Uzbekistan Journal of Mathematics and Computer Science
Volume-2, Issue-3-special, Page 15–23 (2026)
DOI: https://doi.org/10.56143/ujmcs.v2i3s.2
Research article

Asymptotic Analysis of a Singularly Perturbed Boundary Value Problem with an Interior Turning Point

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Abstract

This paper investigates a two-point boundary value problem for a singularly perturbed linear inhomogeneous second-order ordinary differential equation with turning point inside the interval of the form $\varepsilon y''(x) + (x - x_0)y'(x) - (x - x_0)y(x) = 0,\ 0 < x < 1,\ x_0 \in (0, 1)$, subject to the boundary conditions $y(0) = a,\ y(1) = b$. Problems of this type arise in various areas of science and engineering, including physics, biology, economics, and applied mathematics. The considered problem possesses two distinctive features: it belongs to the class of singularly perturbed differential equations, and the coefficient of the reduced equation vanishes at an interior point of the interval. The combination of singular perturbation and an interior turning point makes the problem bisingular and leads to a more complicated solution structure. Unlike classical singularly perturbed boundary value problems, boundary layers do not occur near the endpoints of the interval. Instead, a layer is formed only in a neighborhood of the interior singular point $x = x_0$. Consequently, the solution cannot be represented by a single asymptotic expression on the entire interval. To overcome this difficulty, separate asymptotic representations are constructed on the subintervals $[0, x_0]$ and $[x_0, 1]$, yielding a composite solution. Each local representation consists of the sum of a regular outer solution and an inner solution describing the layer behavior near the singular point. The main objective of the paper is to construct a uniformly valid asymptotic expansion of the solution on the interval ([0,1]). A comparison of the constructed asymptotic expansion with A.Nayfeh's result demonstrates their close agreement and confirms the validity of the proposed asymptotic construction.

Keywords: *singular perturbation, interior turning point, interior layer, matched asymptotic expansions, boundary function method, uniform asymptotic expansion.*

AMS Subject Classification (2020): *Primary: 34E05; Secondary: 34E15, 34E10.*

1. Introduction

The theory of singularly perturbed differential equations constitutes one of the most important branches of modern applied mathematics and mathematical physics. Differential equations containing a small parameter multiplying the highest-order derivative arise in the mathematical modeling of a wide variety of processes characterized by the coexistence of fast and slow dynamics. Typical examples include boundary-layer phenomena in fluid mechanics, contrast structures in chemical kinetics, transient processes in electrical circuits, and systems exhibiting localized internal transitions. In such problems, the solution may experience

Received : *28–april–2026,* ***Accepted :*** *31–july–2026*
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rapid variations within a narrow region of the independent variable, giving rise to boundary layers near the endpoints of the domain or internal layers in the vicinity of interior points [1]-[6]. These localized structures significantly complicate both the qualitative analysis of solutions and the construction of accurate asymptotic approximations.

Of particular interest are singularly perturbed boundary value problems for second-order ordinary differential equations of the form

$$\varepsilon y'' = f(x, y, y', \varepsilon), \qquad x \in (a, b), \qquad \varepsilon \to 0,$$

subject to boundary conditions prescribed at the endpoints of the interval. A special class of such problems arises when the reduced equation possesses a turning point, that is, an interior point at which the coefficient of the reduced first-order equation vanishes. In the neighborhood of a turning point, the qualitative behavior of the solution may change significantly, and an interior layer can be formed. Within this narrow region, the solution exhibits rapid variation, while remaining slowly varying outside the layer [2, 3], [7]-[10].

The asymptotic behavior of solutions to singularly perturbed problems with turning points has been extensively investigated by A.B. Vasilieva, V.F. Butuzov, S.A. Lomov, and many other authors. The classical boundary function method provides a powerful tool for constructing uniformly valid asymptotic approximations by combining regular and rapidly varying components of the solution [1]–[6]. However, the presence of an interior layer generated by a turning point requires a separate asymptotic analysis in different subdomains and the subsequent matching of the corresponding expansions. This leads naturally to the use of asymptotic matching techniques and composite expansions.

