n(s_n,\sigma_n) = Z_n^{-1} \exp\left\{-\beta H_n(s_n,\sigma_n) + \sum_{x\in W_n} h_{s(x),\,\sigma(x),\,x}\right\}, \tag{2.1}$$

where $\beta = 1/T$, and $T > 0$ is the temperature, Z_n^{-1} is the normalizing factor,

$$\left\{h_x = (h_{0,0,x}, h_{0,1,x}, h_{0,2,x}, h_{1,0,x}, h_{1,1,x}, h_{1,2,x}, h_{2,0,x}, h_{2,1,x}, h_{2,2,x}) \in \mathbb{R}^9,\ x\in V\right\} \tag{2.2}$$

is a collection of vectors and

$$H_n(s_n,\sigma_n) = -J \sum_{\langle x,y\rangle\in L_n} \big(\delta_{s(x)s(y)} - |\sigma(x)-\sigma(y)|\big) - \sum_{x\in V_n} p(s(x),\sigma(x)).$$

Definition 2.1. We say that the probability distributions (2.1) are compatible if for all $n \geq 1$ and $s_{n-1} \in \Omega_{n-1},\ \sigma_{n-1} \in \Omega_{n-1}$:

$$\sum_{(w_n,\omega_n)\in\Phi^{W_n}\times\Phi^{W_n}} \mu_n\big(s_{n-1}\vee w_n,\ \sigma_{n-1}\vee\omega_n\big) = \mu_{n-1}(s_{n-1},\sigma_{n-1}), \tag{2.3}$$

where symbol $\vee$ denotes the concatenation (union) of the configurations.

In this case, by Kolmogorov's theorem [20] there exists a unique measure μ on the measurable space $(\Omega\times\Omega, \mathcal{F})$, where $\mathcal{F}$ is the sigma-algebra generated by the cylinder sets of $\Omega\times\Omega$, such that for all n and $s_n, \sigma_n \in \Omega_n$,

$$\mu\big(\{(s,\sigma)\ |_{V_n} = (s_n,\sigma_n)\}\big) = \mu_n(s_n,\sigma_n).$$

Such a measure is called a *splitting Gibbs measure* (SGM) corresponding to the Hamiltonian (1.1) and vector-valued functions (2.2).

Remark 2.1. In this paper, we exclusively study SGMs and thus omit the qualifier "splitting" for brevity.

The following theorem describes conditions on h_x guaranteeing compatibility of $\mu_n(s_n, \sigma_n)$.

Theorem 2.1. *The probability distributions $\mu_n(s_n, \sigma_n)$, $n \in \mathbb{N}$, defined by (2.1) are compatible if and only if, for any $x \in V \setminus \{x^0\}$, the following system of equations holds:*

$$u_{1,x} = a_{0,1} \prod_{y \in S(x)} \frac{1 + ru_{1,y} + u_{2,y} + r^{-1}u_{3,y} + u_{4,y} + r^{-1}u_{5,y} + r^{-1}u_{6,y} + u_{7,y} + r^{-1}u_{8,y}}{r + u_{1,y} + r^{-1}u_{2,y} + u_{3,y} + r^{-1}u_{4,y} + r^{-2}u_{5,y} + u_{6,y} + r^{-1}u_{7,y} + r^{-2}u_{8,y}}, \tag{2.4}$$

$$u_{2,x} = a_{0,2} \prod_{y \in S(x)} \frac{r^{-1} + u_{1,y} + ru_{2,y} + r^{-2}u_{3,y} + r^{-1}u_{4,y} + u_{5,y} + r^{-2}u_{6,y} + r^{-1}u_{7,y} + u_{8,y}}{r + u_{1,y} + r^{-1}u_{2,y} + u_{3,y} + r^{-1}u_{4,y} + r^{-2}u_{5,y} + u_{6,y} + r^{-1}u_{7,y} + r^{-2}u_{8,y}}, \tag{2.5}$$

$$u_{3,x} = a_{1,0} \prod_{y \in S(x)} \frac{1 + r^{-1}u_{1,y} + r^{-2}u_{2,y} + ru_{3,y} + u_{4,y} + r^{-1}u_{5,y} + u_{6,y} + r^{-1}u_{7,y} + r^{-2}u_{8,y}}{r + u_{1,y} + r^{-1}u_{2,y} + u_{3,y} + r^{-1}u_{4,y} + r^{-2}u_{5,y} + u_{6,y} + r^{-1}u_{7,y} + r^{-2}u_{8,y}}, \tag{2.6}$$

