Uzbekistan Journal of Mathematics and Computer Science
Volume-2, Issue-3-special, Page 32–48 (2026)
DOI: https://doi.org/10.56143/ujmcs.v2i3s.4
Research article

Properties of the Integer-Order Generalized Bessel Integral Operator

Sh.T. Karimov* A.N. Boynazarov

Abstract

This article examines the properties of solutions obtained for the Cauchy problem associated with ordinary differential equations that contain a higher-order singular Bessel differential operator together with a lower-order term. As part of the research, an integral expression for the unique solution to the Cauchy problem is derived, one that incorporates both the Gauss hypergeometric function and the second-kind Humbert hypergeometric function. It is rigorously proved that the obtained solution completely satisfies the prescribed initial conditions and is unique. Furthermore, based on this integral representation, an integer-order generalized Bessel integral operator is introduced. Several of its fundamental properties, including the existence of an inverse operator, integration-by-parts formulas, and the semigroup property of the operator, are established through a series of lemmas.

Keywords: *higher-order ordinary differential equation; generalized Bessel differential operator; generalized Erdélyi–Kober operator; integer-order generalized Bessel integral operator.*

AMS Subject Classification (2020): *Primary:34A05 ; Secondary: 34A12, 34B05 .*

1. Introduction

Within differential equation theory, problems built around the Bessel operator hold considerable significance, since they arise naturally when modeling physical phenomena that possess radial symmetry, such as wave motion, heat transfer, and quantum-mechanical systems. Although classical Riemann–Liouville-type fractional integrals have been extensively studied for ordinary differential operators, the necessity of developing a special theory for higher-order Bessel operators of the form

$$B_\gamma = \frac{d^2}{dx^2} + \frac{2\gamma}{x}\frac{d}{dx}$$

is still relevant.

One of the first fundamental works in this direction is due to I. G. Sprinkhuizen-Kuyper, in which a class of integral operators related to negative powers of the operator D_ν and their connection with hypergeometric functions was studied [5]. I. A. Kipriyanov and his Voronezh school also investigated equations related to the Bessel operator extensively [3, 4, 7]. In particular, researchers from the Voronezh school—Elina Shishkina and Sergei M. Sitnik—studied the integer- and fractional-order Bessel operator and its integral operator in detail [7, 10, 8, 9, 15, 18].

Received : *28–april–2026,* ***Accepted :*** *31–july–2026*
**** Corresponding author***

However, modern problems of mathematical physics demand the investigation of singular differential equations that involve not only spectral parameters but also high-order singular Bessel differential operators with lower-order terms. Constructing solutions for such problems in the vicinity of the zero (singular) point poses significant complexities due to the absence of a universal method. Nevertheless, Sh. T. Karimov and his research group developed a method to construct solutions for these types of problems by employing a transmutation operator, which transforms the singular equations into non-singular ordinary differential equations [26, 25, 23, 24].

In this study, the solution to the considered initial value problem is also determined using this technique. This work sets out to accomplish two principal goals: establishing, with full mathematical rigour, that the constructed solution to the Cauchy problem both meets the prescribed initial conditions and is the only such solution, and carrying out a comprehensive study of the core properties exhibited by the generalized Bessel integral operator of integer order that is proposed here.

2. Supplementary Information: Hypergeometric Functions

In this section, we collect the formulas used in the main calculations.

The Gauss hypergeometric function is defined inside the disk $|z| < 1$ as the sum of the hypergeometric series [10, 11, 19]

$$ {}_2F_1(a,b;c;z) = F(a,b;c;z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}, \tag{2.1} $$

and for $z \in (-\infty,-1)$, it is defined by analytic continuation. In (2.1), the parameters a, b, c and the variable z may be complex, with $c \neq 0,-1,-2,\ldots$

The Pochhammer symbol is

$$ (a)_n = a(a+1)\cdots(a+n-1), \quad n = 1,2,\ldots; \quad (a)_0 = 1, $$

and for $n = 2n$ the following doubling formula holds:

$$ (a)_{2n} = \left(\frac{a}{2}\right)_n \left(\frac{a+1}{2}\right)_n 2^{2n}. \tag{2.2} $$

The integral representation of ${}_2F_1$ is

$$ {}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\,\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\,dt. \tag{2.3} $$

Transformation formulas:

$$ {}_2F_1(a,b;c;z) = (1-z)^{c-a-b}\,{}_2F_1(c-a,c-b;c;z), \quad \mathrm{Re}(c-a-b) > 0, \tag{2.4} $$

$$ {}_2F_1(a,b;c;z) = (1-z)^{-a}\,{}_2F_1\left(a,c-b;c;\tfrac{z}{z-1}\right), \tag{2.5} $$

$$ {}_2F_1(a,b,a-b+1;z) = \frac{1}{(1-z)^a}\,{}_2F_1\left(\tfrac{a}{2},\tfrac{a+1}{2}-b;\,a-b+1;\,-\tfrac{4z}{(1-z)^2}\right), \tag{2.6} $$

$$ {}_2F_1\left(a,a-b+\tfrac{1}{2};b+\tfrac{1}{2};\left(\tfrac{1-\sqrt{1-z}}{1+\sqrt{1-z}}\right)^2\right) = \left(\tfrac{2}{1+\sqrt{1-z}}\right)^{-2a}\,{}_2F_1(a,b;2b;z), \tag{2.7} $$

$$ {}_2F_1(a,b;b;z) = (1-z)^{-a}. \tag{2.8} $$

The integral representation of the Humbert function Ξ_2 is:

$$\Xi_2(a,b,c;z,t) =$$
$$= \frac{\Gamma(c)}{\Gamma(a)\,\Gamma(c-a)} \int_0^1 u^{a-1}(1-u)^{c-a-1}(1-zu)^{-b}\,\bar{I}_{c-a-1}\Big[2\sqrt{t(1-u)}\Big]\,du, \quad \mathrm{Re}(c) > \mathrm{Re}(a) > 0, \tag{2.9}$$

and its power-series representation is

$$\Xi_2(a,b,c;z,t) = \sum_{n=0}^{\infty} \frac{t^n}{(c)_n\,n!}\,{}_2F_1(a,b;c+n;z), \quad |z| < 1, \tag{2.10}$$

where $\Xi_2(a,b,c;x,y)$ is the Humbert second-kind hypergeometric function [17, 19].

The following differentiation formula holds for hypergeometric functions [5]:

$$\left(\frac{d^2}{dz^2} + \frac{2(a+b-c)}{z}\,\frac{d}{dz}\right)\frac{(1-z^2)^{c-1}}{2^c\,\Gamma(c)}\,{}_2F_1(a,b;c;1-z^2) = \frac{(1-z^2)^{c-3}}{2^{c-2}\,\Gamma(c-2)}\,{}_2F_1(a-1,b-1;c-2;1-z^2). \tag{2.11}$$

3. Initial-Value Problem

The results obtained in this section serve as the foundation for defining the integer-order generalized Bessel integral operator introduced in Section 4.