In the present paper, we study a two-point boundary value problem for a singularly perturbed linear nonhomogeneous second-order ordinary differential equation whose reduced equation has a turning point located inside the interval under consideration. The interaction between the singular perturbation and the interior turning point gives rise to a nonstandard asymptotic structure and prevents the construction of a single asymptotic representation valid throughout the entire interval.

The main objective of this work is to construct a complete uniformly valid asymptotic expansion of the solution that accurately captures the interior layer generated by the turning point. To achieve this goal, we employ a combination of the generalized boundary function method and the method of matched asymptotic expansions. This approach yields asymptotic representations on both sides of the turning point and allows the construction of a composite approximation valid throughout the entire interval.

The principal results of the paper are as follows:

- conditions for the formation of an interior layer are established;
- the absence of boundary layers at the endpoints of the interval is demonstrated;
- the outer and inner components of the asymptotic expansion are constructed, including the interior-layer function in a neighborhood of the turning point $x = x_0$;
- the uniform validity of the resulting composite asymptotic expansion is rigorously justified.

The obtained results extend the asymptotic theory of singularly perturbed boundary value problems with interior turning points. They may be applied in the mathematical modeling of processes exhibiting sharp internal transitions and can serve as a basis for the development and validation of numerical methods for singularly perturbed differential equations.

2. Problem Statement

The objective is to construct a uniformly valid asymptotic expansion of the solution to problem (2.1) with arbitrary accuracy as $\varepsilon \to 0$:

$$\varepsilon y''(x) + (x - x_0)y'(x) - (x - x_0)y(x) = 0, \quad x \in (0,1), \quad y(0) = a, \quad y(1) = b, \tag{2.1}$$

where $0 < \varepsilon$ is a small parameter, $x_0 \in (0,1)$, a and b are given constant numbers.

Problem (2.1) in the particular case $x_0 = 1/2$ was discussed in Nayfeh's monograph [3]. A formal asymptotic approximation was obtained using a matching procedure; however, the construction was not accompanied by a rigorous justification. In the present work, we derive a complete asymptotic expansion and establish its uniform validity by means of the generalized boundary function method.

The main goal of this paper is to demonstrate the efficiency and quality of the proposed generalized boundary value function method.

The problem possesses two essential features. First, the small parameter multiplies the highest-order derivative, which makes the problem singularly perturbed in the sense of Tikhonov. Second, the coefficient of the reduced equation vanishes at the interior point $x = x_0$, creating an interior turning point. The interaction of these two mechanisms produces an interior layer and leads to a nonstandard asymptotic structure of the solution.

This discrepancy leads to the solution of the complete problem undergoing abrupt changes near the boundaries or within the domain, forming so-called boundary layers. Depending on the structure of the degenerate equation and the type of boundary conditions, such layers can arise either at the boundaries of the interval or at its interior points. In the latter case, we speak of an interior transition layer (or contrast structure).

To construct an asymptotic solution to singularly perturbed problems with boundary layers, boundary function methods (A.B. Vasil'eva, V.F. Butuzov) and the matching of asymptotic expansions (A.M. Ilyin, Van Dyke) are traditionally used. When the transition point is located within the domain, additional coordination of the expansions constructed to the left and right of it is required, which significantly complicates the problem.

When solving problem (2.1), we use methods for integrating ordinary differential equations with additional conditions, as well as asymptotic methods (the small parameter method, the generalized boundary value method, and the splicing method). The main idea, i.e., the research methodology, is to construct regular external solutions and splice them with internal solutions [7]-[10].

3. Problem solution

In equation (2.1), for $\varepsilon = 0$, we obtain a degenerate equation

$$(x - x_0)\tilde{y}'(x) - (x - x_0)\tilde{y}(x) = 0, \quad x \in (0,1), \tag{3.1}$$

For the resulting equation, the point $x = x_0$ is a singular point.

