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Research article

A Vector Field in a Semi-Riemannian Manifold

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Abstract

The study of vector fields in semi-Riemannian manifolds forms a critical component in differential geometry and mathematical physics. Semi-Riemannian manifolds generalize the concept of Riemannian manifolds by allowing the metric tensor to have indefinite signature, thus encompassing both Riemannian and Lorentzian manifolds. This generalization is essential for understanding the geometry underlying General Relativity and various field theories.

Keywords: *manifold; foliation;semi-Riemannian manifold; Minkowski space; spacelike; timelike; vector field.*

AMS Subject Classification (2020): *Primary: 53A35 ; Secondary: 57R10; 57R25.*

1. Introduction

Hawking and Ellis explore the geometric and topological aspects of spacetime, which are crucial to understanding the structure of the universe within general relativity [15]. Semi-Riemannian geometry involves the study of a smooth manifold equipped with a non-degenerate metric of any given signature. [17] examines the concept of duality in semi-Riemannian geometry, offering insights into the geometric structures in general relativity. [18] this textbook introduces advanced geometric methods, including those relevant to semi-Riemannian geometry and dual transformations, with applications in physics. [20] is book provides a clear introduction to semi-Riemannian geometry and its applications in general relativity, covering essential topics like curvature and geodesics.

Ismoilov and Artikbayev studied Monge-Ampère and dual transformations in an isotropic and Galilean space, where the subspace is a semi-Riemannian submanifold [12, 13, 9, 10, 8, 7].

1.1. Preliminaries

Definition 1.1. The metric tensor g on a smooth manifold $M = (M, g)$ is said a symmetric non-degenerate tensor field on M of nonzero constant index. [25]

In other words $g \in \mathrm{T}_p(M)$ smoothly assigns to each point p of M a scalar product g_p in the tangent space $T_p(M)$, and the index of g_p is the same for all p.

Definition 1.2. A semi-Riemannian manifold is a smooth manifold M equipped with a metric tensor g.[26]

A semi-Riemannian manifold is an ordered pair (M, g): two different metric tensors on the same manifold constitute different semi-Riemannian manifolds. However, we usually denote a semi-Riemannian manifold by the name of its smooth manifold M, N, and others.

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Let us consider an $n = dimM$ dimensional semi-Riemannian manifold M. The general value k of index g on M is called the index of M: $0 \le k \le n$. If $k = 0$, M is a Riemannian manifold; then each g is a inner product of $T_p(M)$, g be positive definite. If $k = 1$ and $n \ge 2$, M is a Lorentz manifold.

Semi-Riemannian manifolds are often called pseudo-Riemannian manifolds, or even - in the old terminology - Riemannian manifolds, but we reserve the latter term for a special positive definite case.

We use $(,\ )$ as an alternative notation for g, writing $g(u, v) = (u, v)$ for tangent vectors, and $g(V_1, V_2) = (V_1, V_2) \in F(M)$ for vector fields.

If $x_1, \ldots, x_n$ is a coordinate system on $U \subset M$ the components of the metric tensor g on U are

$$g_{ij} = \left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right),$$

were, $i = \overline{1.n}$, $\ j = \overline{1.n}$.

Thus for vector fields $V_1 = \sum\limits_{i=1}^{n} V_{1,i} \frac{\partial}{\partial x^i}$ and $V_2 = \sum\limits_{i=1}^{n} V_{2,j} \frac{\partial}{\partial x^j}$,

$$g(V_1, V_2) = \sum_{i,j=1}^{n} g_{ij} V_{1,i} V_{2,j}$$

Denoted by matrix $\left(g^{ij}(p)\right)$ is the inverse of matrix $\left(g_{ij}(p)\right)$.

Since g is symmetric, $g_{ij} = g_{ji}$ and hence $g^{ij} = g^{ji}$ for $i = \overline{1.n}$, $\ j = \overline{1.n}$. Finally on U the metric tensor can be written as

$$\boldsymbol{g} = \sum_{i,j=1}^{n} g_{ij} dx^i \otimes dx^j$$

For each $\forall p \in R_n$ there is linear isomorphism from R^n to $T_p\ (R^n)$ that, in terms of natural coordinates, sends v to $v_p = \sum\limits_{i=1}^{n} v^i \frac{\partial}{\partial x^i}$. Thus the scaly product on R^n gives rise to a metric tensor on R^n with

$$\left(v_{1,p}, v_{2,p}\right) = \sum_{i=1}^{n} v_{1,i} v_{2,i}.$$

Henceforth in any geometric context R_n will denote the resulting Riemannian manifold, called n-dimensional Euclidean space.

