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Research article

Theorems on Changing the Order of Integration in Laplace Transform over Classical Domain of the Second Type

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Abstract

The problem of changing the order of integration in the Laplace transform over classical domains of the second type is investigated. These domains belong to the class of Cartan classical domains and consist of complex symmetric matrices satisfying certain positivity conditions. Matrix-valued original functions defined on the class of symmetric matrices and their Laplace transforms with respect to matrix arguments are considered. Sufficient conditions for interchanging the order of integration in the Laplace transform integral are established. The obtained theorems provide a rigorous justification for changing the order of integration in integral representations associated with matrix Laplace transforms and can be applied in the theory of holomorphic functions on classical domains as well as in the study of integral transforms involving symmetric matrices.

Keywords: *classical domains; Hermitian matrix; holomorphic function; Laplace transform; matrix image function; matrix trace.*

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1. Introduction

The Laplace transform is named after the great mathematician and astronomer Pierre-Simon Marquis de Laplace (1749–1827), who used a similar transformation in his work on probability theory [1], and as a result, the integral form of the Laplace transform naturally developed. The theory of the Laplace transform was further developed in the 19th and early 20th centuries by Matthias Lerch, Oliver Heaviside, and Thomas Bromwich. The scientist who brought the theory of the Laplace transform into a complete theory of the now widespread operational calculus is Gustav Doetz. The concept of the Laplace transform was first used in his work published in 1937 [2]. He was the first to apply the Laplace transform to solving engineering problems in his scientific work. After the Laplace transform relations for functions of one variable were studied, the question of constructing Laplace transform relations for functions of many arguments naturally arose. The solution to this problem was found in 1981 by the Russian mathematician L.G. Smishlyayeva in his monograph [3].

The Laplace transform is used to study the properties of dynamical systems and to facilitate the solution of differential equations [4, 5]. For example, the Laplace transform reduces the solution of differential equations to the solution of algebraic equations, and then obtains solutions of equal strength [3,6,7]. After the emergence

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of matrix analysis, direct Laplace transforms were applied to matrix fields and their matrix analogues were obtained [4,8,9]. Until now, the Laplace transforms have been used by leading scientists from major scientific schools of the world for hypergeometric functions with matrix arguments [10], Lager for matrix functions to apply polynomials to the numerical inversion of matrix functions [11]. And for the Bernstein operational matrix [12] and from the latest scientific works for functions with many matrix arguments [13], for functions with symmetric matrix arguments the Laplace transform analogues were obtained [14,18].

2. Preliminaries

The French mathematician E. Cartan showed in 1935 that there are six types of classes of unbounded, symmetric domains [19]. Domains belonging to the first four of these classes are called classical domains [20].

Definition 2.1. If $D \subset \mathbb{C}^n$ the group of automorphisms of a field is transitive, that is, $z_1,\ z_2 \in D$ if there exists an automorphism satisfying the condition $\varphi \in Aut\,(D)$ for an arbitrary $\varphi\,(z_1) = z_2$, then $D \subset \mathbb{C}^n$ the domain is called **a homogeneous domain**.

Definition 2.2. If for an arbitrary point in $\zeta \in D$ a homogeneous field $D \subset \mathbb{C}^n$ there exists an automorphism satisfying the following conditions: $\varphi \in Aut\,(D)$
1)$\varphi\,(\zeta) = \zeta$ the equality is valid, but ζ for points $\varphi\,(z) \neq z$ other than the point $z \in D$;
2) $\varphi \circ \varphi = e$, where $e \in Aut\,(D)$ is the igentity mapping; then $D \subset \mathbb{C}^n$ is called **a symmetric domain**.

Definition 2.3. The domain $D \subset \mathbb{C}^n$ is called **an irreducible domain** if it is not a direct product of bounded symmetric domains of lower dimension.

Definition 2.4. If $D \subset \mathbb{C}^n$ the automorphism group of a bounded field is transitive and forms a Lie group, then $D \subset \mathbb{C}^n$ the domain is called **a classical domain**.

Below are the classical domains classified by E. Cartan [19, 20]:

$$\Re_I\,(m,k) = \left\{Z \in \mathbb{C}\,[m,k] : I^{(m)} - ZZ^* > 0\right\},$$

$$\Re_{II}\,(m) = \left\{Z \in \mathbb{C}\,[m,m] : I^{(m)} - Z\bar{Z} > 0,\ \forall Z' = Z\right\},$$

$$\Re_{III}\,(m) = \left\{Z \in \mathbb{C}\,[m,m] : I^{(m)} + Z\bar{Z} > 0,\ \forall Z' = -Z\right\},$$

$$\Re_{IV}\,(n) = \left\{Z \in \mathbb{C}^n : |\langle z,z\rangle|^2 - 2|z|^2 + 1 > 0,\ |\langle z,z\rangle| < 1\right\},$$

where $I^{(m)}$ m-order unit is a matrix, Z^* and the matrix is the union of the matrix and its transpose (H for a Hermitian matrix, $H > 0$ the sign: indicates that it is a positive definite matrix, i.e., all its eigenvalues are positive: $\det|\lambda I - H| = 0 \Rightarrow \forall\lambda_i > 0$) [20, 21].