$$u_{4,x} = a_{1,1} \prod_{y \in S(x)} \frac{r^{-1} + u_{1,y} + r^{-1}u_{2,y} + u_{3,y} + ru_{4,y} + u_{5,y} + r^{-1}u_{6,y} + u_{7,y} + r^{-1}u_{8,y}}{r + u_{1,y} + r^{-1}u_{2,y} + u_{3,y} + r^{-1}u_{4,y} + r^{-2}u_{5,y} + u_{6,y} + r^{-1}u_{7,y} + r^{-2}u_{8,y}}, \tag{2.7}$$

$$u_{5,x} = a_{1,2} \prod_{y \in S(x)} \frac{r^{-2} + r^{-1}u_{1,y} + u_{2,y} + r^{-1}u_{3,y} + u_{4,y} + ru_{5,y} + r^{-2}u_{6,y} + r^{-1}u_{7,y} + u_{8,y}}{r + u_{1,y} + r^{-1}u_{2,y} + u_{3,y} + r^{-1}u_{4,y} + r^{-2}u_{5,y} + u_{6,y} + r^{-1}u_{7,y} + r^{-2}u_{8,y}}, \tag{2.8}$$

$$u_{6,x} = a_{2,0} \prod_{y \in S(x)} \frac{1 + r^{-1}u_{1,y} + r^{-2}u_{2,y} + u_{3,y} + r^{-1}u_{4,y} + r^{-2}u_{5,y} + ru_{6,y} + u_{7,y} + r^{-1}u_{8,y}}{r + u_{1,y} + r^{-1}u_{2,y} + u_{3,y} + r^{-1}u_{4,y} + r^{-2}u_{5,y} + u_{6,y} + r^{-1}u_{7,y} + r^{-2}u_{8,y}}, \tag{2.9}$$

$$u_{7,x} = a_{2,1} \prod_{y \in S(x)} \frac{r^{-1} + u_{1,y} + r^{-1}u_{2,y} + r^{-1}u_{3,y} + u_{4,y} + r^{-1}u_{5,y} + u_{6,y} + ru_{7,y} + u_{8,y}}{r + u_{1,y} + r^{-1}u_{2,y} + u_{3,y} + r^{-1}u_{4,y} + r^{-2}u_{5,y} + u_{6,y} + r^{-1}u_{7,y} + r^{-2}u_{8,y}}, \tag{2.10}$$

$$u_{8,x} = a_{2,2} \prod_{y \in S(x)} \frac{r^{-2} + r^{-1}u_{1,y} + u_{2,y} + r^{-2}u_{3,y} + r^{-1}u_{4,y} + u_{5,y} + r^{-1}u_{6,y} + u_{7,y} + ru_{8,y}}{r + u_{1,y} + r^{-1}u_{2,y} + u_{3,y} + r^{-1}u_{4,y} + r^{-2}u_{5,y} + u_{6,y} + r^{-1}u_{7,y} + r^{-2}u_{8,y}}, \tag{2.11}$$

where the quantities $u_{i,x}$, $i = 1, \dots, 8$, are defined by

$$u_{1,x} = a_{0,1}\, z_{0,1,x}, \quad u_{2,x} = a_{0,2}\, z_{0,2,x}, \quad u_{3,x} = a_{1,0}\, z_{1,0,x}, \quad u_{4,x} = a_{1,1}\, z_{1,1,x},$$

$$u_{5,x} = a_{1,2}\, z_{1,2,x}, \quad u_{6,x} = a_{2,0}\, z_{2,0,x}, \quad u_{7,x} = a_{2,1}\, z_{2,1,x}, \quad u_{8,x} = a_{2,2}\, z_{2,2,x}.$$

Here,

$$a_{\epsilon,\xi} = \exp\big(\beta\big(p(\epsilon, \xi) - p(0,0)\big)\big), \quad z_{\epsilon,\xi,x} = \exp\left(h_{\epsilon,\xi,x} - h_{0,0,x}\right),$$

for $\epsilon, \xi \in \{0, 1, 2\}$, and $r = \exp(J\beta)$. Moreover, $S(x)$ denotes the set of direct successors of x.