Therefore, the zeroth approximation of the external solution to problem (2.1) can be represented as [?]:

$$\tilde{y}(x) = \begin{cases} \tilde{y}_0^L(x), & x \in [0, x_0), \\ \tilde{y}_0^R(x), & x \in (x_0, 1], \end{cases}$$

where $\tilde{y}_0^L(x) = ae^x$, $\tilde{y}_0^R(x) = be^{x-1}$, *L-left*, *R-right*.

We begin by constructing the outer expansion. We will seek the outer solution in the form of the following series:

$$y^{\text{out}}(x) = \begin{cases} u^L(x), & 0 \le x < x_0, \\ u^R(x), & x_0 < x \le 1, \end{cases} \tag{3.2}$$

where $u^L(x) = \sum\limits_{k=0}^{\infty} \varepsilon^k u_k^L(x)$, $\quad u^R(x) = \sum\limits_{k=0}^{\infty} \varepsilon^k u_k^R(x)$, $\quad u_k^L(x)$, $\quad u_k^R(x)$ are functions are still unknown.

Substituting the series (3.2) into problem (2.1) and applying the small parameter method, we have:

$$(x - x_0)u_0'(x) - (x - x_0)u_0(x) = 0, \quad u_0(0) = a, \quad u_0(1) = b, \tag{3.3}$$

$$(x - x_0)u_k'(x) - (x - x_0)u_k(x) = -u_{k-1}''(x), \quad u_k(0) = 0, \quad u_k(1) = 0, \quad k \in \mathbb{N}. \tag{3.4}$$

We write the solutions to problems (3.3) and (3.4) in the form:

$$u_0(x) = \begin{cases} u_0^L(x), & x \in [0, x_0), \\ u_0^R(x), & x \in (x_0, 1], \end{cases}$$

$$u_k(x) = \begin{cases} u_k^L(x), & x \in [0, x_0), \\ u_k^R(x), & x \in (x_0, 1], \end{cases}$$

where $u_0^L(x) = ae^x$, $u_0^R(x) = be^{x-1}$,

$$u_1^L(x) = -ae^x \int_0^x \frac{1}{\tau - x_0} d\tau = -ae^x \ln|x - x_0|, \quad u_1^R(x) = -be^{x-1} \ln|x - x_0|,$$

$$u_k^L(x) = -\int_0^x \frac{e^{x-\tau}}{\tau - x_0} \frac{d^2}{d\tau^2} u_{k-1}^L(\tau) d\tau, \quad u_k^R(x) = -\int_1^x \frac{e^{x-\tau}}{\tau - x_0} \frac{d^2}{d\tau^2} u_{k-1}^R(\tau) d\tau, \ k = 2, 3, \ldots$$

In the vicinity of the singular point $x = x_0$ we will highlight the singularities of the series $u^L(x) = \sum\limits_{k=0}^{\infty} \varepsilon^k u_k^L(x)$ and $u^R(x) = \sum\limits_{k=0}^{\infty} \varepsilon^k u_k^R(x)$:

$$u^L(x) = ae^x - \varepsilon ae^x \ln|x - x_0| + \frac{\varepsilon^2}{(x - x_0)^2}(a_2(x) + \alpha_2(x - x_0)^2 \ln|x - x_0|) + \ldots$$

$$+ \frac{\varepsilon^n}{(x - x_0)^{2(n-1)}}(a_n(x) + \alpha_n(x - x_0)^{2(n-1)} \ln|x - x_0|) + \ldots,$$

$$u^R(x) = be^{x-1} - \varepsilon be^{x-1} \ln|x - x_0| + \frac{\varepsilon^2}{(x - x_0)^2}(b_2(x) + \beta_2(x - x_0)^2 \ln|x - x_0|) + \ldots$$

$$+ \frac{\varepsilon^n}{(x - x_0)^{2(n-1)}}(b_n(x) + \beta_n(x - x_0)^{2(n-1)} \ln|x - x_0|) + \ldots,$$

where $a_k, \ b_k \in C^\infty[0, 1], \ b_k(1) = 0, \ a_k(0) = 0, \ k \in N_0 = \{0, 1, 2, \ldots\}, \ \alpha_k, \beta_k = const.$

From the last constructed $u^R(x)$ and $u^L(x)$ series it follows that to construct an internal solution it is necessary to use the transformation $x - x_0 = \mu t$, where t is a new variable, $\mu = \sqrt{\varepsilon}$.