Each of these classical domains is a homogeneous, symmetric, irreducible, convex complete circular domains centered at a point O (O is the zero matrix of m-order). These domains do not have a biholomorphic equivalence relation with each other, so a complex analysis is constructed for each of them separately.

We conduct this research work in **classical domain of the second type**:

$$\Re_{II}\,(m) = \left\{Z \in \mathbb{C}[m \times m] : I^{(m)} - Z\overline{Z} > 0,\ \forall Z = Z'\right\}.$$

Us $f : S_m \rightarrow S_m$ $\left(A \in S_m \subset \mathbb{R}\,[m \times m] : f\ (A) = f(A)'\right)$ be given a symmetric matrix-function with matrix argument (here, S_m – the class of real symmetric matrices) [10].

Definition 2.5. A function that satisfies the following conditions is called $f(A)$ **a matrix original**:

I. $f(A) \equiv 0$ for $A < 0$, (here $A < 0$ the relation A is understood as each element of the matrix being less than 0);

II. $f(A)$ symmetric matrix-function is continuous in the real symmetric matrix right half-plane $\Upsilon_{S_m} = \left\{A = (a_{ij}) \in \mathbb{R}[m \times m] : \forall a_{ij} \geq 0, i, j = \overline{1, m}\right\}$, (that is, $\forall X_0 \in \Upsilon_{S_m}$: $\lim_{X \to X_0} f(X) = f(X_0)$);

III. $\forall A \in S_m$ there exist constant matrices $M = M' > 0$ and $X_0 > 0$, $X_0 = X_0[\alpha]$ whose entries are equal to $\alpha \geq 0$, such that the inequality $|f(A)| \leq M \cdot e^{Sp(X_0 A)}$ holds (here, $|f(A)|$ each element of which $f(A)$ is understood to be equal to the modulus of the element of the symmetric matrix-function).

Now, based on *Definition 2.5* above, we give the definition of a matrix image.

Definition 2.6. The matrix original is defined as **a matrix image** in the classical domains of second type, $Z = X + iY$ $(Z \in \Re_{II}(m))$ variable

$$F(Z) = \mathcal{L}_Z\{f(A)\} = \int\limits_{A>0} e^{-Sp(ZA)} f(A)\, dA \tag{2.1}$$

a matrix function defined by its integral is called a function with an argument. Here, is defined as $A = (a_{ij})$ for $dA = \prod\limits_{i \leq j} da_{ij}$, $Sp(ZA) = \sum\limits_{i \geq j} z_{ij} \cdot a_{ij}$– the trace of the matrix [10,21,22].

Remark 2.1. The integral in (2.1), $Z = (\eta_{ij} z_{ij})$ is a complex parametric matrix defined as, and the leading coefficients (η_{ij} invariant coefficients) of the matrix elements are defined as follows [22]:

$$\eta_{ij} = \begin{cases} \frac{1}{2}, & i \neq j. \\ 1, & i = j. \end{cases} \tag{2.2}$$

Definition 2.7. The transformation from a $f(A)$ matrix original to a $F(Z)$ matrix image by formula (2.1) is called the **Laplace transform for $\Re_{II}(m)$ classical domain of the second type**.

Remark 2.2. The matrix original $f(A)$ and the matrix image $F(Z)$ is defined as $F(Z) \stackrel{\cdot}{\rightarrow} f(A)$ or $f(A) \stackrel{\cdot}{\leftarrow} F(Z)$. Here, "$\stackrel{\cdot}{\rightarrow}$" the direction of the sign is always from the matrix image to the matrix original, without loss of generality [14,18]. In some references, this relation is also denoted by $f(t) \risingdotseq F(p)$ or $F(p) \fallingdotseq f(t)$ [3,6,7]. Also, $\mathcal{L}_Z\{f(A)\} = F(Z)$ the designation relation is also used [2,4,5,8-13,22-25,31].