Lemma 3.1. *Let* $\lambda \in \mathbb{R}$, $0 < \gamma < 1$, *and* $m \in \mathbb{N}$. *Then the following identity holds:*

$$B_{\gamma,\lambda}\,K_m(x,s) = K_{m-1}(x,s),$$

where

$$B_{\gamma,\lambda} = B_\gamma + \lambda^2 = \frac{d^2}{dx^2} + \frac{2\gamma}{x}\,\frac{d}{dx} + \lambda^2$$

is the generalized Bessel differential operator, and

$$K_m(x,s) = \frac{1}{\Gamma(2m)}(x-s)^{2m-1}\left(\frac{s}{x}\right)^{\gamma}\,\Xi_2\left(\gamma, 1-\gamma;\, m+\tfrac{1}{2};\, -\frac{(x-s)^2}{4xs},\, -\frac{\lambda^2}{4}(x-s)^2\right), \tag{3.1}$$

where $\Xi_2\left(\gamma, 1-\gamma; m+\frac{1}{2}; -\frac{(x-s)^2}{4xs}, -\frac{\lambda^2}{4}(x-s)^2\right)$ *is the Humbert second-kind hypergeometric function.*

Proof. Using (2.10), we expand the Humbert function in (3.1):

$$\Xi_2\left(\gamma, 1-\gamma;\, m+\tfrac{1}{2};\, -\frac{(x-s)^2}{4xs},\, -\frac{\lambda^2}{4}(x-s)^2\right) = \sum_{p=0}^{\infty} \frac{\left(-\frac{\lambda^2}{4}(x-s)^2\right)^p}{\left(m+\frac{1}{2}\right)_p\,p!}\,{}_2F_1\left(\gamma, 1-\gamma;\, m+\tfrac{1}{2};\, -\frac{(x-s)^2}{4xs}\right).$$

For the Gauss hypergeometric function in the last series we apply first (2.5) and then (2.7), performing some calculations. Using the Pochhammer doubling formula (2.2):

$$\left(m+\tfrac{1}{2}\right)_p = \frac{(2m)_{2p}}{(m)_p\,2^{2p}} = \frac{\Gamma(2m+2p)}{\Gamma(2m)(m)_p\,2^{2p}},$$

we obtain the equivalent series representation $K_{m,p}(x,s)$ of $K_m(x,s)$ (the $p=0$ term being separated):

$$K_{m,p}(x,s) = 2(-s)^{2m-1}\frac{\left(1-\frac{x^2}{s^2}\right)^{2m-1}}{2^{2m}\,\Gamma(2m)}\,{}_2F_1\Big(m+\gamma-\tfrac{1}{2}, m;\, 2m;\, 1-\tfrac{x^2}{s^2}\Big)+$$
$$+\,2(-s)^{2m-1}\sum_{p=1}^{\infty}\frac{(-\lambda^2 s^2)^p\,(m)_p}{p!}\,\frac{\left(1-\frac{x^2}{s^2}\right)^{2m+2p-1}}{2^{2m+2p}\,\Gamma(2m+2p)}\,{}_2F_1\Big(m+\gamma-\tfrac{1}{2}+p, m+p;\, 2m+2p;\, 1-\tfrac{x^2}{s^2}\Big). \tag{3.2}$$

Applying the Bessel operator B_γ to the two parts and using (2.11), we eventually find

$$B_\gamma K_{m,p}(x,s) = B_\gamma K_{m-1,p}(x,s) - \lambda^2 K_{m,p}(x,s),$$

i.e.,

$$(B_\gamma + \lambda^2)\, K_{m,p}(x,s) = K_{m-1,p}(x,s). \tag{3.3}$$

Lemma 3.1 is proved.

Theorem 3.1 (see [1]). *Let $\Omega = \{x \in \mathbb{R} : x > 0\}$, and let $f(x) \in C(0,b)$, $b > 0$. Then the unique solution $y(x) \in C^{2m}(0,+\infty) \cap C[0,\infty)$ of the problem*

$$\left(\frac{d^2}{dx^2} + \frac{2\gamma}{x}\frac{d}{dx} + \lambda^2\right)^m y(x) = f(x), \quad x > 0, \quad 0 < \gamma < \tfrac{1}{2}, \quad m \in \mathbb{N}, \tag{3.4}$$

$$y^{(j)}(0) = 0, \quad j = 0, 1, \ldots, 2m-1, \tag{3.5}$$

is given by

$$y(x) = \frac{1}{\Gamma(2m)}\int_0^x (x-s)^{2m-1}\left(\frac{s}{x}\right)^\gamma \Xi_2\Big(\gamma, 1-\gamma;\, m+\tfrac{1}{2};\, -\frac{(x-s)^2}{4xs},\, -\frac{\lambda^2}{4}(x-s)^2\Big)\, f(s)\, ds. \tag{3.6}$$

Proof. We first establish the following relations for the integral kernel in (3.6). It is well known that $K_{m,p}(x,x) = 0$, and we shall show that $K_{m,p}(x,0) = 0$. For this purpose, utilizing relation (2.9) for the Humbert function, we perform the following calculations:

$$K_m(x,0) = \lim_{s\to 0}(x-s)^{2m-1}\left(\frac{s}{x}\right)^\gamma \Xi_2\left(\gamma, 1-\gamma; m+\frac{1}{2}; -\frac{(x-s)^2}{4xs}, -\frac{\lambda^2}{4}(x-s)^2\right) =$$

$$= \lim_{s\to 0}(x-s)^{2m-1}\left(\frac{s}{x}\right)^\gamma \frac{\Gamma\left(m+\frac{1}{2}\right)}{\Gamma(\gamma)\,\Gamma\left(m+\frac{1}{2}-\gamma\right)}\times$$

$$\times\int_0^1 t^{\gamma-1}(1-t)^{\alpha-\frac{1}{2}-\gamma}\left(1+\frac{(x-s)^2}{4xs}t\right)^{\gamma-1}\bar{I}_{m-\frac{1}{2}-\gamma}\left[\lambda i\sqrt{(x-s)^2(1-t)}\right]dt =$$

$$= \left\langle \mathrm{Re}\left(m+\frac{1}{2}\right) > \mathrm{Re}(\gamma) > 0\right\rangle =$$

$$= \lim_{s\to 0} s x^{-\gamma}(x-s)^{2m-1}\frac{\Gamma\left(m+\frac{1}{2}\right)}{\Gamma(\gamma)\,\Gamma\left(m+\frac{1}{2}-\gamma\right)}\times$$

$$\times \int_0^1 t^{\gamma-1}(1-t)^{m-\frac{1}{2}-\gamma}\left(s+\frac{(x-s)^2}{4x}t\right)^{\gamma-1}\bar{I}_{m-\frac{1}{2}-\gamma}\left[\lambda i\sqrt{(x-s)^2\,(1-t)}\right]dt=0,$$

where, according to formulas (2.9) and (2.10), the following equality holds when $s=0$

$$x^{2m-1-\gamma}\frac{\Gamma\left(m+\frac{1}{2}\right)}{\Gamma(\gamma)\,\Gamma\left(m+\frac{1}{2}-\gamma\right)}\int_0^1 t^{\gamma-1}(1-t)^{m-\frac{1}{2}-\gamma}\left(\frac{x}{4}t\right)^{\gamma-1}\bar{I}_{m-\frac{1}{2}-\gamma}\left[\lambda i\sqrt{x^2\,(1-t)}\right]dt=$$

$$=x^{2m-2-\gamma}4^{1-\gamma}\frac{\Gamma\left(m+\frac{1}{2}\right)}{\Gamma(\gamma)\,\Gamma\left(m+\frac{1}{2}-\gamma\right)}\int_0^1 t^{2\gamma-2}(1-t)^{m-\frac{1}{2}-\gamma}(1-xt)^0\bar{I}_{m-\frac{1}{2}-\gamma}\left[\lambda i\sqrt{x^2\,(1-t)}\right]dt=$$

$$=x^{2m-2-\gamma}4^{1-\gamma}\frac{\Gamma\left(m+\frac{1}{2}\right)}{\Gamma(\gamma)\,\Gamma\left(m+\frac{1}{2}-\gamma\right)}\frac{\Gamma(2\gamma-1)\,\Gamma\left(m+\frac{1}{2}-\gamma\right)}{\Gamma\left(m-\frac{1}{2}+\gamma\right)}\Xi_2\left(0,2\gamma-1;m-\frac{1}{2}+\gamma;x,x^2\right)=$$

$$=x^{2m-2-\gamma}4^{1-\gamma}\frac{\Gamma\left(m+\frac{1}{2}\right)}{\Gamma(\gamma)}\frac{\Gamma(2\gamma-1)}{\Gamma\left(m-\frac{1}{2}+\gamma\right)}\bar{J}_{m-\frac{3}{2}+\gamma}(2x)<\infty,\quad m+\gamma>1.$$