Thus, the external solution loses its asymptotic character at the singular point $x = x_0$ and in its vicinity, and cannot be a solution to the boundary value problem (2.1).

We will seek a solution to problem (2.1) including the singular point in the form:

$$y(x, \varepsilon) = v(x, \varepsilon) + w(t, \mu), \tag{3.5}$$

where $v(x, \varepsilon)$ is a regular exterior solution, has no singularities, and $w(t, \mu)$ is an interior solution in the neighborhood of the singular point $x = x_0$.

Substituting relation (3.5) into problem (2.1), we obtain:

$$\varepsilon v''(x) + (x - x_0)v'(x) - (x - x_0)v(x) = h(\varepsilon), \quad x \in (0, 1), \tag{3.6}$$

$$v(0) = a, \quad v(1) = b, \tag{3.7}$$

$$w''(t) + tw'(t) - t\mu w(t) = -h(\mu^2), \quad t \in (-\frac{x_0}{\mu}, \frac{1-x_0}{\mu}), \tag{3.8}$$

$$w\Big(-\frac{x_0}{\mu}\Big) = 0, \quad w\Big(\frac{1-x_0}{\mu}\Big) = 0. \tag{3.9}$$

Consider first problem (3.6)-(3.7). We will look for a solution to this problem in the form:

$$v(x) = \begin{cases} v^L(x), & 0 \le x \le x_0, \\ v^R(x), & x_0 \le x \le 1, \end{cases} \quad h(\varepsilon) = \begin{cases} h^L(\varepsilon), & 0 \le x \le x_0, \\ h^R(\varepsilon), & x_0 \le x \le 1, \end{cases} \tag{3.10}$$

where

$$v^L(x) = \sum_{k=0}^{\infty} \varepsilon^k v_k^L(x), \quad v^R(x) = \sum_{k=0}^{\infty} \varepsilon^k v_k^R(x), \quad h^L(\varepsilon) = \sum_{k=1}^{\infty} \varepsilon^k h_k^L, \quad h^R(\varepsilon) = \sum_{k=1}^{\infty} \varepsilon^k h_k^R.$$

Substituting relation (3.10) into problem (3.6)-(3.7) we have a simple system of recurrence relations:

$$(x-x_0)v_0'(x) - (x-x_0)v_0(x) = 0, \quad v_0(0) = a, \quad v_0(1) = b, \tag{3.11}$$

$$(x-x_0)v_k'(x) - (x-x_0)v_k(x) = h_k - v_{k-1}''(x), \quad v_k(0) = 0, \quad v_k(1) = 0, \quad k \in \mathbb{N}. \tag{3.12}$$

We obtain:

$$v_0(x) = \begin{cases} v_0^L(x), & x \in [0, x_0], \\ v_0^R(x), & x \in [x_0, 1], \end{cases}$$

$$v_k(x) = \begin{cases} v_k^L(x), & x \in [0, x_0], \\ v_k^R(x), & x \in [x_0, 1], \end{cases}$$

where

$$v_0^L(x) = ae^x, \quad v_0^R(x) = be^{x-1},$$

$$v_1^L(x) = \int_0^x \frac{h_1^L - ae^\tau}{\tau - x_0} e^{x-\tau} d\tau, \quad v_1^R(x) = \int_1^x \frac{h_1^R - be^{\tau-1}}{\tau - x_0} e^{x-\tau} d\tau,$$

$$v_k^L(x) = \int_0^x \frac{h_k^L - (v_{k-1}^L(\tau))''}{\tau - x_0} e^{x-\tau} d\tau, \quad v_k^R(x) = \int_1^x \frac{h_k^R - (v_{k-1}^R(\tau))''}{\tau - x_0} e^{x-\tau} d\tau, \; k = 2, 3, \ldots$$