Example 2.1. Suppose we are given the following real symmetric matrix function:

$$f(A) = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}.$$

Using the Laplace transform for classical domain of the second type (2.1), we determine the corresponding matrix image function of $f(A)$. To compute the matrix image function, we first determine the quantity $Sp(ZA)$, which plays a fundamental role in the integral representation (2.1). For this purpose, we consider the complex parametric matrix

$$Z = \begin{pmatrix} z_{11} & \frac{1}{2} z_{12} \\ \frac{1}{2} z_{12} & z_{22} \end{pmatrix}, \quad Z \in \Re_{II}(m)$$

whose elements are difined according to relation (2.2), and it left-multiply the real parametric matrix A. We obtain the following result:

$$ZA = \begin{pmatrix} z_{11} & \frac{1}{2} z_{12} \\ \frac{1}{2} z_{12} & z_{22} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} z_{11} \cdot a_{11} + \frac{1}{2} z_{12} \cdot a_{12} & z_{11} \cdot a_{12} + \frac{1}{2} z_{12} \cdot a_{22} \\ \frac{1}{2} z_{12} \cdot a_{11} + z_{22} \cdot a_{12} & \frac{1}{2} z_{12} \cdot a_{12} + z_{22} \cdot a_{22} \end{pmatrix}.$$

Taking the trace of the resulting matrix:

$$Sp\,(ZA) = z_{11}\cdot a_{11} + \frac{1}{2}z_{12}\cdot a_{12} + \frac{1}{2}z_{12}\cdot a_{12} + z_{22}\cdot a_{22} =$$
$$= z_{11}\cdot a_{11} + z_{12}\cdot a_{12} + z_{22}\cdot a_{22}.$$

Substituting this expression $Sp\,(ZA)$ into the Laplace transform formula (2.1), we obtain:

$$F\,(Z) = \mathcal{L}_Z\,\{f\,(A)\} = \int\limits_{A>0} e^{-Sp(ZA)} f\,(A)\,dA =$$

$$= \int\limits_0^{+\infty}\int\limits_0^{+\infty}\int\limits_0^{+\infty} e^{-(z_{11}\cdot a_{11}+z_{12}\cdot a_{12}+z_{22}\cdot a_{22})}\cdot f\,(a_{11},\, a_{12},\, a_{22})\,da_{11}da_{12}da_{22} =$$

$$= \int\limits_0^{+\infty}\int\limits_0^{+\infty}\int\limits_0^{+\infty} e^{-z_{11}\cdot a_{11}}\cdot e^{-z_{12}\cdot a_{12}}\cdot e^{-z_{22}\cdot a_{22}}\cdot f\,(a_{11},\, a_{12},\, a_{22})\,da_{11}da_{12}da_{22}.$$

For computational convenience, we evaluate the multiple integral separately with respect to each variable.

$$1)\quad \int\limits_0^{+\infty} e^{-z_{11}\cdot a_{11}}\cdot a_{11}\,da_{11} = \lim_{t\to+\infty}\int\limits_0^{t} e^{-z_{11}\cdot a_{11}}\cdot a_{11}\,da_{11} = \lim_{t\to+\infty}\left[-\frac{1}{z_{11}}\int\limits_0^{t} a_{11}\,d\left(e^{-z_{11}\cdot a_{11}}\right)\right] =$$

$$= \lim_{t\to+\infty}\left[-\frac{1}{z_{11}}\left(a_{11}\cdot e^{-z_{11}a_{11}}\Big|_0^t - \int\limits_0^t e^{-z_{11}a_{11}}da_{11}\right)\right] = \lim_{t\to+\infty}\left[-\frac{1}{z_{11}}\left(t\cdot e^{-z_{11}t} - 0 + \frac{1}{z_{11}}e^{-z_{11}a_{11}}\Big|_0^t\right)\right] =$$

$$= \lim_{t\to+\infty}\left[-\frac{t\cdot e^{-z_{11}t}}{z_{11}} - \frac{e^{-z_{11}t}}{z_{11}^2} + \frac{1}{z_{11}^2}\right] = \frac{1}{z_{11}^2} = z_{11}^{-2}.$$

In the same way as this integral, we also compute the following:

$$2)\quad \int\limits_0^{+\infty} e^{-z_{12}\cdot a_{12}}\cdot a_{12}\,da_{12} = \lim_{t\to+\infty}\int\limits_0^{t} e^{-z_{12}\cdot a_{12}}\cdot a_{12}\,da_{12} = \frac{1}{z_{12}^2} = z_{12}^{-2}.$$

$$3)\quad \int\limits_0^{+\infty} e^{-z_{22}\cdot a_{22}}\cdot a_{22}\,da_{22} = \lim_{t\to+\infty}\int\limits_0^{t} e^{-z_{22}\cdot a_{22}}\cdot a_{22}\,da_{22} = \frac{1}{z_{22}^2} = z_{22}^{-2}.$$