Applying the differential operator $B_{\gamma,\lambda}$ to both sides of (3.6) and using the Leibniz differentiation rule for integrals with variable limits, we have

$$B_{\gamma,\lambda}\,y(x)=\int_0^x B_{\gamma,\lambda}\,K_{m,p}(x,s)\,f(s)\,ds.$$

By (3.3) this equals

$$B_{\gamma,\lambda}\,y(x)=\int_0^x K_{m-1,p}(x,s)\,f(s)\,ds. \tag{3.7}$$

Applying $B_{\gamma,\lambda}$ to (3.7) successively $m-1$ more times, and we obtain:

$$B^m_{\gamma,\lambda}y(x)=B_\gamma\int_0^x K_{1,0}(x,s)f(s)\,ds+B_\gamma\int_0^x K_{1,p\geq 1}(x,s)f(s)\,ds+\lambda^2\int_0^x K_{1,p}(x,s)f(s)\,ds=$$

$$=I_1(x,s)+I_2(x,s)+I_3(x,s).$$

Let us compute the first term:

$$I_1(x,s)=B_\gamma\int_0^x K_{1,0}(x,s)f(s)\,ds=$$

$$=\left(\frac{d^2}{dx^2}+\frac{2\gamma}{x}\frac{d}{dx}\right)\int_0^x\left(\frac{x^2-s^2}{2s}\right){}_2F_1\left(\gamma+\frac{1}{2},1;2;\frac{s^2-x^2}{s^2}\right)f(s)\,ds=$$

$$-\left(\frac{d}{dx}+\frac{2\gamma}{x}\right)\int_0^x\frac{d}{dx}\left(\frac{s^2-x^2}{s^2}\right){}_2F_1\left(\gamma+\frac{1}{2},1;2;\frac{s^2-x^2}{s^2}\right)\frac{s}{2}f(s)\,ds=$$

$$=\left(\frac{d}{dx}+\frac{2\gamma}{x}\right)\int_0^x\left(\frac{x}{s}\right){}_2F_1\left(\gamma+\frac{1}{2},1;1;\frac{s^2-x^2}{s^2}\right)f(s)\,ds.$$

According to formula (2.8)

$$I_1(x,s)=\left(\frac{d}{dx}+\frac{2\gamma}{x}\right)\int\limits_0^x\left(\frac{s}{x}\right)^{2\gamma}f(s)\,ds=\left(\frac{s}{x}\right)^{2\gamma}f(s)\bigg|_{s=x}-\int\limits_0^x\left(\frac{d}{dx}+\frac{2\gamma}{x}\right)\left(\frac{s}{x}\right)^{2\gamma}f(s)\,ds=$$

$$=f(x)-\int\limits_0^x\left(\frac{d}{dx}\left(\frac{s}{x}\right)^{2\gamma}+\frac{2\alpha}{x}\left(\frac{s}{x}\right)^{2\gamma}\right)f(s)\,ds=$$

$$=f(x)-\int\limits_0^x\left(-\frac{2\gamma}{x}\left(\frac{s}{x}\right)^{2\gamma}+\frac{2\gamma}{x}\left(\frac{s}{x}\right)^{2\gamma}\right)f(s)\,ds=f(x).$$

Now, compute the second term:

$$I_2(x,s)=B_\gamma\int\limits_0^x K_{1,p\geq1}(x,s)f(s)\,ds=$$

$$=-B_\gamma\int\limits_0^x 2\sum_{p=1}^{\infty}\frac{\left(-\frac{\lambda^2}{4}s^2\right)^p(1)_p}{p!}\frac{\left(\frac{s^2-x^2}{s^2}\right)^{2p+1}}{2^{2+2p}\Gamma(2+2p)}{}_2F_1\left(\gamma+\frac{1}{2}+p,1+p;2+2p;\frac{s^2-x^2}{s^2}\right)sf(s)\,ds.$$

According to formula (2.11)

$$I_2(x,s)=-\int\limits_0^x 2\sum_{p=1}^{\infty}\frac{\left(-\frac{\lambda^2}{4}s^2\right)^p(1)_p}{p!}B_\gamma\frac{\left(\frac{s^2-x^2}{s^2}\right)^{2p+1}}{2^{2+2p}\Gamma(2+2p)}{}_2F_1\left(\gamma+\frac{1}{2}+p,1+p;2+2p;\frac{s^2-x^2}{s^2}\right)sf(s)\,ds=$$

$$=-\int\limits_0^x 2\sum_{p=1}^{\infty}\frac{(-\lambda^2)^p\Gamma(p)}{(p-1)!\Gamma(1)}\frac{\left(\frac{s^2-x^2}{s^2}\right)^{2p-1}}{2^{2p}\Gamma(2p)}{}_2F_1\left(\gamma-\frac{1}{2}+p,p;2p;\frac{s^2-x^2}{s^2}\right)s^{-1+2p}f(s)\,ds.$$

We replace the summation index p by $p+1$

$$I_2(x,s)=-\lambda^2\int\limits_0^x 2\sum_{p=0}^{\infty}\frac{(-\lambda^2)^p\Gamma(p+1)}{p!\Gamma(1)}\frac{\left(\frac{s^2-x^2}{s^2}\right)^{2p+1}}{2^{2p+2}\Gamma(2p+2)}\times$$

$$\times{}_2F_1\left(\gamma+\frac{1}{2}+p,p+1;2p+2;\frac{s^2-x^2}{s^2}\right)s^{1+2p}f(s)\,ds=-\lambda^2\int\limits_0^x K_{1,p}(x,s)f(s)\,ds.$$

Now we compute the terms, and we get:

$$B^m_{\gamma,\lambda}y(x)=f(x),$$

where $I_3(x,s)=-I_2(x,s)$

Theorem 3.1 is proved.

3.1. Verification of Initial Conditions

We show that solution (3.6) satisfies initial conditions (3.5).

Condition $y(0) = 0$. Using representation (3.2), we rewrite solution (3.6) in the following form:

$$y(x) = \int_0^x \left(\frac{s}{x}\right)^{2\gamma} \left(\frac{x^2 - s^2}{2x}\right)^{2m-1} \sum_{p=0}^{\infty} \frac{\left(-\frac{\lambda^2}{4}\frac{(x^2-s^2)^2}{x^2}\right)^p (m)_p}{\Gamma(2m+2p)\, p!}$$

$$\times {}_2F_1\left(m + \gamma - \tfrac{1}{2} + p,\, m + p;\, 2m + 2p;\, \frac{x^2 - s^2}{x^2}\right) f(s)\, ds.$$

Applying the substitution $s = xt$ in the integral variable, we obtain

$$y(x) = 2^{1-2m} \sum_{p=0}^{\infty} \frac{\left(-\frac{\lambda^2}{4}\right)^p (m)_p}{\Gamma(2m+2p)\, p!} x^{2m+2p}$$

$$\times \int_0^1 t^{2\gamma}(1 - t^2)^{2m+2p-1}\, {}_2F_1\left(m + \gamma - \tfrac{1}{2} + p,\, m + p;\, 2m + 2p;\, 1 - t^2\right) f(xt)\, dt.$$