Proof. Using the same approach as in the proof of Theorem 1 in [7], one obtains the result. □

From Theorem 2.1, it follows that for any $u = \{u_{i,x},\ x \in V\}$ that satisfies the system of functional equations (2.4)–(2.11), there exists a unique SGM μ, and conversely. Therefore, the core problem of describing Gibbs measures reduces to solving the system of functional equations (2.4)–(2.11). However, solving this system is particularly challenging due to its non-linear nature, multidimensional structure, and the fact that the unknown functions are defined on a tree. Even the task of determining all constant functions (which are independent of the tree's vertices) is complex. In this paper, we present a class of such constant solutions and explore how the corresponding Gibbs measures can be applied in machine learning.

Our Theorem 2.1 ensures that the locally defined probability kernels derived from the Hamiltonian (1.1) correspond to a consistent global Gibbs measure. This guarantees the mathematical coherence of the model, stabilizes probabilistic inference procedures, and validates the correctness of methods used to estimate hidden states.

3. Translation-invariant Gibbs measures

In this section, we study translation-invariant Gibbs measures corresponding to solutions satisfying

$$u_{i,x} = u_i, \qquad x \in V. \tag{3.1}$$

Substituting condition (3.1) into equations (2.4)–(2.11), we obtain

$$u_1 = a_{0,1}\left(\frac{1 + ru_1 + u_2 + r^{-1}u_3 + u_4 + r^{-1}u_5 + r^{-1}u_6 + u_7 + r^{-1}u_8}{r + u_1 + r^{-1}u_2 + u_3 + r^{-1}u_4 + r^{-2}u_5 + u_6 + r^{-1}u_7 + r^{-2}u_8}\right)^k, \tag{3.2}$$

$$u_2 = a_{0,2}\left(\frac{r^{-1} + u_1 + ru_2 + r^{-2}u_3 + r^{-1}u_4 + u_5 + r^{-2}u_6 + r^{-1}u_7 + u_8}{r + u_1 + r^{-1}u_2 + u_3 + r^{-1}u_4 + r^{-2}u_5 + u_6 + r^{-1}u_7 + r^{-2}u_8}\right)^k, \tag{3.3}$$

$$u_3 = a_{1,0}\left(\frac{1 + r^{-1}u_1 + r^{-2}u_2 + ru_3 + u_4 + r^{-1}u_5 + u_6 + r^{-1}u_7 + r^{-2}u_8}{r + u_1 + r^{-1}u_2 + u_3 + r^{-1}u_4 + r^{-2}u_5 + u_6 + r^{-1}u_7 + r^{-2}u_8}\right)^k, \tag{3.4}$$

$$u_4 = a_{1,1}\left(\frac{r^{-1} + u_1 + r^{-1}u_2 + u_3 + ru_4 + u_5 + r^{-1}u_6 + u_7 + r^{-1}u_8}{r + u_1 + r^{-1}u_2 + u_3 + r^{-1}u_4 + r^{-2}u_5 + u_6 + r^{-1}u_7 + r^{-2}u_8}\right)^k, \tag{3.5}$$

$$u_5 = a_{1,2}\left(\frac{r^{-2} + r^{-1}u_1 + u_2 + r^{-1}u_3 + u_4 + ru_5 + r^{-2}u_6 + r^{-1}u_7 + u_8}{r + u_1 + r^{-1}u_2 + u_3 + r^{-1}u_4 + r^{-2}u_5 + u_6 + r^{-1}u_7 + r^{-2}u_8}\right)^k, \tag{3.6}$$

$$u_6 = a_{2,0}\left(\frac{1 + r^{-1}u_1 + r^{-2}u_2 + u_3 + r^{-1}u_4 + r^{-2}u_5 + ru_6 + u_7 + r^{-1}u_8}{r + u_1 + r^{-1}u_2 + u_3 + r^{-1}u_4 + r^{-2}u_5 + u_6 + r^{-1}u_7 + r^{-2}u_8}\right)^k, \tag{3.7}$$

$$u_7 = a_{2,1}\left(\frac{r^{-1} + u_1 + r^{-1}u_2 + r^{-1}u_3 + u_4 + r^{-1}u_5 + u_6 + ru_7 + u_8}{r + u_1 + r^{-1}u_2 + u_3 + r^{-1}u_4 + r^{-2}u_5 + u_6 + r^{-1}u_7 + r^{-2}u_8}\right)^k, \tag{3.8}$$

$$u_8 = a_{2,2}\left(\frac{r^{-2} + r^{-1}u_1 + u_2 + r^{-2}u_3 + r^{-1}u_4 + u_5 + r^{-1}u_6 + u_7 + ru_8}{r + u_1 + r^{-1}u_2 + u_3 + r^{-1}u_4 + r^{-2}u_5 + u_6 + r^{-1}u_7 + r^{-2}u_8}\right)^k. \tag{3.9}$$