According to the obtained equalities 1), 2) and 3), we pass from a multivariable function to a symmetric matrix function. That is,

$$F\,(Z) := \begin{pmatrix} z_{11}^{-2} & z_{12}^{-2} \\ z_{12}^{-2} & z_{22}^{-2} \end{pmatrix}.$$

Therefore,

$$\mathcal{L}_Z\left\{\begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}\right\} = \begin{pmatrix} z_{11}^{-2} & z_{12}^{-2} \\ z_{12}^{-2} & z_{22}^{-2} \end{pmatrix}.$$

Thus, the Laplace transform relation holds. We observe that the obtained matrix Laplace transform relation constitutes the matrix analogue of the classical one-dimensional Laplace transform formula

$$\mathcal{L}\,\{t\} = \frac{1}{p^2},\quad (t\in\mathbb{R}_+,\; p\in\mathbb{C} : p = s + i\sigma)$$

as defined for functions of a single variable [6, 7].

2.1. Matrix Analogues of the Fundamental Properties of the Laplace Transform

Theorem 2.1 (*theorem on the holomorphism of an image*)**.** ***[31]*** *If there exists (2.1) the Laplace transform is defined $F(Z)$ is a matrix image function for $f(A)$ a matrix original: $\mathcal{L}_Z\{f(A)\} = F(Z)$, then this $F(Z)$ function is a holomorphic function of the Z matrix variable taken from the matrix right half-plane: $\Upsilon_{\Re_{II}} = \{Z \in \Re_{II}(m) : \operatorname{Re} Z = X > X_0 > 0\}$.*

Theorem 2.2 (*The inverse Laplace transform*)**.** ***[31]*** *If $F(Z) \in \Re_{II}(m)$ matrix image function*

$$\int\limits_{-\infty}^{+\infty} |F(X+iY)\,dY| < +\infty, \tag{2.3}$$

for arbitrary $X > X_0 > 0$

$$\lim_{X\to 0}\int\limits_{-\infty}^{+\infty} |F(X+iY)\,dY| = 0, \tag{2.4}$$

and $\mathcal{L}_Z\{f(A)\} = F(Z)$, then at every point where $f(A)$ is differentiable, the following unique inverse Laplace transform relation hold:

$$\mathcal{L}^{-1}\{F(Z)\} = f(A) = \frac{2^{\frac{1}{2}m(m-1)}}{(2\pi i)^{\frac{1}{2}m(m+1)}}\int\limits_{\operatorname{Re} Z>0} e^{Sp(ZA)}F(Z)\,dZ\,. \tag{2.5}$$

Here $Z = \left(\eta_{ij}\, z_{ij}\right)$, $\eta_{ij} = 1$. It is defined as follows [5]: $\mathcal{L}^{-1}(F(Z)) = f(A)$.

Remark 2.3. It should also be noted that $\mathcal{L}^{-1}(F(Z)) = f(A)$ attitude with $\mathcal{L}_Z\{f(A)\} = F(Z)$ the relationships are equally strong:

$$\mathcal{L}_Z\{f(A)\} = F(Z) \Leftrightarrow \mathcal{L}^{-1}(F(Z)) = f(A)\,.$$

Lemma 2.1. ***[14]*** *$F(Z) \in \Re_{II}(m)$ the following linearity relations hold for a matrix representation:*

I. if $c = const$ and $\mathcal{L}_Z\{f(A)\} = F(Z)$ then

$$\mathcal{L}_Z\{c\,f(A)\} = c\,F(Z). \tag{2.6}$$

II. if $\mathcal{L}_Z\{f_1(A)\} = F_1(Z)$ and $\mathcal{L}_Z\{f_2(A)\} = F_2(Z)$ then

$$\mathcal{L}_Z\{f_1(A) + f_2(A)\} = F_1(Z) + F_2(Z). \tag{2.7}$$

Theorem 2.3 (*The uniqueness of the original*)**.** ***[14]*** *If $F(Z) \in \Re_{II}(m)$ is the matrix image function of two matrix original functions $f_1(A)$ and $f_2(A)$, then these originals coincide at all their continuous points. That is, $f_1(A) \equiv f_2(A)$ the equality holds.*

Theorem 2.4 (*Analogs of general properties of the Laplace transform*)**.** ***[14]*** *If $\mathcal{L}_Z\{f(A)\} = F(Z)$ the Laplace transform $F(Z) \in \Re_{II}(m)$ relation holds, then $f(A)$ the following properties hold between the matrix image and the matrix original:*