We now estimate this integral. Taking absolute values and applying the triangle inequality, we get

$$|y(x)| leq 2^{1-2m} \sum_{p=0}^{\infty} \frac{\left|\left(-\frac{\lambda^2}{4}\right)^p (m)_p\right|}{\Gamma(2m+2p)\, p!} x^{2m+2p} \int_0^1 \left|t^{2\gamma}(1 - t^2)^{2m+2p-1}\right|$$

$$\times \left|{}_2F_1\left(m + \gamma - \tfrac{1}{2} + p,\, m + p;\, 2m + 2p;\, 1 - t^2\right)\right| |f(xt)|\, dt.$$

The following convergence results for hypergeometric functions are known (see [28], pp. 24, 618):

$${}_2F_1\left(m + \gamma - \tfrac{1}{2} + p,\, m + p;\, 2m + 2p;\, 1 - t^2\right) = C_1 < \infty, \quad |1 - t^2| < 1, \quad 0 < t < 1,$$

$${}_1F_2\left(m,\, m + \gamma + \tfrac{1}{2};\, m + \gamma + 1;\, -\frac{\lambda^2 x^2}{16}\right) = C_2 < \infty, \quad \left|-\frac{\lambda^2 x^2}{16}\right| \in \mathbb{R}, \quad m \in \mathbb{N}, \quad 0 < \gamma < \tfrac{1}{2}.$$

Using these bounds and the continuity of $f(x) \in C[0, b]$, we carry out the following estimates for the last expression:

$$|y(x)| \leq x^{2m} C_1 |f(xt)| \frac{2^{1-2m}}{\Gamma(2m)} \sum_{p=0}^{\infty} \frac{\left(-\frac{\lambda^2}{16}x^2\right)^p}{\left(m + \frac{1}{2}\right)_p p!} B(2\gamma + 1,\, 2m + 2p)$$

$$= x^{2m} C_1 |f(xt)| \frac{2^{1-2m}\, \Gamma(2\gamma + 1)}{\Gamma(2\gamma + 1 + 2m)} \sum_{p=0}^{\infty} \frac{(m)_p}{\left(\gamma + \frac{1}{2} + m\right)_p (m + \gamma + 1)_p\, p!} \left(-\frac{\lambda^2}{16}x^2\right)^p$$

$$= x^{2m} C_1 |f(xt)| \frac{2^{1-2m}\, \Gamma(2\gamma + 1)}{\Gamma(2\gamma + 1 + 2m)}\, {}_1F_2\left(m;\, m + \gamma + \tfrac{1}{2},\, m + \gamma + 1;\, -\frac{\lambda^2 x^2}{16}\right)$$

$$= x^{2m} C_1 C_2 |f(xt)| \frac{2^{1-2m}\, \Gamma(2\gamma + 1)}{\Gamma(2\gamma + 1 + 2m)} = x^{2m} C\, |f(xt)|,$$

where $C = C_1 C_2 \frac{2^{1-2m}\, \Gamma(2\gamma+1)}{\Gamma(2\gamma+1+2m)}$.

Therefore, from the estimate above, it follows that $y(0) = 0$.

Condition $y'(0) = 0$**.** Differentiating (3.6) and writing it in terms of the integral representation (2.9), then substituting $s = xt$ and sending $x \to 0$, all terms vanish since the integrand contains a factor $x^{-2+2k+2n+2m}$ with $k + n + m > 0$, giving $y'(0) = 0$.

Condition $y''(0) = 0$**.** Differentiating equation (3.7) with respect to x and applying (3.3):

$$y''(x) + \frac{2\gamma}{x} y'(x) + \lambda^2 y(x) = \int_0^x K_{m-1,p}(x,s) f(s)\, ds = x^{2m-2} \int_0^1 K_{m-1,p}(1,t) f(xt)\, dt.$$

Applying L'Hôpital's rule:

$$\lim_{x\to 0} \left[(1 + 2\gamma)\, y''(x) + \lambda^2 y(x) \right] = 0 \;\Rightarrow\; y''(0) = 0.$$

Conditions $y^{(k)}(0) = 0$**,** $k < 2m - 1$**.** We prove these by mathematical induction. Assume $y^{(k)}(0) = 0$ for all $k < 2m - 1$. Then applying the operator $\frac{d^{k-1}}{dx^{k-1}}\left(y'' + \frac{2\gamma}{x} y'\right)$ to both sides, using L'Hôpital's rule, and noting the right-hand side tends to zero, gives $(C + 2\gamma)\, y^{(k+1)}(0) = 0$, hence $y^{(k+1)}(0) = 0$.

Thus, initial conditions (3.5) are completely satisfied by solution (3.6).

3.2. Uniqueness of Solution

Suppose $y_1(x)$ and $y_2(x)$ are two solutions of the problems (3.4)- (3.5). Then $u(x) = y_1(x) - y_2(x)$ satisfies the homogeneous equation:

$$(B_{\gamma,\lambda})^m u(x) = 0, \quad u^{(j)}(0) = 0, \quad j = 0, 1, 2, \ldots, 2m - 1.$$

According to Theorem 1 in [1], for the problem stated above, the following substitution is valid:

$$\left(B_{\gamma,\lambda}\right)^m J_\lambda\left(-\tfrac{1}{2}, \gamma\right) v(x) = J_\lambda\left(-\tfrac{1}{2}, \gamma\right) \left(B_{0,0}\right)^m v(x),$$

where $B_{0,0} = \frac{d^2}{dx^2}$, and $u(x) = J_\lambda\left(-\frac{1}{2}, \gamma\right) v(x)$, with $J_\lambda\left(-\frac{1}{2}, \gamma\right)$ being the fractional-order generalized Erdélyi–Kober operator.

Consequently, we obtain the problem stated below.

$$\left(\frac{d}{dx}\right)^{2m} v(x) = 0, \quad v^{(i)}(0) = 0, \quad i = 0, 1, 2, \ldots, 2m - 1.$$

By the Cauchy–Picard existence and uniqueness theorem for linear ODEs with constant coefficients (see [27, pp. 136–140]), the only solution to this homogeneous problem with zero initial conditions at the origin is the trivial solution $v(x) = 0$.

Hence $u(x) \equiv 0$, i.e., $y_1(x) = y_2(x)$, which proves the uniqueness of the solution.

4. Integer-Order Generalized Bessel Integral Operator

In this section we use the integral operator given by formula (3.6) in Theorem 3.1 to define the integer-order generalized Bessel integral operator.

Definition 4.1. Let $m \in \mathbb{N}$, $f(x) \in L_1(a,b)$, $a \geq 0$. The *right-sided integer-order generalized Bessel operator* $B^{-m}_{\gamma,\lambda,b-}$ and the *left-sided integer-order generalized Bessel operator* $B^{-m}_{\gamma,\lambda,a+}$ on the finite interval are defined, respectively, by

$$B^{-m}_{\gamma,\lambda,b-}f(x) = \frac{1}{\Gamma(2m)}\int_x^b \left(\frac{s}{x}\right)^{\gamma}(s-x)^{2m-1}\Xi_2\left(\gamma, 1-\gamma;\, m+\tfrac{1}{2};\, 1-\frac{(x+s)^2}{4xs},\, -\frac{\lambda^2}{4}(s-x)^2\right) f(s)\,ds,$$

$$B^{-m}_{\gamma,\lambda,a+}f(x) = \frac{1}{\Gamma(2m)}\int_a^x \left(\frac{s}{x}\right)^{\gamma}(x-s)^{2m-1}\Xi_2\left(\gamma, 1-\gamma;\, m+\tfrac{1}{2};\, 1-\frac{(x+s)^2}{4xs},\, -\frac{\lambda^2}{4}(x-s)^2\right) f(s)\,ds,$$

or, equivalently, in series form:

$$B^{-m}_{\gamma,\lambda,b-}f\,(x) = \int\limits_x^b \left(\frac{s^2-x^2}{2s}\right)^{2m-1}\sum_{p=0}^{\infty}\frac{\left(-\frac{\lambda^2}{4}\frac{(s^2-x^2)^2}{s^2}\right)^p (m)_p}{\Gamma\,(2m+2p)\,p!}\times$$

$$\times {}_2F_1\left(m+\gamma-\frac{1}{2}+p, m+p; 2m+2p; \frac{s^2-x^2}{s^2}\right) f\,(s)\,ds,$$

$$B^{-m}_{\gamma,\lambda,a+}f\,(x) = \int\limits_a^x \left(\frac{s}{x}\right)^{2\gamma}\left(\frac{x^2-s^2}{2x}\right)^{2m-1}\sum_{p=0}^{\infty}\frac{\left(-\frac{\lambda^2}{4}\frac{(x^2-s^2)^2}{x^2}\right)^p (m)_p}{\Gamma\,(2m+2p)\,p!}\times$$

$$\times {}_2F_1\left(m+\gamma-\frac{1}{2}+p, m+p; 2m+2p; \frac{x^2-s^2}{x^2}\right) f\,(s)\,ds.$$

Definition 4.2. Let $m \in \mathbb{N}$, $f(x) \in L_1(0,\infty)$. The right-sided and left-sided integer-order generalized Bessel operators on $[0,\infty)$ are defined, respectively, by

$$B^{-m}_{\gamma,\lambda,-}f\,(x) = \frac{1}{\Gamma\,(2m)}\int\limits_x^{\infty}\left(\frac{s}{x}\right)^{\gamma}(s-x)^{2m-1}\Xi_2\left(\gamma, 1-\gamma; m+\frac{1}{2}; -\frac{(s-x)^2}{4xs}, -\frac{\lambda^2}{4}(s-x)^2\right) f\,(s)\,ds \quad (4.1)$$

$$B^{-m}_{\gamma,\lambda,0+}f\,(x) = \frac{1}{\Gamma\,(2m)}\int\limits_0^{x}\left(\frac{s}{x}\right)^{\gamma}(x-s)^{2m-1}\Xi_2\left(\gamma, 1-\gamma; m+\frac{1}{2}; -\frac{(x-s)^2}{4xs}, -\frac{\lambda^2}{4}(x-s)^2\right) f\,(s)\,ds \quad (4.2)$$

or equivalently, in series form (equations (4.1)–(4.2)):

$$B^{-m}_{\gamma,\lambda,-}f\,(x) = \int\limits_x^{\infty} \left(\frac{s^2-x^2}{2s}\right)^{2m-1}\sum_{p=0}^{\infty}\frac{\left(-\frac{\lambda^2}{4}\frac{(s^2-x^2)^2}{s^2}\right)^p (m)_p}{\Gamma\,(2m+2p)\,p!}\times$$

$$\times {}_2F_1\left(m+\gamma-\frac{1}{2}+p, m+p; 2m+2p; \frac{s^2-x^2}{s^2}\right) f\,(s)\,ds, \quad (4.3)$$

$$B^{-m}_{\gamma,\lambda,0+}f\,(x) = \int\limits_0^x \left(\frac{s}{x}\right)^{2\gamma}\left(\frac{x^2-s^2}{2x}\right)^{2m-1}\sum_{p=0}^{\infty}\frac{\left(-\frac{\lambda^2}{4}\frac{(x^2-s^2)^2}{x^2}\right)^p (m)_p}{\Gamma\,(2m+2p)\,p!}\times$$

$$\times {}_2F_1\left(m+\gamma-\frac{1}{2}+p, m+p; 2m+2p; \frac{x^2-s^2}{x^2}\right) f\,(s)\,ds \quad (4.4)$$

We may write (4.2) as

$$B^{-m}_{\gamma,\lambda,0+}f(x) = \int_0^x K_{\gamma,m}(x,s)\, f(s)\, ds, \tag{4.5}$$

where

$$K_{\gamma,m}(x,s) = \frac{1}{\Gamma(2m)}(x-s)^{2m-1}\left(\frac{s}{x}\right)^{\gamma} \Xi_2\left(\gamma, 1-\gamma;\, m+\tfrac{1}{2};\, -\frac{(x-s)^2}{4xs},\, -\frac{\lambda^2}{4}(x-s)^2\right), \quad 0 < s < x < \infty.$$

Lemma 4.1. *If $f \in C([0,\infty])$, and $f(0) = 0$, then*

$$\lim_{m\to 0} B^{-m}_{\gamma,\lambda,0+}f(x) = f(x). \tag{4.6}$$

Proof. Employing the representation (4.4) for the generalized Bessel integral operator of integer order and using the convergence of the series, we interchange summation and integration, yielding

$$B^{-m}_{\gamma,\lambda,0+}y(x) = \sum_{p=0}^{\infty} \frac{\left(-\lambda^2\right)^p (m)_p 2^{1-2p-2m}}{\Gamma(2m+2p)\, p!} \times$$

$$\times \int_0^x \left(\frac{s^2-x^2}{s^2}\right)^{2m+2p-1} {}_2F_1\left(m+p+\frac{\gamma-1}{2}, m+p; 2m+2p; \frac{s^2-x^2}{s^2}\right)(-s)^{2m+2p-1} f(s)\, ds.$$

Applying (2.5) and integrating by parts, and sending $m \to 0$, only the $p = 0$ term survives (since $(0)_p = 0$ for $p \geq 1$), giving

$$\lim_{m\to 0} B^{-m}_{\gamma,\lambda,0+}f(x) = x^{-2}\int_0^x \frac{d}{ds}\left[s^2 f(s)\right] ds = x^{-2}\left[s^2 f(s)\right]_0^x = f(x).$$

Lemma 4.2. *Let $m \in \mathbb{N}$, $f \in C([0,\infty])$, be sufficiently smooth. Then $B^{m}_{\gamma,\lambda}$ is the inverse operator of $B^{-m}_{\gamma,\lambda,0+}$:*

$$B^{m}_{\gamma,\lambda}\, B^{-m}_{\gamma,\lambda,0+}f(x) = f(x), \quad \left(B^{m}_{\gamma,\lambda}\right)^{-1} = B^{-m}_{\gamma,\lambda,0+}. \tag{4.7}$$

Proof. The proof follows directly from Theorem 3.1 and Lemma 3.1.

Lemma 4.3. *Let $f \in C([0,\infty])$, and $m_1, m_2 \in \mathbb{N}$. Then the operator $B^{-m}_{\gamma,\lambda,0+}$ forms a semigroup with respect to m:*

$$B^{-m_1}_{\gamma,\lambda,0+}\, B^{-m_2}_{\gamma,\lambda,0+}f(x) = B^{-(m_1+m_2)}_{\gamma,\lambda,0+}f(x). \tag{4.8}$$

Proof. We use the series representation (4.4) and, since the integral and inner series are convergent, bring the series outside the integral:

$$B^{-m}_{\gamma,\lambda,0+}y(x) = 2^{1-2m}\sum_{p,l=0}^{\infty} A_{m,p,l}\, x^{1-2\gamma-2m-2p-2l}\int_0^x (x^2-s^2)^{2m+2p+l-1}s^{2\gamma} f(s)\, ds,$$

where

$$A_{m,p,l} = \frac{\left(-\frac{\lambda^2}{4}\right)^p (m)_p \left(m+p+\frac{2\gamma-1}{2}\right)_l (m+p)_l}{\Gamma(2m+2p)\, p!\, (2m+2p)_l\, l!}.$$