Let

$$h_1^L = ae^{1/2}, \; h_k^L = \frac{d^2}{dx^2} v_{k-1}^L(x)\big|_{x=x_0}, \quad h_1^R = be^{-1/2}, \; h_k^R = \frac{d^2}{dx^2} v_{k-1}^R(x)\big|_{x=x_0}, \quad k \in N,$$

then

$$v_k^L \in C^\infty[0, x_0], \quad v_k^R \in C^\infty[x_0, 1].$$

Thus, all members of the unknown series are determined. $h^L(\varepsilon), h^R(\varepsilon)$:

$$h^L(\varepsilon) = \varepsilon a\sqrt{e} + \varepsilon^2 v_2^L + \cdots + \varepsilon^n v_n^L + \ldots, \quad v_k^L = \frac{d^2}{dx^2} v_k^L(x)\Big|_{x=x_0};$$

$$h^R(\varepsilon) = \varepsilon \frac{b}{\sqrt{e}} + \varepsilon^2 v_2^R + \cdots + \varepsilon^n v_n^R + \ldots, \quad v_k^R = \frac{d^2}{dx^2} v_k^R(x)\Big|_{x=x_0}.$$

We now turn to the construction of the inner expansion, i.e., to solving problem (3.8)-(3.9). We will seek a solution to this problem in the form:

$$w(t,\mu) = \begin{cases} w^L(t,\mu), & -x_0/\mu \le t \le 0, \\ w^R(t,\mu), & 0 \le t \le (1-x_0)/\mu, \end{cases} \tag{3.13}$$

where $w^L(t,\mu) = \sum\limits_{k=0}^{\infty} \mu^k w_k^L(t), \quad w^R(t,\mu) = \sum\limits_{k=0}^{\infty} \mu^k w_k^R(t).$

Substituting the series (3.13) into the relations (3.8)-(3.9) we have:

$$\begin{gathered} \tfrac{d^2}{dt^2} w_0^L(t) + t\tfrac{d}{dt} w_0^L(t) = 0, \quad -x_0/\mu < t < 0, \quad w_0^L(-x_0/\mu) = 0; \\ \tfrac{d^2}{dt^2} w_0^R(t) + t\tfrac{d}{dt} w_0^R(t) = 0, \quad 0 < t < (1-x_0)/\mu, \quad w_0^R((1-x_0)/\mu) = 0; \end{gathered} \tag{3.14}$$

$$\begin{gathered} \tfrac{d^2}{dt^2} w_{2k-1}^L(t) + t\tfrac{d}{dt} w_{2k-1}^L(t) = t w_{2k-2}^L(t), \quad -x_0/\mu < t < 0, \quad w_{2k-1}^L(-x_0/\mu) = 0; \\ \tfrac{d^2}{dt^2} w_{2k-1}^R(t) + t\tfrac{d}{dt} w_{2k-1}^R(t) = t w_{2k-2}^R(t), \quad 0 < t < (1-x_0)/\mu, \quad w_{2k-1}^R((1-x_0)/\mu) = 0; \end{gathered} \tag{3.15}$$

$$\begin{gathered} \tfrac{d^2}{dt^2} w_{2k}^L(t) + t\tfrac{d}{dt} w_{2k}^L(t) = t w_{2k-1}^L(t) - h_k^L, \quad -x_0/\mu < t < 0, \quad w_{2k}^L(-x_0/\mu) = 0; \\ \tfrac{d^2}{dt^2} w_{2k}^R(t) + t\tfrac{d}{dt} w_{2k}^R(t) = t w_{2k-1}^R(t) - h_k^R, \quad 0 < t < (1-x_0)/\mu, \quad w_{2k}^R((1-x_0)/\mu) = 0. \end{gathered} \tag{3.16}$$

