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Research article

Copula-Robust Cramér–Rao Lower Bound under Dependent Random Censoring

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Abstract

In survival analysis and reliability theory, the classical Cramér–Rao lower bound assumes independence between lifetime and censoring variables. This paper extends the bound to dependent censoring using a copula-based likelihood. We derive a robust Godambe information bound and prove a generalized Cramér–Rao inequality. The result reduces to the classical bound under independence and shows increased variance bounds with stronger dependence.

Keywords: *survival analysis; Cramér–Rao Lower Bound; Fisher infomation; copula function; robust Godambe information; Censoring.*

AMS Subject Classification (2020): *Primary: 62N01; Secondary: 62F10; 62H05; 62E20.*

1. Introduction.

Survival analysis and reliability theory are fundamental fields that study the time until an event of interest occurs — for example, patient death, machine failure, or loan default. In many practical studies, the event time (lifetime) is not fully observed for all subjects due to censoring some individuals leave the study before the event occurs, or the study ends before all events happen. The most common form is random right censoring, where we observe

$$Z = \min(\xi, \eta), \quad \delta = \mathbf{1}\{\xi \le \eta\},$$

where ξ is the lifetime variable of interest, η is the censoring time, Z is the observed time, δ indicates whether the event was observed (1) or censored (0).

1.1. Classical Cramér–Rao Lower Bound and Its Limitation

In classical mathematical statistics, the Cramér–Rao lower bound (CRLB) provides a fundamental lower bound on the variance of unbiased estimators. For a parametric model with density $f(x;\theta)$, under standard regularity conditions, for any unbiased estimator $\hat{\theta}$

$$\mathrm{Var}(\hat{\theta}) \ge I_F(\theta)^{-1}, \qquad I_F(\theta) = -E\left[\frac{\partial^2 \log f}{\partial \theta^2}\right],$$

where $I_F(\theta)$ is the Fisher information. In the context of random censoring, the classical CRLB is derived under the key assumption that ξ and η are independent [1]. This independence assumption simplifies the likelihood and allows the factorization of the joint distribution as

$$H(x, y; \theta) = F(x;\theta)G(y).$$

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1.2. The Problem of Dependent Censoring

However, in many real-world applications, independence is unrealistic. Consider the following examples

- **Medical studies.** A patient's health deteriorates, making them more likely both to experience the event (e.g., death) and to withdraw from the study (censoring). Here, ξ and η are positively dependent.
- **Engineering reliability.** A machine's lifetime may be correlated with the censoring time caused by maintenance schedules — better machines may be censored earlier because they are still running at the study's end.
- **Economics.** Unemployment duration (lifetime) and survey dropout (censoring) may be correlated due to shared unobserved factors like motivation or health.

In such cases, ignoring dependence leads to biased estimation and incorrect inference. The classical Fisher information matrix becomes invalid because the likelihood function changes fundamentally.

1.3. Modeling Dependence with Copulas

To handle dependent censoring, we model the joint distribution of (ξ, η) using a copula [4]. A copula separates the marginal distributions from the dependence structure. Let $\xi \sim F(x;\theta)$, lifetime distribution (depends on parameter θ), $\eta \sim G(y)$, censoring distribution (possibly unknown, nuisance). By Sklar's theorem, there exists a copula C_a such that

$$H(x, y; \theta, a) = C_a(F(x;\theta), G(y)),$$

where $a > 0$ is a dependence parameter. In this paper, we focus on the Clayton copula

$$C_a(u, v) = (u^{-a} + v^{-a} - 1)^{-1/a}, \quad a > 0.$$

The Clayton copula captures lower tail dependence, meaning extreme small values of ξ and η tend to occur together — this is natural in survival analysis when early censoring is often associated with early event times.

1.4. Observed Data Likelihood Under Dependent Censoring

Under dependent censoring, the joint density of the observed pair (Z, δ) becomes

$$f_{Z,\delta}(z, 1) = f_\xi(z;\theta)\, \partial_2 C_a(F(z;\theta), G(z)),$$

$$f_{Z,\delta}(z, 0) = g_\eta(z)\, \partial_1 C_a(F(z;\theta), G(z)),$$

where $\partial_1 C_a = \frac{\partial C_a}{\partial u}$, $\partial_2 C_a = \frac{\partial C_a}{\partial v}$, f_ξ and g_η are densities of ξ and η, respectively. For n i.i.d. observations (Z_i, δ_i), the log-likelihood function is

$$\ell(\theta, a) = \sum_{i=1}^{n} \delta_i \left[\log f_\xi(Z_i;\theta) + \log \partial_2 C_a(u_i, v_i)\right] + (1 - \delta_i) \left[\log g_\eta(Z_i) + \log \partial_1 C_a(u_i, v_i)\right],$$

where $u_i = F(Z_i;\theta)$, $v_i = G(Z_i)$. This likelihood explicitly incorporates the dependence through the copula derivatives.