For an integer k with $0 \le k \le n$, changing the first k plus signs above to minus gives a metric tensor

$$\left(v_{1,p}, v_{2,p}\right) = -\sum_{i=1}^{k} v_{1,i} v_{2,i} + \sum_{j=k+1}^{n} v_{1,j} v_{2,j}$$

of index k. The resulting semi-Euclidean space ${}^kR_n$ reduces to R_n if $k = 0$. For $n \ge 2$,^{1} R_n is called Minkowski n-space; if $n = 4$ it is the simplest example of a relativistic spacetime.

Fix the notation

$$\varepsilon_i = \begin{cases} -1 & \text{for} \quad 1 \le i \le v \\ +1 & \text{for} \quad v + 1 \le i \le n \end{cases}$$

Then the metric tensor of ${}^kR_n$ can be written

$$\boldsymbol{g} = \sum_{i=1}^{n} \varepsilon_i \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^i}$$

The geometric significance of the index of a semi-Riemannian manifold derives from the following trichotomy.

Definition 1.3. A tangent vector v to M is [1]

$$\begin{aligned} \text{spacelike} \ \ &\text{if } (v, v) > 0 \quad \text{or} \quad v = 0 \\ \text{null} \ \ &\text{if } (v, v) = 0 \quad \text{and} \quad v \neq 0 \\ \text{timelike} \ \ &\text{if } (v, v) < 0. \end{aligned}$$

By an orthonormal frame on a pseudo-Riemannian $n-$manifold M, we mean a set consists of n mutually orthogonal unit vector fields $e_1, \ldots, e_n$ on M.

The mean curvature H of a submanifold M' in a semi-Riemannian manifold (M, g) is related to the Laplace-Beltrami operator Δ acting on the embedding functions of M'. For a submanifold M' with an embedding $X : M' \longrightarrow M$, the mean curvature vector H can be expressed as:

$$H = \Delta X$$

where Δ is the Laplace-Beltrami operator on M'.

Let V and W be vector fields on a semi-Riemannian manifold M. The goal of this section is to show how to define a new vector field $\nabla_V W$ on M whose value at each point p is the vector rate of change of W in the V_p direction. There is a natural way to do this on $\boldsymbol{R}^n_v$.

Definition 1.4. Let $u^1, \ldots, u^n$ be the natural coordinates on $\boldsymbol{R}^n_v$. If $V = \sum\limits_{i=1}^{n} V^i \frac{\partial}{\partial x^i}$ and $Y = \sum\limits_{i=1}^{n} Y^i \frac{\partial}{\partial x^i}$ are vector fields on $\boldsymbol{R}^n_v$, the vector field

$$\nabla_V W = \sum_{i=1}^{n} V\left(Y^i\right) \frac{\partial}{\partial x^i}$$

is called the natural covariant derivative of Y with respect to V.

The second fundamental form II is given by:

$$\mathrm{II}(X, Y) = (\nabla_X Y)^{\perp}$$

where X, Y are tangent vectors to M'. The perpendicular component of the covariant derivative is given by: $(\nabla_X Y)^{\perp} = \nabla_X Y - \langle \nabla_X Y, X \rangle X$.

The mean curvature vector h is:

$$H = \frac{1}{m} \sum_{i=1}^{m} (\nabla_{e_i} e_i)^{\perp},$$

where $\{e_i\}_{i=1}^{m}$ is an orthonormal basis of the tangent space of M and $\nabla_{e_i} e_i = e_i^j \frac{\partial e_i^k}{\partial x^j} + \Gamma^k_{jk} e_i^j e_i^k$, where $\Gamma^i_{jk} = \frac{1}{2} g^{im} \left(\frac{\partial g_{mj}}{\partial x^k} + \frac{\partial g_{mk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^m} \right)$.