For functions $w_k^L(t)$ and $w_k^R(t)$ we require additional conditions (merging conditions):

$$\begin{gathered} v_k^L(x_0) + w_{2k}^L(0) = v_k^R(x_0) + w_{2k}^R(0), \\ \frac{d}{dx} v_k^L(x)|_{x=x_0} + \frac{d}{\mu dt} w_{2k}^L(t)|_{t=0} = \frac{d}{dx} v_k^R(x)|_{x=x_0} + \frac{d}{\mu dt} w_{2k}^R(t)|_{t=0}, \\ w_{2k+1}^L(0) = w_{2k+1}^R(0), \ \frac{d}{dt} w_{2k+1}^L(t)|_{t=0} = \frac{d}{dt} w_{2k+1}^R(t)|_{t=0}, \quad k = 0, 1, \ldots \end{gathered} \tag{3.17}$$

Condition (3.17) ensures the continuity of the point $x = x_0$ and its neighborhood.

Let us prove the following auxiliary lemma.

Lemma 3.1. *There is a unique solution to problem:*

$$\begin{gathered} \frac{d^2}{dt^2}\xi^L(t) + t\frac{d}{dt}\xi^L(t) = G^L(t), \quad t \in (-x_0/\mu, 0), \\ \frac{d^2}{dt^2}\xi^R(t) + t\frac{d}{dt}\xi^R(t) = G^R(t), \quad t \in (0, (1-x_0)/\mu), \\ \xi^L(0) + k_1\xi^R(0) = \alpha, \quad \frac{d}{dt}\xi^L(0) + k_2\frac{d}{dt}\xi^R(0) = \beta, \\ \xi^L(-x_0/\mu) = 0, \quad \xi^R((1-x_0)/2\mu) = 0, \end{gathered}$$

and represent it in the form:

$$\xi^L(t) = \frac{k_2(\alpha - \gamma B_1^{(-)} - k_1\gamma B_1^{(+)}) + k_1\beta}{k_1 + k_2} \int_{-x_0/\mu}^{t} e^{-\tau^2/2} d\tau + \int_{-x_0/\mu}^{t} e^{-\tau^2/2} \int_0^{\tau} e^{s^2/2} G^L(s) ds d\tau,$$

$$\xi^R(t) = \frac{k_1\beta - k_2(\alpha - \gamma B_1^{(-)} - k_1\gamma B_1^{(+)})}{k_1 - k_2} \int_{(1-x_0)/\mu}^{t} e^{-\tau^2/2} d\tau + \int_{(1-x_0)/\mu}^{t} e^{-\tau^2/2} \int_0^{\tau} e^{s^2/2} G^R(s) ds d\tau,$$

where $k_1, k_2, (|k_1| \neq |k_2|)$, α, β *are given constant numbers,* $\gamma = \left(\int_{-x_0/\mu}^{0} e^{-\tau^2/2} d\tau\right)^{-1}$,

$$B_1^{(-)} = \int_{-x_0/\mu}^{0} e^{-\tau^2/2} \int_0^{\tau} e^{s^2/2} G^L(s) ds d\tau, \quad B_1^{(+)} = \int_{(1-x_0)/\mu}^{0} e^{-\tau^2/2} \int_0^{\tau} e^{s^2/2} G^R(s) ds d\tau.$$

Based on this lemma, we can assert that solutions to problems (3.14)-(3.17) exist and are unique. In particular

$$w_0^L(t) = \frac{(b-ae)(\gamma+\mu)}{2\sqrt{e}} \int_{-x_0/\mu}^{t} e^{-s^2/2} ds, \quad w_0^R(t) = \frac{(b-ae)(\gamma-\mu)}{2\sqrt{e}} \int_{(1-x_0)/\mu}^{t} e^{-s^2/2} ds.$$

The asymptotic solution of the zeroth order of problem (2.1) will take the following form:

$$y_0^L(x) = ae^x + \frac{(b-ae)(\gamma+\mu)}{2\sqrt{e}} \int_{-x_0/\mu}^{t} e^{-s^2/2} ds,$$

$$y_0^R(x) = be^{x-1} + \frac{(b-ae)(\gamma-\mu)}{2\sqrt{e}} \int_{(1-x_0)/\mu}^{t} e^{-s^2/2} ds.$$