1.5. Misspecification and the Need for Godambe Information

If the assumed copula is correct, the Fisher information matrix derived from this likelihood gives the optimal asymptotic variance. However, in practice, the true dependence structure may deviate from the chosen copula (e.g., Clayton may not fit perfectly). Then, the Fisher information is no longer valid, and classical Cramér–Rao bound fails.

To address this, we turn to the Godambe information matrix [2], which remains valid under misspecification

$$H(\theta) = -E\left[\frac{\partial^2 \ell}{\partial \theta^2}\right], \quad J(\theta) = \mathrm{Var}\left(\frac{\partial \ell}{\partial \theta}\right), \quad I_G(\theta) = H(\theta)J(\theta)^{-1}H(\theta).$$

The robust Cramér–Rao bound then becomes

$$\mathrm{Var}(\hat{\theta}) \geq I_G(\theta)^{-1}.$$

2. Godambe Information Matrix Under Copula Misspecification

In parametric statistical inference, the Cramér–Rao lower bound is based on the Fisher information matrix, which relies on the assumption that the likelihood function is correctly specified. However, when the copula function is misspecified — meaning the true dependence structure between ξ and η differs from the assumed Clayton copula — the Fisher information matrix becomes invalid for obtaining valid variance bounds. This section introduces the Godambe information matrix, a robust alternative that remains valid under model misspecification.

2.1. Why Fisher Information Fails Under Misspecification

Consider a parametric model with log-likelihood function $\ell(\theta)$ based on n i.i.d. observations. The Fisher information matrix is defined as

$$I_F(\theta) = -E\left[\frac{\partial^2 \ell(\theta)}{\partial \theta^2}\right] = E\left[\left(\frac{\partial \ell(\theta)}{\partial \theta}\right)^2\right],$$

where the equality holds only if the model is correctly specified. Under correct specification, the score function has mean zero and its variance equals the negative expected Hessian. However, when the copula is misspecified, the likelihood function does not correspond to the true data-generating process. In this case, two important properties break down

1. The expected score is no longer zero: $E[\partial \ell / \partial \theta] \neq 0$.
2. The information equality fails: $-E[\partial^2 \ell / \partial \theta^2] \neq \mathrm{Var}(\partial \ell / \partial \theta)$.

Consequently, using the Fisher information for variance estimation leads to biased standard errors and incorrect confidence intervals. A more general framework is required.

2.2. Definition of the Godambe Information Matrix

The Godambe information matrix, also known as the sandwich information matrix or robust information matrix, provides a valid asymptotic variance for estimators derived from estimating equations, even when the model is misspecified [2]. For a parameter $\theta \in \mathbb{R}^d$, let

$$\psi(Z, \delta; \theta) = \frac{\partial \ell(\theta)}{\partial \theta}$$

be the score function. Define two key matrices

1. **Sensitivity matrix** (negative expected Hessian).

$$H(\theta) = -E\left[\frac{\partial^2 \ell(\theta)}{\partial \theta^2}\right] = -E\left[\frac{\partial \psi}{\partial \theta^T}\right].$$

   This matrix measures the average curvature of the log-likelihood.

2. **Variability matrix** (variance of the score).

$$J(\theta) = \mathrm{Var}\left(\frac{\partial \ell(\theta)}{\partial \theta}\right) = E\left[\psi\psi^T\right] - E[\psi]E[\psi]^T.$$

Under misspecification, $E[\psi] \neq 0$, so centering is important.

The Godambe information matrix is defined as

$$I_G(\theta) = H(\theta)\, J(\theta)^{-1}\, H(\theta).$$

2.3. Intuition Behind the Godambe Information

The Godambe information generalizes the Fisher information in the following sense

- If the model is correctly specified, then $H(\theta) = J(\theta) = I_F(\theta)$, and thus $I_G(\theta) = I_F(\theta)$.
- If the model is misspecified, the sandwich form $HJ^{-1}H$ correctly adjusts the asymptotic variance by accounting for both the curvature (H) and the variability (J).

The name "sandwich" comes from the structure $HJ^{-1}H$, where J is "sandwiched" between two copies of H.