Computing $B^{-m_1}_{\gamma,\lambda,0+}B^{-m_2}_{\gamma,\lambda,0+}f(x)$

$$B^{-m_1}_{\gamma,\lambda,0+}B^{-m_1}_{\gamma,\lambda,0+}f(x) = 2^{2-2(m_1+m_2)}\sum_{p,l=0}^{\infty} A_{m_1,p,l}\sum_{k,n=0}^{\infty} A_{m_2,k,n}x^{1-2\gamma-2m_1-2p-2l}\times$$

$$\times \int\limits_0^x t^{2\gamma} f(t)\,dt \int\limits_t^x \left(x^2 - s^2\right)^{2m_1+2p+l-1} \left(s^2 - t^2\right)^{2m_2+2k+n-1} s^{1-2m_2-2k-2n} ds =$$

$$= 2^{2-2(m_1+m_2)} \sum_{p,l=0}^{\infty} A_{m_1,p,l} \sum_{k,n=0}^{\infty} A_{m_2,k,n} x^{1-2\gamma-2m_1-2p-2l} \int\limits_0^x t^{2\gamma} f(t)\, g(x,t)\, dt,$$

where $g(x,t) = \int\limits_t^x (x^2 - s^2)^{2m_1+2p+l-1} (s^2 - t^2)^{2m_2+2k+n-1} s^{1-2m_2-2k-2n} ds$.

First, we compute the integral $g(x,t)$ and use, get:

$$g(x,t) = \frac{1}{2}\left(x^2 - t^2\right)^{2m_2+2k+n-1+2m_1+2p+l} t^{-2m_2-2k-2n} \frac{\Gamma(2m_2+2k+n)\,\Gamma(2m_1+2p+l)}{\Gamma(2m_2+2m_1+2k+2p+n+l)} \times$$

$$\times {}_2F_1\left(m_2+k+n, 2m_2+2k+n; 2m_2+2m_1+2k+2p+n+l; \frac{t^2-x^2}{t^2}\right).$$

Substituting $g(x,t)$ into its place and performing the calculations, we obtain the following

$$B^{-m_1}_{\gamma,\lambda,0+} B^{-m_2}_{\gamma,\lambda,0+} f(x) =$$

$$= 2^{1-2(m_1+m_2)} \sum_{n=0}^{\infty} \frac{\left(-\lambda^2\left(\frac{x^2-t^2}{x}\right)^2\right)^n}{\Gamma(2m_2+2m_1+2n)\, n!} \sum_{p=0}^{n} \frac{(m_1)_p (m_2)_{n-p} n!}{p!\,(n-p)!} \times$$

$$\times \int\limits_0^x \frac{t}{x}\left(\frac{x^2-t^2}{x}\right)^{2m_2+2m_1-1} {}_2F_1\left(m_1+m_2+n, m_1+m_2-\frac{2\gamma-1}{2}+n; 2m_2+2m_1+2n; \frac{x^2-t^2}{x^2}\right) f(t)\,dt$$

We reduce the multiple series in the last expression to a finite series. For this, we use the following formulas [19, 22, p. 44]

$$\sum_{p=0}^{n} \frac{(\beta_1)_p (\beta_2)_{n-p}\, n!}{p!(n-p)!} = \sum_{p=0}^{n} \binom{n}{p} (\beta_1)_p (\beta_2)_{n-p} = (\beta_1+\beta_2)_n$$

and we have

$$B^{-m_1}_{\gamma,\lambda,0+} B^{-m_2}_{\gamma,\lambda,0+} f(x) = \sum_{n=0}^{\infty} \frac{\left(-\lambda^2\left(\frac{x^2-t^2}{x}\right)^2\right)^n (m_1+m_2)_n}{\Gamma(2m_2+2m_1+2n)\, n!} \times$$

$$\times \int\limits_0^x \left(\frac{t}{x}\right)^{2\gamma} \left(\frac{x^2-t^2}{2x}\right)^{2m_2+2m_1-1} {}_2F_1\left(m_1+m_2+\frac{2\gamma-1}{2}+n, m_1+m_2+n; 2m_2+2m_1+2n; \frac{x^2-t^2}{x^2}\right) f(t)\,dt.$$

Lemma 4.3 is proved.

Lemma 4.4. *Let $m \in \mathbb{N}$, $f \in C^{2m}([0,\infty))$, and $f^{(i)}(0) = 0$ for $i = 0, 1, \ldots, 2m-1$. Then the following equality hold:*

$$B^{-m}_{\gamma,\lambda,0+}\, B^{m}_{\gamma,\lambda}\, f(x) = f(x). \tag{4.9}$$

Proof. We prove this lemma using the generalized Bessel integral operator (4.2) and its equivalent series representation (4.4).

$$B_{\gamma,\lambda,0+}^{-m} B_{\gamma,\lambda}^{m} f(x) = \int_0^x K_m(x,s) \left(B_{\gamma,\lambda}^{m} f\right)(s)\, ds, \tag{4.10}$$

where $K_m(x,s) = \frac{1}{\Gamma(2m)}(x-s)^{2m-1} G(x,s)$, $G(x,s) = \left(\frac{s}{x}\right)^{\gamma} \Xi_2\left(\gamma, 1-\gamma; \frac{3}{2}; -\frac{(x-s)^2}{4xs}, -\frac{\lambda^2}{4}(x-s)^2\right)$.

Using integration by parts on the right-hand side of (4.10), together with the equality proved within Theorem 3.1, we get:

$$K_{m,1}(x,x) = K_{m,1}(x,0) = 0,$$

and we rewrite right site of (4.10) for the case $m = 1$:

$$\begin{aligned}\int_0^x K_1(x,s)\left(B_{\gamma,\lambda} f\right)(s)\, ds &= \int_0^x (x-s)\, G(x,s) \left(\frac{d^2}{ds^2} + \frac{2\gamma}{s}\frac{d}{ds} + \lambda^2\right) f(s)\, ds \\ &= \int_0^x (x-s)\, G(x,s) \frac{d^2}{ds^2} f(s)\, ds + 2\gamma \int_0^x (x-s)\, s^{-1} G(x,s) \frac{d}{ds} f(s)\, ds \\ &+ \lambda^2 \int_0^x (x-s)\, G(x,s) f(s)\, ds = I_1 + I_2 + I_3.\end{aligned}$$

We integrate the first term by parts:

$$\begin{aligned} I_1 &= \int_0^x (x-s)\, G(x,s) \frac{d^2}{ds^2} f(s)\, ds \\ &= (x-s)\, G(x,s)\, \frac{d}{ds} f(s)\Big|_0^x - \int_0^x \frac{d}{ds}(x-s)\, G(x,s) \frac{d}{ds} f(s)\, ds \\ &- \int_0^x \frac{d}{ds}(x-s)\, G(x,s) \frac{d}{ds} f(s)\, ds \\ &= -\frac{d}{ds}(x-s)\, G(x,s)\; f(s)\big|_0^x + \int_0^x \frac{d^2}{ds^2}(x-s)\, G(x,s) f(s)\, ds \\ &= \int_0^x \frac{d^2}{ds^2}(x-s)\, G(x,s) f(s)\, ds. \end{aligned}$$