Definition 1.5. A pseudo-Riemannian submanifold N is called minimal if the mean curvature vector H vanishes identically, i.e., $H \equiv 0$. [27]

If the submanifold is given by the formula $x_{n+1} = f(x_1,\ x_2,\ \ldots,\ x_n)$ in a curvilinear coordinate system. For a function $\varphi : M \longrightarrow M^*$ its Dual transform is defined to curved coordinates

$$\begin{cases} x_i^* = f'_{x_i}(x_1, x_2, ..., x_n) \quad i = \overline{1.n} \\ x_{n+1}^* = \sum\limits_{i=1}^{n} x_i \cdot f'_{x_i}(x_1, x_2, ..., x_n) - f(x_1, x_2, ..., x_n) \end{cases} .$$

Lemma 1.1. *Properties of the Dual Transform:*

1. Involutory Property: The dual transform is an involution; that is, the dual transform of f^* is f itself.

2. Duality: The transform maps the problem of finding the extremum of f to a dual problem involving f^*.

3. Smoothness: If f is smooth and strictly convex, then f^* is also smooth.[12]

2. Results

Theorem 2.1. *: If (M, g) is a semi-Riemannian manifold and $N \subset M$ is a minimal submanifold, then the dual transformation of N into $N^* \subset M$ will be conformal. That is, the dual transformation $\phi : N \longrightarrow N^*$ results in metrics g and $\phi^* g$ being related as follows:*

$$\phi^* g = e^{2\lambda} g$$

where $\lambda : M \longrightarrow \mathbb{R}$ is a smooth function.

Proof. To prove this theorem, we will consider the concepts of minimal submanifolds, dual transformations, and the properties of conformal changes in detail.

To understand how the metric changes under dual transformation, we first consider the pull-back of the metric via the dual transformation:

$$(\phi^* g)(X, Y) = g(d\phi(X), d\phi(Y)),$$

where X, Y are tangent vectors on M.

Conformal transformations scale the metric by a scalar factor. If $\phi : M \longrightarrow M$ is a conformal transformation, then the metrics $\phi^* g$ and g are related by:

$$\phi^* g = e^{2\lambda} g,$$

where $\lambda : M \longrightarrow \mathbb{R}$ is a smooth function.

This implies that the metric under dual transformation is scaled by a factor. The conformal change of the mean curvature vector is given by:

$$\mathbf{H}^* = e^{-\lambda} \mathbf{H}.$$

Since $\mathbf{H} = 0$ for a minimal submanifold:

$$\mathbf{H}^* = e^{-\lambda} \cdot 0 = 0.$$

This confirms that the dual transformation is conformal.

Theorem 2.2. *If M two dimensional is a minimal pseudo-Riemannian supmanifold in the Null submanifold, then the dual mapping will be conformal.*

Proof. A mapping is conformal if the first fundamental form is

$$g^* = \lambda(x^1, x^2) \left(\left(\frac{\partial}{\partial x^1} \right)^2 + \left(\frac{\partial}{\partial x^2} \right)^2 \right).$$

Let us determine the conditions that satisfy this equality. Calculate the first fundamental form of the dual submanifold M^* under the condition that M is the minimal submanifold:

$$g^* = \left(\frac{\partial}{\partial (x^*)^1} \right)^2 + \left(\frac{\partial}{\partial (x^*)^2} \right)^2 = \left(\frac{\partial^2 f}{\partial (x^1)^2} \frac{\partial}{\partial x^1} + \frac{\partial^2 f}{\partial x^1 \partial x^2} \frac{\partial}{\partial x^2} \right)^2 + \left(\frac{\partial^2 f}{\partial x^1 \partial x^2} \frac{\partial}{\partial x^1} + \frac{\partial^2 f}{\partial (x^2)^2} \frac{\partial}{\partial x^2} \right)^2 =$$

$$\left[\left(\frac{\partial^2 f}{\partial (x^1)^2} \right)^2 + \left(\frac{\partial^2 f}{\partial x^1 \partial x^2} \right)^2 \right] \left(\frac{\partial}{\partial x^1} \right)^2 + 2 \frac{\partial^2 f}{\partial x^1 \partial x^2} \left(\frac{\partial^2 f}{\partial (x^1)^2} + \frac{\partial^2 f}{\partial (x^2)^2} \right) \frac{\partial}{\partial x^1} \frac{\partial}{\partial x^2} +$$

$$+\left[\left(\frac{\partial^2 f}{\partial x^1 \partial x^2}\right)^2+\left(\frac{\partial^2 f}{\partial (x^2)^2}\right)^2\right]\left(\frac{\partial}{\partial x^2}\right)^2.$$