3.1. Case: $a_{0,1} = a_{0,2} = a_{1,0} = a_{1,1} = a_{1,2} = a_{2,0} = a_{2,1} = a_{2,2} = 1$

Under this assumption, the system (3.2)–(3.9) can be represented as a fixed-point equation $F(t) = t$, where $t = (u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8) \in \mathbb{R}^8_+$ and the operator $F : \mathbb{R}^8_+ \to \mathbb{R}^8_+$ is defined by

$$\begin{cases} u'_1 = \left(\dfrac{1 + ru_1 + u_2 + r^{-1}u_3 + u_4 + r^{-1}u_5 + r^{-1}u_6 + u_7 + r^{-1}u_8}{r + u_1 + r^{-1}u_2 + u_3 + r^{-1}u_4 + r^{-2}u_5 + u_6 + r^{-1}u_7 + r^{-2}u_8}\right)^k, \\ u'_2 = \left(\dfrac{r^{-1} + u_1 + ru_2 + r^{-2}u_3 + r^{-1}u_4 + u_5 + r^{-2}u_6 + r^{-1}u_7 + u_8}{r + u_1 + r^{-1}u_2 + u_3 + r^{-1}u_4 + r^{-2}u_5 + u_6 + r^{-1}u_7 + r^{-2}u_8}\right)^k, \\ u'_3 = \left(\dfrac{1 + r^{-1}u_1 + r^{-2}u_2 + ru_3 + u_4 + r^{-1}u_5 + u_6 + r^{-1}u_7 + r^{-2}u_8}{r + u_1 + r^{-1}u_2 + u_3 + r^{-1}u_4 + r^{-2}u_5 + u_6 + r^{-1}u_7 + r^{-2}u_8}\right)^k, \\ u'_4 = \left(\dfrac{r^{-1} + u_1 + r^{-1}u_2 + u_3 + ru_4 + u_5 + r^{-1}u_6 + u_7 + r^{-1}u_8}{r + u_1 + r^{-1}u_2 + u_3 + r^{-1}u_4 + r^{-2}u_5 + u_6 + r^{-1}u_7 + r^{-2}u_8}\right)^k, \\ u'_5 = \left(\dfrac{r^{-2} + r^{-1}u_1 + u_2 + r^{-1}u_3 + u_4 + ru_5 + r^{-2}u_6 + r^{-1}u_7 + u_8}{r + u_1 + r^{-1}u_2 + u_3 + r^{-1}u_4 + r^{-2}u_5 + u_6 + r^{-1}u_7 + r^{-2}u_8}\right)^k, \\ u'_6 = \left(\dfrac{1 + r^{-1}u_1 + r^{-2}u_2 + u_3 + r^{-1}u_4 + r^{-2}u_5 + ru_6 + u_7 + r^{-1}u_8}{r + u_1 + r^{-1}u_2 + u_3 + r^{-1}u_4 + r^{-2}u_5 + u_6 + r^{-1}u_7 + r^{-2}u_8}\right)^k, \\ u'_7 = \left(\dfrac{r^{-1} + u_1 + r^{-1}u_2 + r^{-1}u_3 + u_4 + r^{-1}u_5 + u_6 + ru_7 + u_8}{r + u_1 + r^{-1}u_2 + u_3 + r^{-1}u_4 + r^{-2}u_5 + u_6 + r^{-1}u_7 + r^{-2}u_8}\right)^k, \\ u'_8 = \left(\dfrac{r^{-2} + r^{-1}u_1 + u_2 + r^{-2}u_3 + r^{-1}u_4 + u_5 + r^{-1}u_6 + u_7 + ru_8}{r + u_1 + r^{-1}u_2 + u_3 + r^{-1}u_4 + r^{-2}u_5 + u_6 + r^{-1}u_7 + r^{-2}u_8}\right)^k. \end{cases} \tag{3.10}$$

Consider the following set

$$I = \left\{ t \in \mathbb{R}^8_+ : u_1 = u_4 = u_7,\ u_2 = u_3 = u_5 = u_6 = u_8 = 1 \right\}.$$

The following statement follows directly from the definition of F.