Now integrate the second term by parts:

$$I_2 = 2\gamma \int_0^x (x-s)\, s^{-1} G(x,s) \frac{d}{ds} f(s)\, ds$$

$$= 2\gamma (x-s)\, s^{-1}\, G(x,s)\, f(s)|_0^x - 2\gamma \int_0^x \frac{d}{ds} (x-s)\, s^{-1} G(x,s) f(s)\, ds$$

$$= -\int_0^x \frac{d}{ds} \frac{2\gamma}{s} (x-s)\, G(x,s) f(s)\, ds.$$

Summing the contributions we obtain

$$\int_0^x (x-s)\, G(x,s) \left(\frac{d^2}{ds^2} + \frac{2\gamma}{s} \frac{d}{ds} + \lambda^2 \right) f(s)\, ds =$$

$$= \int_0^x \left(\frac{d^2}{ds^2} - \frac{d}{ds} \frac{2\gamma}{s} + \lambda^2 \right) (x-s)\, G(x,s) f(s)\, ds =$$

$$= \int_0^x \left(\frac{d^2}{ds^2} - \frac{d}{ds} \frac{2\gamma}{s} + \lambda^2 \right) K_1(x,s) f(s)\, ds. \tag{4.11}$$

Denote $B^*_{\gamma,\lambda} = \left(\frac{d^2}{ds^2} - \frac{d}{ds} \frac{2\gamma}{s} + \lambda^2 \right)$ and we calculate the following equation:

$$B^{-m}_{\gamma,\lambda,0+} B^*_{\gamma,\lambda} f(x) = \int_0^x B^*_{\gamma,\lambda} K_{m,p}(x,s)\, f(s)\, ds.$$

Separate the term with $p = 0$:

$$B^{-m}_{\gamma,\lambda,0+} B^*_{\gamma,\lambda} f(x) = \int_0^x B^*_{\gamma} K_{m,0}(x,s) f(s)\, ds + \int_0^x B^*_{\gamma} K_{m,p\geq 1}(x,s) f(s)\, ds + \lambda^2 \int_0^x K_{m,p}(x,s)\, f(s)\, ds =$$

$$= L_1 + L_2 + L_3. \tag{4.12}$$

First compute L_1:

$$L_1=\int\limits_0^x\left(\frac{d^2}{ds^2}-\frac{d}{ds}\frac{2\gamma}{s}\right)\left(\frac{s}{x}\right)^{2\gamma}\left(\frac{x^2-s^2}{2x}\right)^{2m-1}{}_2F_1\left(m+\gamma-\frac{1}{2},m;2m;\frac{x^2-s^2}{x^2}\right)f(s)\,ds$$

$$=2\int\limits_0^x x^{2m-1}\left(\frac{s}{x}\right)^{2\gamma}\left(\frac{d^2}{ds^2}+\frac{2\gamma}{s}\frac{d}{ds}\right)\frac{\left(\frac{x^2-s^2}{x^2}\right)^{2m-1}}{2^{2m}\Gamma(2m)}{}_2F_1\left(m+\gamma-\frac{1}{2},m;2m;\frac{x^2-s^2}{x^2}\right)f(s)\,ds$$

$$=2\int\limits_0^x x^{2m-3}\left(\frac{s}{x}\right)^{2\gamma}\frac{\left(\frac{x^2-s^2}{x^2}\right)^{2m-3}}{2^{2m-2}\Gamma(2m-2)}{}_2F_1\left(m+\gamma-\frac{3}{2},m-1;2m-2;\frac{x^2-s^2}{x^2}\right)f(s)\,ds$$

$$=\frac{1}{\Gamma(2(m-1))}\int\limits_0^x\left(\frac{s}{x}\right)^{2\gamma}\left(\frac{x^2-s^2}{2x}\right)^{2(m-1)-1}{}_2F_1\left(m-1+\gamma-\frac{1}{2},m-1;2(m-1);\frac{x^2-s^2}{x^2}\right)f(s)\,ds$$

$$=\int\limits_0^x K_{m-1,0}(x,s)\,f(s)\,ds.$$

Now compute L_2:

$$L_2=2\sum_{p=1}^{\infty}\frac{(-\lambda^2x^2)^p\,(m)_p}{p!}\int_0^x x^{2m-1}\left(\frac{s}{x}\right)^{2\gamma}\left(\frac{d^2}{ds^2}+\frac{2\gamma}{s}\frac{d}{ds}\right)\frac{\left(\frac{x^2-s^2}{x^2}\right)^{2m+2p-1}}{2^{2m+2p}\Gamma(2m+2p)}$$

$$\times{}_2F_1\left(m+\gamma-\frac{1}{2}+p,\ m+p;\ 2m+2p;\ \frac{x^2-s^2}{x^2}\right)f(s)\,ds$$

$$=2\sum_{p=1}^{\infty}\frac{(-\lambda^2x^2)^p\,(m)_p}{p!}\int_0^x x^{2m-3}\left(\frac{s}{x}\right)^{2\gamma}\frac{\left(\frac{x^2-s^2}{x^2}\right)^{2m+2p-3}}{2^{2m+2p-2}\Gamma(2m+2p-2)}$$

$$\times{}_2F_1\left(m+\gamma-\frac{3}{2}+p,\ m+p-1;\ 2m+2p-2;\ \frac{x^2-s^2}{x^2}\right)f(s)\,ds$$

$$=\int_0^x\left(\frac{s}{x}\right)^{2\gamma}\left(\frac{x^2-s^2}{2x}\right)^{2m-3}\sum_{p=1}^{\infty}\frac{\left(-\frac{\lambda^2}{4}x^2\left(\frac{x^2-s^2}{x^2}\right)^2\right)^p(m)_p}{\Gamma(2m+2p-2)\,p!}$$

$$\times{}_2F_1\left(m+\gamma-\frac{3}{2}+p,\ m+p-1;\ 2m+2p-2;\ \frac{x^2-s^2}{x^2}\right)f(s)\,ds.$$

Using the identity

$$(m)_p=(m-1)_p+p\,(m)_{p-1},$$

we split L_2 into two parts:

$$L_2 = \int_0^x \left(\frac{s}{x}\right)^{2\gamma} \left(\frac{x^2 - s^2}{2x}\right)^{2m-3} \sum_{p=1}^{\infty} \frac{\left(-\frac{\lambda^2}{4} x^2 \left(\frac{x^2-s^2}{x^2}\right)^2\right)^p (m-1)_p}{\Gamma(2m+2p-2)\, p!}$$
$$\times {}_2F_1\left(m + \gamma - \frac{3}{2} + p, m + p - 1; 2m + 2p - 2; \frac{x^2 - s^2}{x^2}\right) f(s)\, ds$$
$$+ \int_0^x \left(\frac{s}{x}\right)^{2\gamma} \left(\frac{x^2 - s^2}{2x}\right)^{2m-3} \sum_{p=1}^{\infty} \frac{\left(-\frac{\lambda^2}{4} x^2 \left(\frac{x^2-s^2}{x^2}\right)^2\right)^p (m)_{p-1}}{\Gamma(2m+2p-2)\,(p-1)!}$$
$$\times {}_2F_1\left(m + \gamma - \frac{3}{2} + p, m + p - 1; 2m + 2p - 2; \frac{x^2 - s^2}{x^2}\right) f(s)\, ds.$$

The first part corresponds to replacing m by $m-1$; in the second part we shift the summation index $p \to p+1$. This yields

$$L_2 = \int_0^x K_{m-1,p\geq 1}(x,s)\, f(s)\, ds - \lambda^2 \int_0^x K_{m-1,p}(x,s)\, f(s)\, ds.$$

Substituting the obtained expressions into (4.12) we finally get

$$B^{-m}_{\gamma,\lambda,0+} B^*_{\gamma,\lambda} f(x) = \int_0^x K_{m-1,p}(x,s)\, f(s)\, ds.$$

The last formula, together with Lemmas 4.1 and 4.3, proves equality (4.9)
This completes the proof.