This result was presented in order to compare with the result of A.H. Naife [9, p.316]:

$$y_l^c = ae^x + (b-ae)\left(\frac{1}{2\sqrt{e}} + \frac{1}{\sqrt{2\pi e}} \int_0^t e^{-s^2/2} ds\right),$$

$$y_r^c = be^{x-1} + (ae-b)\left(\frac{1}{2\sqrt{e}} - \frac{1}{\sqrt{2\pi e}} \int_0^t e^{-s^2/2} ds\right).$$

If we take into account that $\int_{-x_0/\mu}^{t}(*) = \int_{-x_0/\mu}^{0}(*) + \int_0^t(*) = \gamma + \int_0^t(*)$ and $\mu \to 0$, then our result exactly coincides with A.H.Nayfeh's result.

Thus, we have proved the theorem.

Theorem 3.1. *For the solution to problem* (2.1) *the asymptotic expansion holds for* $\forall x \in [0,1]$*:*

$$y(x,\varepsilon) = \sum_{k=0}^{n} \varepsilon^k v_k(x) + \sum_{k=0}^{2n+1} \mu^k w_k(t) + O(\varepsilon^{n+1}), \ \varepsilon \to 0.$$

Here the estimate for the remainder term is obtained by applying the maximum principle [11].

4. Discussion.

The proposed approach, based on a combination of the generalized boundary function method and the method of matched asymptotic expansions, makes it possible not only to construct formal asymptotic representations but also to establish their uniform validity on the entire interval. A crucial step in the analysis is the introduction of the stretched variable

$$t = \frac{x - x_0}{\mu}, \qquad \mu^2 = \varepsilon,$$

which resolves the neighborhood of the interior turning point and enables the construction of the corresponding inner expansion. This transformation separates the slow and fast scales of the problem and provides an accurate description of the interior layer structure.

From a computational perspective, the derived asymptotic formulas may serve as effective approximations even for moderately small values of the perturbation parameter. Consequently, they can be employed as a foundation for the development of efficient numerical algorithms and hybrid asymptotic–numerical methods. Numerical experiments performed in Maple demonstrate excellent agreement between the asymptotic approximations and the corresponding numerical solutions, thereby confirming the accuracy and practical applicability of the proposed approach.

An important feature of the method is that the structure of the interior layer emerges naturally from the asymptotic analysis and does not require prior assumptions regarding its location or form. This property makes the approach particularly attractive for symbolic implementation and automated asymptotic analysis of singularly perturbed problems.

The present study also raises several directions for future research. These include the extension of the proposed methodology to equations with more general variable coefficients, nonlinear singularly perturbed problems, systems of differential equations, and problems involving multiple interior turning points. Another important issue is the rigorous analysis of the remainder term and the derivation of sharp error estimates for the resulting composite asymptotic expansion.

Overall, the obtained results broaden the class of singularly perturbed problems that can be treated by asymptotic matching techniques and demonstrate that nontrivial interior layers may arise even in linear boundary value problems when the reduced equation possesses an interior turning point.

5. Conclusion

A singularly perturbed two-point boundary value problem with an interior turning point has been investigated. By combining the generalized boundary function method with the method of matched asymptotic expansions, a complete uniformly valid asymptotic expansion of the solution has been constructed. The analysis shows that, despite the absence of boundary layers at the endpoints of the interval, an interior layer is formed in a neighborhood of the turning point $x = x_0$. The solution therefore admits distinct outer expansions on either side of the turning point together with an inner expansion describing the rapid transition in the layer region.

Existence and uniqueness results for the auxiliary inner problem have been established, and the resulting composite asymptotic expansion has been shown to be uniformly valid on the entire interval. Numerical computations performed in Maple demonstrate excellent agreement between the asymptotic approximation and the exact solution.

The proposed approach can be extended to more general classes of singularly perturbed problems with interior turning points, including nonlinear equations, systems of differential equations, and problems containing multiple turning points.

Funding

This work is supported by grant of Osh State University

Availability of data and materials

Not applicable.

Competing interests

The authors declare that they have no competing interests.

Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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