Lemma 3.1. *The set I is invariant under the operator F, that is, $F(I) \subset I$.*

Next, we determine the fixed points of F restricted to the invariant set I. In this case, the fixed-point equation reduces to

$$u_1 = \left(\frac{2r + r^2u_1}{r^2 + 1 + ru_1}\right)^k. \tag{3.11}$$

The case $k = 1$

For $k = 1$, the system (3.11) takes the form

$$ru_1^2 + u_1 - 2r = 0.$$

Taking into account the condition $u_1 > 0$, we obtain the unique positive solution $u_1 = \frac{-1+\sqrt{8r^2+1}}{2r}$. Consequently, for $k = 1$ and arbitrary $r > 0$, the system (3.2)–(3.13) admits the unique solution

$$(u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8) = (u_1, 1, 1, u_1, 1, 1, u_1, 1). \tag{3.12}$$

Combining Lemma 3.1 with (3.12), we get

Lemma 3.2. *Let $k = 1$. Then the operator F restricted to the invariant set I has a unique fixed point $(u_1, 1, 1, u_1, 1, 1, u_1, 1)$, for every $r > 0$.*

Applying Theorem 2.1, we obtain the following statement.

Theorem 3.1. *Let $k = 1$ and assume that $p(\epsilon, \xi) = p(0,0)$, $\forall \epsilon, \xi \in \Phi$. Then the Hamiltonian* (1.1) *admits at least one splitting Gibbs measure for all $r > 0$.*

Remark 3.1. Even in the special case $a_{0,1} = a_{0,2} = a_{1,0} = a_{1,1} = a_{1,2} = a_{2,0} = a_{2,1} = a_{2,2} = 1$, the complete analysis of the system (3.2)–(3.9) outside the invariant set I remains highly nontrivial.

Phase transitions in hidden Markov models correspond to qualitative changes in the behavior of the system caused by variations in model parameters or the structure of hidden states. Such transitions may appear through the loss of uniqueness of Gibbs measures, instability of reconstruction procedures, or the emergence of multiple equilibrium regimes.

Understanding these phenomena is important for identifying parameter regions where the model exhibits stable probabilistic behavior. In the next section, we investigate this issue through the non-uniqueness of Gibbs measures and the existence of multiple translation-invariant states.

The case $k = 2$

For $k = 2$, equation (3.11) reduces to the cubic equation

$$r^2 u_1^3 + \big(2r(r^2+1) - r^4\big)u_1^2 + \big((r^2+1)^2 - 4r^3\big)u_1 - 4r^2 = 0. \tag{3.13}$$

Analyzing equation (3.13), one can determine all its positive solutions. Combining these solutions with the invariance property established in Lemma 3.1, we obtain the following result.

Lemma 3.3. *Let $k = 2$. Then there exists a critical value $r_c \approx 7.0725$ such that the operator F, restricted to the invariant set I, exhibits the following behavior.*

(i) If $0 < r < r_c$, then F has a unique fixed point $\big(u_1^{(1)}(r), 1, 1, u_1^{(1)}(r), 1, 1, u_1^{(1)}(r), 1\big)$.

(ii) If $r = r_c$, then F has exactly two fixed points $\big(u_1^{(1)}(r), 1, 1, u_1^{(1)}(r), 1, 1, u_1^{(1)}(r), 1\big)$ and $\big(u_1^{(2)}(r), 1, 1, u_1^{(2)}(r), 1, 1, u_1^{(2)}(r), 1\big)$.

(iii) If $r > r_c$, then F has three fixed points $\big(u_1^{(1)}(r), 1, 1, u_1^{(1)}(r), 1, 1, u_1^{(1)}(r), 1\big)$, $\big(u_1^{(2)}(r), 1, 1, u_1^{(2)}(r), 1, 1, u_1^{(2)}(r), 1\big)$, and $\big(u_1^{(3)}(r), 1, 1, u_1^{(3)}(r), 1, 1, u_1^{(3)}(r), 1\big)$.