5. Conclusion

In this study, a unified analytical approach for higher-order ordinary differential equations involving the generalized Bessel differential operator with a spectral parameter has been developed. By employing the transmutation operator method together with the analytical properties of the Gauss and Humbert hypergeometric functions, an explicit integral representation of the solution to the corresponding Cauchy problem has been derived. The obtained integral representation has been rigorously verified to satisfy both the differential equation and the prescribed initial conditions, and the uniqueness of the solution has been established. These results provide a solid mathematical foundation for the analysis of singular differential equations associated with generalized Bessel operators. Based on the derived solution, a new integer-order generalized Bessel integral operator has been introduced. In contrast to the classical Erdélyi–Kober and Bessel-type integral operators, the proposed operator combines a higher-order generalized Bessel differential structure with a spectral parameter through a kernel represented by the Humbert hypergeometric function. Furthermore, its fundamental properties, including the inverse operator, left- and right-inverse identities, semigroup property, limiting behavior, and integration-by-parts formulas, have been rigorously established. The obtained results extend the existing theory of Bessel and Erdélyi–Kober integral operators and provide a theoretical basis for further investigations of generalized transmutation operators and fractional generalized Bessel integral operators. Future work will focus on the construction of fractional-order generalized Bessel integral operators, the investigation of their mapping properties in functional spaces, the derivation of composition and factorization formulas, and their applications to boundary value problems, inverse problems, and singular differential equations arising in mathematical physics.

Acknowledgements

The authors have no acknowledgements to declare.

Funding

The authors received no financial support for this research.

Availability of data and materials

No datasets were generated or analyzed during the current study.

Competing interests

The authors declare that they have no competing interests.

Author’s contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

References

[1] Sitnik, S., Karimov, Sh., & Boynazarov, A. (2025). Solving the Cauchy problem for an ordinary differential equation with an integer power of the Bessel operator using transmutation operators. *Journal of Mathematical Analysis and Applications*. Received: 30 July 2025; Accepted: 7 October 2025. https://doi.org/10.1007/s10958-025-xxxx-x

[2] Samko, S. G., Kilbas, A. A., & Marichev, O. I. (1987). *Fractional integrals and derivatives: Theory and applications*. Minsk: Nauka i Tekhnika. 688 p.

[3] Urinov, A. K., Sitnik, S. M., Shishkina, E. L., & Karimov, Sh. T. (2022). *Fractional integrals and derivatives (Generalizations and applications)*. [In Russian]. Fergana: Publishing House “Farg’ona”. 192 p.

[4] Urinov, A. K., & Karimov, Sh. T. (2021). *Erdélyi–Kober operators and applications to partial differential equations*. Fergana: Publishing House “Farg’ona”. 202 p.

[5] Sprinkhuizen-Kuyper, I. G. (1979). A fractional integral operator corresponding to negative powers of a certain second-order differential operator. *Journal of Mathematical Analysis and Applications*, 72(2), 674–702. https://doi.org/10.1016/0022-247X(79)90278-9

[6] Lowndes, J. S. (1970). A generalization of the Erdélyi–Kober operators. *Proceedings of the Edinburgh Mathematical Society*, 17(2), 139–148. https://doi.org/10.1017/S001309150000856X

[7] Kiryakova, V. (1986). Applications of the generalized Poisson transformation for solving hyper-Bessel differential equations. *Godishnik VUZ. Applied Mathematics*, 22(4), 129–140.

[8] Kiryakova, V. (1994). *Generalized fractional calculus and applications*. Harlow: Longman Scientific & Technical. 360 p.

[9] Kiryakova, V. (2008). Transmutation method for solving hyper-Bessel differential equations based on the Poisson–Dimovski transformation. *Fractional Calculus and Applied Analysis*, 11(3), 299–316.

[10] Kiryakova, V. S. (1994). *Generalized fractional calculus and applications*. New York: John Wiley & Sons. 360 p.

[11] Urinov, A. K. (2012). *Special functions and special operators*. [In Uzbek]. Fergana: “Farg’ona” nashriyoti. 112 p.

[12] Lebedev, N. N. (1972). *Special functions and their applications*. New York: Dover Publications. 358 p.

[13] Sitnik, S. M., & Skoromnik, O. V. (2020). Transmutation operators that generalize the Sonine and Poisson transforms. *Journal of Mathematical Sciences*, 249(4), 543–558. https://doi.org/10.1007/s10958-020-04987-2

[14] Luchko, Y., & Gorenflo, R. (1998). Fractional derivatives of Erdélyi–Kober type and some of their applications. *Fractional Calculus and Applied Analysis*, 1(1), 63–78.

[15] Kiryakova, V. (2021). The interplay of generalized fractional calculus and integral transforms. *Mathematics*, 9(15), 1789. https://doi.org/10.3390/math9151789

[16] Karimov, Sh. T. (2024). On a method for solving the Cauchy problem for a high-order equation with a Bessel operator. *Izvestiya Vysshikh Uchebnykh Zavedeniy. Matematika*.

[17] Hasanov, A. S., & Ergashev, T. G. (2023). New properties of the Humbert functions and their applications. *Integral Transforms and Special Functions*, 34(5), 412–428. https://doi.org/10.1080/10652469.2022.2156789
[18] Shishkina, E., & Sitnik, S. M. (2017). On fractional powers of the Bessel operator on a semiaxis. *arXiv preprint* arXiv:1706.01928. https://doi.org/10.48550/arXiv.1706.01928
[19] Sneddon, I. N. (1972). *The use of integral transforms*. New York: McGraw-Hill. 540 p.
[20] Bateman, H., & Erdélyi, A. (1954). *Tables of integral transforms* (Vol. 1). New York: McGraw-Hill. 391 p.
[21] Erdélyi, A., Magnus, W., Oberhettinger, F., & Tricomi, F. G. (1953). *Higher transcendental functions* (Vol. I). New York: McGraw-Hill. 302 p.
[22] Comtet, L. (1974). *Advanced Combinatorics: The Art of Finite and Infinite Expansions*. D. Reidel Publishing Company, Dordrecht.
[23] Sh. T. Karimov, "The Cauchy Problem for the Iterated Klein–Gordon Equation with the Bessel Operator," *Lobachevskii Journal of Mathematics*, vol. 41, no. 5, pp. 772–784, 2020.
[24] Sh. T. Karimov and Sh. A. Oripov, "On a Method for Constructing the Riemann Function for Partial Differential Equations with a Singular Bessel Operator," *Lobachevskii Journal of Mathematics*, vol. 41, no. 6, pp. 1087–1093, 2020.
[25] Sh. T. Karimov and E. L. Shishkina, "Some methods of solution to the Cauchy problem for a inhomogeneous equation of hyperbolic type with a Bessel operator," *Journal of Physics: Conference Series*, vol. 1203, no. 1, art. no. 012096, 2019.
[26] Sh. T. Karimov, "On One Method for the Solution of an Analog of the Cauchy Problem for a Polycaloric Equation with Singular Bessel Operator," *Ukrainian Mathematical Journal*, vol. 69, no. 10, pp. 1593–1606, 2018.
[27] M. S. Salokhiddinov and G. N. Nasriddinov, *Oddiy differensial tenglamalar* [Ordinary Differential Equations], Tashkent: Ozbekiston, 1994 (in Uzbek).
[28] Dunaev A.S., Shlychkov V.I. Gipergeometricheskie Funktsii (Hypergeometric Functions).*Educational Electronic Text Resource.* Yekaterinburg, 2017.

Affiliations

Karimov Shakhobiddin Tuychiboyevich
Address: Fergana State University. Fergana, Uzbekistan
e-mail: shaxkarimov@gmail.com
ORCID ID: https://orcid.org/0000-0001-8325-998X

Boynazarov Akhrorjon Numonjonovich
Address: Fergana State University. Fergana, Uzbekistan
e-mail: ahror010185@gmail.com
ORCID ID: https://orcid.org/0009-0002-7422-4368