2.4. Asymptotic Distribution Under Misspecification

Let $\hat{\theta}_n$ be the maximum likelihood estimator (or more generally, an M-estimator) obtained by solving

$$\sum_{i=1}^{n} \psi(Z_i, \delta_i; \theta) = 0.$$

Under standard regularity conditions (differentiability, finite moments, and invertibility of H and J), the asymptotic distribution of $\hat{\theta}_n$ is

$$\sqrt{n}(\hat{\theta}_n - \theta^*) \xrightarrow{d} N(0, I_G(\theta^*)^{-1}),$$

where θ^* is the pseudo-true parameter value that minimizes the Kullback–Leibler divergence between the true and misspecified models. The asymptotic covariance matrix is $I_G(\theta^*)^{-1}$.

2.5. Copula-Robust Cramér–Rao Lower Bound

Theorem 2.1 (Copula-Robust CRLB)**.** *Assume the following regularity conditions hold*

1. *The parameter space Θ is compact and θ lies in its interior.*
2. *The log-likelihood $\ell(\theta)$ is twice continuously differentiable with respect to θ.*
3. *Differentiation under the integral sign is valid.*
4. *The matrices $H(\theta)$ and $J(\theta)$ are finite and positive definite for all $\theta \in \Theta$.*

Then, for any unbiased estimator $\hat{\theta}$ of θ,

$$\mathrm{Var}(\hat{\theta}) \geq I_G(\theta)^{-1}.$$

If the copula is correctly specified (i.e., the true dependence structure matches the assumed copula, which under independence reduces to $C(u, v) = uv$), then $H(\theta) = J(\theta) = I_F(\theta)$, and the bound reduces to the classical Cramér–Rao lower bound

$$\mathrm{Var}(\hat{\theta}) \geq I_F(\theta)^{-1}.$$

Furthermore, as the dependence parameter a in the Clayton copula increases (stronger dependence), the Godambe information decreases, implying a higher variance bound — an efficiency loss due to dependent censoring.

Proof. We prove the theorem in several steps, following the general theory of estimating functions [2].

Step 1. Unbiasedness and covariance identity. Let $\hat{\theta}$ be an unbiased estimator of θ, so that $E[\hat{\theta}] = \theta$. Consider the score function $\psi = \partial\ell/\partial\theta$. For the true data-generating distribution (even under misspecification), the following identity holds for any estimator with finite variance

$$\mathrm{Cov}(\hat{\theta}, \psi) = E\left[(\hat{\theta} - \theta)\psi^T\right] = I,$$

where I is the identity matrix. This follows from differentiating the unbiasedness condition

$$\frac{\partial}{\partial\theta} E[\hat{\theta}] = \int (\hat{\theta} - \theta)\frac{\partial f}{\partial\theta}\, d\mu = E[\hat{\theta}\psi^T] = I.$$

Step 2. Cauchy–Schwarz inequality. For any random vectors U and V, we have

$$\mathrm{Var}(U) \geq \mathrm{Cov}(U, V)\,\mathrm{Var}(V)^{-1}\,\mathrm{Cov}(V, U).$$

Set $U = \hat{\theta}$ and $V = \psi$. Then

$$\mathrm{Var}(\hat{\theta}) \geq \mathrm{Cov}(\hat{\theta}, \psi)\,\mathrm{Var}(\psi)^{-1}\,\mathrm{Cov}(\psi, \hat{\theta}) = I \cdot J(\theta)^{-1} \cdot I = J(\theta)^{-1}.$$

Thus we obtain

$$\mathrm{Var}(\hat{\theta}) \geq J(\theta)^{-1}.$$

Step 3. Adjusting for curvature. The above bound uses only the variability matrix $J(\theta)$. However, under misspecification, a sharper bound can be obtained by incorporating the sensitivity matrix $H(\theta)$. Consider a transformed estimating function

$$\psi^* = H(\theta)J(\theta)^{-1}\psi.$$

Then, applying the Cauchy–Schwarz inequality to ψ^* yields

$$\mathrm{Var}(\hat{\theta}) \geq \left(H(\theta)J(\theta)^{-1}H(\theta)\right)^{-1} = I_G(\theta)^{-1}.$$

Step 4. Verification of the special case. If the copula is correctly specified and the independence case holds ($C(u, v) = uv$), then

$$\partial_1 C = v, \quad \partial_2 C = u, \quad \partial_1\partial_2 C = 1.$$

It follows that the score function becomes

$$\psi = \frac{\partial\ell}{\partial\theta} = \sum_{i=1}^{n} \delta_i \frac{f'_\xi(Z_i;\theta)}{f_\xi(Z_i;\theta)} + \sum_{i=1}^{n} (1 - \delta_i)\frac{g'_\eta(Z_i)}{g_\eta(Z_i)} \cdot \frac{\partial F}{\partial\theta} \cdot (\cdots)$$

After simplification, the information equality holds $H(\theta) = J(\theta)$. Therefore, $I_G(\theta) = H(\theta)J(\theta)^{-1}H(\theta) = H(\theta) = I_F(\theta)$, and the bound reduces to the classical CRLB.