Here the functions $u_1^{(i)}(r)$, $i = 1, 2, 3$, are defined by

$$u_1^{(1)}(r) = \frac{M^2 + 4r(r^6 - 4r^5 + r^4 + 8r^3 + 2r^2 + 1) + 2M(r^3 - 2r^2 - 2)}{6Mr},$$

$$u_1^{(2,3)}(r) = \frac{-M^2 - 4(r^6 - 4r^5 + r^4 + 8r^3 + 2r^2 + 1) + 4M(r^3 - 2r^2 - 2)}{12Mr}$$
$$\pm \frac{i\sqrt{3}\Big(M^2 - 4(r^6 - 4r^5 + r^4 + 8r^3 + 2r^2 + 1)\Big)}{12Mr},$$

where

$$M = \Bigg(8r^9 - 48r^8 + 60r^7 + 104r^6 - 168r^5 + 24r^4 + 204r^3 + 24r^2 + 8$$
$$+12r^2\sqrt{-\frac{3\left(r^{11} - 12r^{10} + 36r^9 + 12r^8 - 146r^7 + 180r^5 + 8r^4 - 71r^3 - 4r^2 - 4\right)}{r}}\Bigg)^{1/3}.$$

Remark 3.2. The formulas for $u_1^{(2)}(r)$ and $u_1^{(3)}(r)$ are obtained from Cardano's representation of the roots of the cubic equation (3.13). Although these expressions involve complex quantities, the corresponding roots are real for all $r \geq r_c$. Furthermore, numerical and analytical verification shows that both roots are positive in their domain of existence. Hence, all three solutions $u_1^{(1)}(r)$, $u_1^{(2)}(r)$, and $u_1^{(3)}(r)$ correspond to admissible positive fixed points of the operator F.

Combining Lemmas 3.1 and 3.3 with Theorem 2.1, we arrive at the following classification of splitting Gibbs measures.

Theorem 3.2. *Let $k = 2$ and assume that $p(\epsilon, \xi) = p(0,0)$, $\forall, \epsilon, \xi \in \Phi$. Then the Hamiltonian (1.1) admits*

- *at least one splitting Gibbs measure μ_1 corresponding to the fixed point $\big(u_1^{(1)}(r), 1, 1, u_1^{(1)}(r), 1, 1, u_1^{(1)}(r), 1\big)$, for $0 < r < r_c$;*
- *at least two splitting Gibbs measures μ_1 and μ_2 corresponding to the fixed points $\big(u_1^{(1)}(r), 1, 1, u_1^{(1)}(r), 1, 1, u_1^{(1)}(r), 1\big)$ and $\big(u_1^{(2)}(r), 1, 1, u_1^{(2)}(r), 1, 1, u_1^{(2)}(r), 1\big)$, respectively, when $r = r_c$;*
- *at least three splitting Gibbs measures μ_1, μ_2, and μ_3 corresponding to the fixed points $\big(u_1^{(1)}(r), 1, 1, u_1^{(1)}(r), 1, 1, u_1^{(1)}(r), 1\big)$, $\big(u_1^{(2)}(r), 1, 1, u_1^{(2)}(r), 1, 1, u_1^{(2)}(r), 1\big)$, and $\big(u_1^{(3)}(r), 1, 1, u_1^{(3)}(r), 1, 1, u_1^{(3)}(r), 1\big)$, respectively, for $r > r_c$.*

4. Prediction of Hidden Potts States from Observable SOS Configurations

One of the central inference problems in the proposed two-layer hidden Potts-observed SOS model is the reconstruction of the hidden Potts configuration from the observed SOS layer. More precisely, given an observed configuration $\sigma = (\sigma(x))_{x \in V}$, our objective is to determine the conditional distribution of the hidden configuration $s = (s(x))_{x \in V}$ conditioned on the observed data.

Let $\mu_n(s_n, \sigma_n)$ denote the finite-volume Gibbs distribution corresponding to the Hamiltonian (1.1). The posterior distribution of the hidden Potts configuration is obtained from Bayes' formula:

$$\mu(s_n \mid \sigma_n) = \frac{1}{Z(\sigma_n)} \exp\Bigg(\beta J \sum_{\langle x,y\rangle \in L_n} \delta_{s(x)s(y)} + \sum_{x \in V_n} \beta p(s(x), \sigma(x))$$
$$+ \sum_{x \in W_n} h_{s(x),\sigma(x),x}\Bigg) = \frac{1}{Z(\sigma_n)} \prod_{\langle x,y\rangle \in L_n} r^{\delta_{s(x)s(y)}} \prod_{x \in V_n} \exp\big(\beta p(s(x), \sigma(x))\big) \tag{4.1}$$
$$\prod_{x \in W_n} z_{s(x),\sigma(x),x},$$

where $Z(\sigma_n)$ is the normalization factor dependent on σ_n.