The expression $\frac{\partial^2 f}{\partial (x^1)^2}+\frac{\partial^2 f}{\partial (x^2)^2}=0$ follows from the definition of a minimal submanifold M and the mean curvature formula. Hence, if we consider the expression $\frac{\partial^2 f}{\partial (x^1)^2}=-\frac{\partial^2 f}{\partial (x^2)^2}$:

$$g^*=\left[\left(\frac{\partial^2 f}{\partial (x^1)^2}\right)^2+\left(\frac{\partial^2 f}{\partial x^1 \partial x^2}\right)^2\right]\left(\frac{\partial}{\partial x^1}\right)^2+\left[\left(\frac{\partial^2 f}{\partial x^1 \partial x^2}\right)^2+\left(\frac{\partial^2 f}{\partial (x^2)^2}\right)^2\right]\left(\frac{\partial}{\partial x^2}\right)^2=$$

$$=\left[\left(\frac{\partial^2 f}{\partial (x^1)^2}\right)^2+\left(\frac{\partial^2 f}{\partial x^1 \partial x^2}\right)^2\right]\left((\frac{\partial}{\partial x^1})^2+(\frac{\partial}{\partial x^2})^2\right)=\lambda(x^1,x^2)g.$$

We get from here that $\lambda(x^1,x^2)=(\frac{\partial^2 f}{\partial (x^1)^2})^2+(\frac{\partial^2 f}{\partial x^1 \partial x^2})^2$.

This proves that the dual mapping of the minimal submanifold is conformal.

Corollary 2.1. *If the translation submanifold satisfies the condition* $\frac{\partial^2 f}{\partial (x^1)^2}=\frac{\partial^2 f}{\partial (x^2)^2}$, *then its dual mapping will be conformal.*

To understand the mechanical meaning of the dual transformation, we need to examine the physical and geometric implications of such mathematical transformations. A dual transformation typically reflects a change in certain structures of the original submanifold, which can be crucial for studying physical mechanisms or geometric properties.

The dual transformation $\phi : N \longrightarrow N^*$ generally involves mapping the new submanifold N^* such that it retains or reinterprets the specific characteristics of the original submanifold N. Mechanically, this can imply the following:

Minimal submanifolds represent minimized potential energy states in physical terms. A dual transformation re-expresses these states in a new coordinate system or geometric context, preserving energy minimization principles in a new setting.

As a result of the dual transformation, the metric changes conformally, meaning the shape is preserved while the size or scale may change:

$$\phi^* g = e^{2\lambda} g$$

This is analogous to symmetry transformations in physics, such as changes related to light propagation under heating or expansion. Conformal transformations preserve shapes and angles, but alter dimensions.

Dual transformations preserve or reinterpret geometric properties in a new geometry. For instance, minimal surfaces, elastic materials, or field theories can be re-expressed with new physical or mathematical insights through dual transformations.

In mechanical systems, dual transformations allow for the study of phenomena in the original system within a new framework. For example, studying structures through conformal transformations can be beneficial for analyzing the elastic properties of materials.

Examples.

The tension and equilibrium states of an elastic membrane can be analyzed under new conditions via dual transformation. For example, the deformation of an elastic material can be studied in a new conformal coordinate system.

The propagation of light in different media can be analyzed through dual transformations in new media. Conformal optics allows for the study of new directions and curvatures of light propagation.

Conclusion

The Dual transform is a versatile mathematical tool that offers significant advantages in the analysis of vector fields on semi-Riemannian manifolds and in the solution of the Monge-Ampère equation. By transforming the original problem into a dual formulation, we can exploit the properties of convex functions and their transforms to find solutions to complex geometric and physical equations. This approach not only simplifies the mathematical treatment but also provides deeper insights into the structure of semi-Riemannian manifolds and their applications in various fields of mathematics and physics.

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