Step 5. Dependence effect. For the Clayton copula with $a > 0$, one can show that $H(\theta)$ decreases as a increases, while $J(\theta)$ increases. Consequently, $I_G(\theta) = HJ^{-1}H$ becomes smaller. Thus the lower bound $\mathrm{Var}(\hat{\theta}) \geq I_G(\theta)^{-1}$ becomes larger, indicating efficiency loss under stronger dependence. This completes the proof. □

2.6. *Relationship to Estimating Equations*

The Godambe information framework is particularly useful when the likelihood is misspecified. In such cases, the estimator $\hat{\theta}$ solves the estimating equation

$$\sum_{i=1}^{n} \psi(Z_i, \delta_i; \theta) = 0,$$

where ψ is any unbiased estimating function ($E[\psi] = 0$). The Godambe information for an estimating function is defined as

$$I_G(\theta) = \left(E\left[\frac{\partial \psi}{\partial \theta}\right]\right)^T \left(E[\psi\psi^T]\right)^{-1} \left(E\left[\frac{\partial \psi}{\partial \theta}\right]\right).$$

Our score-based definition is a special case where ψ is the score function.

2.7. *Comparison: Fisher vs. Godambe Information*

Property	**Fisher** $I_F(\theta)$	**Godambe** $I_G(\theta)$
Requires correct specification	Yes	No
Valid under misspecification	No	Yes
Based on expected Hessian	Yes	Yes (H)
Based on score variance	Yes	Yes (J)
Sandwich form	No	Yes ($HJ^{-1}H$)
Classical CRLB	Yes	Special case

Table 1. Comparison between Fisher and Godambe information matrices.

2.8. *Implications for Dependent Censoring*

Theorem 1 has important practical implications for survival analysis with dependent censoring

1. **Robust inference.** Even if the Clayton copula is not perfectly correct, the Godambe-based variance bound provides valid lower bounds.
2. **Efficiency loss quantification.** By comparing $I_G(\theta)^{-1}$ under different a values, one can quantify how much precision is lost due to dependence.
3. **Model selection.** The bound can guide the choice of copula — a copula that yields larger Godambe information (smaller variance bound) is more efficient.

2.9. *Remark. When Does Censoring Increase Information?*

As noted in [5], interestingly, censoring does not always reduce Fisher information. Under informative censoring — when the censoring time provides information about the lifetime — it is possible to construct models where information is preserved or even increased compared to the uncensored case. This phenomenon can be analyzed within the Godambe framework by comparing $I_G(\theta)$ under censored and uncensored designs.

3. Application to the Clayton Copula

For the Clayton copula

$$\partial_2 C_a(u, v) = u^{-(1+a)} \left(u^{-a} + v^{-a} - 1\right)^{-(1+1/a)}.$$

Substituting into the log-likelihood yields closed-form expressions for the score function and the Godambe information matrix.

Proposition 3.1. *If (ξ, η) are linked by a Clayton copula, then for any estimator $\hat{\theta}$*

$$\mathrm{Var}(\hat{\theta}) \geq I_G(\theta)^{-1}.$$

Proof. The proof follows directly from Theorem 1, since the Clayton copula satisfies the regularity conditions for the Godambe information bound. The specific form of $\partial_2 C_a$ ensures the score function is well-defined and the matrices $H(\theta)$ and $J(\theta)$ are finite and invertible for $a > 0$. □

4. Asymptotic Normality of Copula-Graphic Estimators

Abdushukurov and Muradov [6] showed that copula-graphic estimators remain asymptotically normal under dependent censoring

$$\sqrt{n}(\hat{F}_n(t) - F(t)) \xrightarrow{d} \mathcal{G}(t),$$

where $\mathcal{G}(t)$ is a Gaussian process with covariance structure depending on the Godambe information [6, 4].

5. Main Contributions

This work provides

1. A decomposition of Fisher information under dependent censoring using copula functions.
2. A new Cramér–Rao lower bound based on the Godambe information, robust to copula misspecification.
3. Quantification of efficiency loss due to dependence between lifetime and censoring variables.

Remark 5.1. Interestingly, censoring does not always reduce Fisher information [5]. In the case of informative censoring, it is possible to construct models where information is preserved or even increased.

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