Formula (4.1) shows that, conditioned on the observed SOS configuration, the hidden Potts variables form an effective Gibbs field with modified local interactions determined by the observable layer. Consequently, the observed configuration acts as additional information that influences the probability distribution of hidden states.

Remark 4.1. For translation-invariant Gibbs measures obtained in the previous section, the boundary laws satisfy $z_{\epsilon,\xi,x} = z_{\epsilon,\xi}$, $\quad \epsilon, \xi \in \Phi$, and therefore do not depend on the vertex x. In this case the posterior distribution (4.1) is completely determined by the translation-invariant solution of the corresponding consistency equations.

Recall that $z_{\epsilon,\xi} = \exp\big(h_{\epsilon,\xi}\big)$, $\quad \epsilon, \xi \in \Phi$. Assuming without loss of generality that $h_{0,0} = 0$, we obtain

$$h_{\epsilon,\xi} = \log z_{\epsilon,\xi}, \quad \epsilon, \xi \in \Phi. \tag{4.2}$$

Table 1. Posterior probabilities of hidden Potts states for the observed configuration $\sigma_{\ell_0} = (0,0)$

s_{ℓ_0}	σ_{ℓ_0}	$\mu(s_{\ell_0} \mid \sigma_{\ell_0})$
$(0,0)$	$(0,0)$	$\frac{r}{r+a_{1,0}+a_{2,0}+(1+ra_{1,0}+a_{2,0})a_{1,0}z_{1,0,y}+(1+a_{1,0}+ra_{2,0})a_{2,0}z_{2,0,y}}$
$(0,1)$	$(0,0)$	$\frac{a_{1,0}z_{1,0,y}}{r+a_{1,0}+a_{2,0}+(1+ra_{1,0}+a_{2,0})a_{1,0}z_{1,0,y}+(1+a_{1,0}+ra_{2,0})a_{2,0}z_{2,0,y}}$
$(0,2)$	$(0,0)$	$\frac{a_{2,0}z_{2,0,y}}{r+a_{1,0}+a_{2,0}+(1+ra_{1,0}+a_{2,0})a_{1,0}z_{1,0,y}+(1+a_{1,0}+ra_{2,0})a_{2,0}z_{2,0,y}}$
$(1,0)$	$(0,0)$	$\frac{a_{1,0}}{r+a_{1,0}+a_{2,0}+(1+ra_{1,0}+a_{2,0})a_{1,0}z_{1,0,y}+(1+a_{1,0}+ra_{2,0})a_{2,0}z_{2,0,y}}$
$(1,1)$	$(0,0)$	$\frac{ra_{1,0}^2z_{1,0,y}}{r+a_{1,0}+a_{2,0}+(1+ra_{1,0}+a_{2,0})a_{1,0}z_{1,0,y}+(1+a_{1,0}+ra_{2,0})a_{2,0}z_{2,0,y}}$
$(1,2)$	$(0,0)$	$\frac{a_{1,0}a_{2,0}z_{1,0,y}}{r+a_{1,0}+a_{2,0}+(1+ra_{1,0}+a_{2,0})a_{1,0}z_{1,0,y}+(1+a_{1,0}+ra_{2,0})a_{2,0}z_{2,0,y}}$
$(2,0)$	$(0,0)$	$\frac{a_{2,0}}{r+a_{1,0}+a_{2,0}+(1+ra_{1,0}+a_{2,0})a_{1,0}z_{1,0,y}+(1+a_{1,0}+ra_{2,0})a_{2,0}z_{2,0,y}}$
$(2,1)$	$(0,0)$	$\frac{a_{1,0}a_{2,0}z_{1,0,y}}{r+a_{1,0}+a_{2,0}+(1+ra_{1,0}+a_{2,0})a_{1,0}z_{1,0,y}+(1+a_{1,0}+ra_{2,0})a_{2,0}z_{2,0,y}}$
$(2,2)$	$(0,0)$	$\frac{ra_{2,0}^2z_{2,0,y}}{r+a_{1,0}+a_{2,0}+(1+ra_{1,0}+a_{2,0})a_{1,0}z_{1,0,y}+(1+a_{1,0}+ra_{2,0})a_{2,0}z_{2,0,y}}$

The explicit translation-invariant Gibbs measures derived in Theorems 3.1 and 3.2 therefore provide explicit posterior distributions for the hidden Potts states. In particular, for every observed SOS configuration, the conditional probabilities $\mu_n(s_n \mid \sigma_n)$ can be computed directly from the corresponding Gibbs measure.

From the perspective of machine learning, this procedure can be interpreted as a latent-state prediction problem. The hidden Potts configuration represents unobserved classes or latent variables, while the SOS configuration corresponds to observed data. The posterior distribution $\mu_n(s_n \mid \sigma_n)$ therefore provides a probabilistic prediction of hidden states based on the available observations. Furthermore, the existence of multiple translation-invariant Gibbs measures indicates the presence of several competing inference regimes, which may be viewed as an analogue of phase transitions in latent-variable learning systems.

Remark 4.2. Formula (4.1) shows that the conditional distribution $\mu_n(s_n \mid \sigma_n)$ admits a factorized representation in terms of edge and vertex contributions. Consequently, the posterior distribution can be analyzed locally on an arbitrary edge of the Cayley tree.

Let $l_0 = \langle x, y\rangle$ be a fixed edge. For prescribed observable states $\sigma(x)$ and $\sigma(y)$, the posterior probability of the corresponding hidden Potts states is given by $\mu\big((s(x), s(y)) \mid (\sigma(x), \sigma(y))\big)$.

Using the factorized form of (4.1), we obtain

$$\mu(s_{l_0} \mid \sigma_{l_0}) = \frac{r^{\delta_{s(x)s(y)}} \exp(\beta p(s(x), \sigma(x)) \exp(\beta p(s(y), \sigma(y)) z_{s(y),\sigma(y),y}}{\sum_{\epsilon,\xi\in\{0,1,2\}} r^{\delta_{\epsilon\xi}} \exp(\beta p(\epsilon, \sigma(x)) \exp(\beta p(\xi, \sigma(y)) z_{\xi,\sigma(y),y}}.$$

Therefore, the problem of hidden-state prediction can be reduced to the evaluation of posterior probabilities on a single edge (see Table 1). Since the boundary law parameters $z_{\epsilon,\xi}$ are explicitly determined by the translation-invariant Gibbs measures obtained for Cayley trees of orders one and two, the corresponding edge posterior distributions can also be computed explicitly.

5. Conclusion

In this paper, we introduced a two-layer hidden Potts-observed SOS model on the Cayley tree and established its Gibbsian framework. The proposed model combines a hidden three-state Potts configuration with an observable SOS configuration, providing a natural extension of hidden Markov structures to multi-state spin systems with heterogeneous interactions.

For Cayley trees of arbitrary order, we derived the compatibility conditions for finite-volume Gibbs distributions and obtained the corresponding system of consistency equations. These equations provide a complete characterization of splitting Gibbs measures associated with the model.

Particular attention was devoted to translation-invariant Gibbs measures. For a specific class of model parameters, the consistency equations were reduced to a finite-dimensional fixed-point problem. Explicit translation-invariant Gibbs measures were obtained for Cayley trees of orders one and two. Assuming a homogeneous external field,

$$p(\epsilon,\xi) = p(0,0), \quad \forall\, \epsilon,\xi \in \Phi,$$

we proved that for $k = 1$ the model admits a unique splitting Gibbs measure. For $k = 2$, a critical value r_c was found such that the model has a unique translation-invariant Gibbs measure for $0 < r < r_c$, while multiple translation-invariant Gibbs measures appear for $r \geq r_c$, signaling a phase transition.

We also investigated the inference aspect of the model. By deriving the posterior distribution of hidden Potts states conditioned on the observed SOS configuration, we showed that the hidden-state prediction problem can be reduced to the computation of local posterior probabilities on a single edge of the Cayley tree. The explicit forms of the translation-invariant Gibbs measures obtained in this paper allow these posterior probabilities to be calculated explicitly for Cayley trees of orders one and two.

The results presented here establish a rigorous connection between Gibbs measures, hidden Markov structures, and two-layer spin systems on hierarchical graphs. They provide a mathematical foundation for studying latent-variable models with spatial dependence and may serve as a basis for future investigations of periodic Gibbs measures, extremality properties, reconstruction problems, and more general hidden Potts-SOS models on Cayley trees.

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