1°. *If $\forall\beta > 0$, then the following relation holds:*

$$\mathcal{L}_Z\{f(\beta A)\} = \left(\frac{1}{\beta}\right)^{\frac{m(m+1)}{2}} F\left(\frac{1}{\beta}Z\right); \tag{2.8}$$

2°. *If $\forall B[c] \in S_m \subset \mathbb{R}[m\times m]$, $(c = const > 0)$, then the following relation is satisfied:*

$$\mathcal{L}_Z\{f(A - B[c])\} = e^{-Sp(ZB[c])}F(Z); \tag{2.9}$$

3°. *If* $\forall\Lambda \in \Upsilon_{S_m}$, *then the following relation is valid:*

$$\mathcal{L}_Z\left\{e^{Sp(\Lambda A)}f(A)\right\} = F(Z-\Lambda); \tag{2.10}$$

4°. *If* $f'(A)$ *exists and forms a matrix original function, then*

$$\mathcal{L}_Z\{f'(A)\} = ZF(Z) - f(O) \tag{2.11}$$

the relationship will be appropriate. Here O *is* m*-order zero matrix.*
5°. *If* $Z \in \Re_{II}(m)$ *for* $\det(Z) \neq 0$ *and* $\int\limits_{0<B<A} f(B)\,dB$ *is convergent, then the following relation is valid:*

$$\mathcal{L}_Z\left(\int\limits_{0<B<A} f(B)\,dB\right) = Z^{-1}F(Z). \tag{2.12}$$

6°. *If* $F'(Z)$ *available* $F'(Z) \in \Re_{II}(m)$ *if so, then the following relationship is valid:*

$$\mathcal{L}^{-1}(F'(Z)) = -Af(A). \tag{2.13}$$

7°. *If* $\det(A) \neq 0$ *it were* $A^{-1}f(A)$ *forms a matrix original, then the following relation holds:*

$$\mathcal{L}^{-1}\left(\int\limits_{W>Z} F(W)\,dW\right) = A^{-1}f(A). \tag{2.14}$$

3. Changing the Order of Integration for Matrix Laplace transform

Us be given A a positive definite $(\det|\lambda I - A| = 0 \Rightarrow \forall\lambda_i > 0)$ real symmetric matrix and its symmetric square root $A^{1/2}$ matrices. Let the Laplace transform $\mathcal{L}_Z\{f(A)\} = F(Z)$ relation hold.

Theorem 3.1. *Let* $f(A) \in S_m \subset \mathbb{R}[m\times m]$ *be a matrix original function. Assume that there exists a matrix* $Z \in \Re_{II}(m)$, $\det(Z) \neq 0$, *such that the Laplace transform relation* $\mathcal{L}_Z\{f(A)\} = F(Z)$ *holds. Suppose further that the integral*

$$\int_{0<B<A} (\det(B))^{-\frac{m+1}{2}} f(B)\,dB$$

is convergent. Then the following equality holds:

$$\mathcal{L}_Z\left\{\int_{0<B<A} (\det(B))^{-\frac{m+1}{2}} f(B)\,dB\right\} = (\det(Z))^{-\frac{m+1}{2}} \int_{W>Z} \mathcal{L}_W\{f(B)\}\,dW. \tag{3.1}$$

Proof. To prove relation (3.1), we first write its left-hand side:

$$\mathcal{L}_Z\left\{\int_{0<B<A} (\det(B))^{-\frac{m+1}{2}} f(B)\,dB\right\} = \int_{A>0} e^{-Sp(ZA)}\left[\int_{0<B<A} (\det(B))^{-\frac{m+1}{2}} f(B)\,dB\right] dA. \tag{3.2}$$

The integration domain in (3.2) is determined by the conditions $A = A' > 0$, $B = B' > 0$, $A - B > 0$. Therefore, (3.2) can be rewritten as

$$\int_{A>0}\int_{0<B<A} e^{-Sp(ZA)}(\det(B))^{-\frac{m+1}{2}} f(B)\,dB\,dA. \tag{3.3}$$

Since the condition $0 < B < A$ is equivalent to $B > 0$ and $A > B$, we obtain

$$\int_{B>0}\int_{A>B} e^{-Sp(ZA)}(\det(B))^{-\frac{m+1}{2}} f(B)\,dA\,dB. \tag{3.4}$$

By Fubini's theorem, relation (3.4) becomes

$$\int_{B>0} (\det(B))^{-\frac{m+1}{2}} f(B) \left[\int_{A>B} e^{-Sp(ZA)}\, dA\right] dB. \tag{3.5}$$

Denote

$$J(B) = \int_{A>B} e^{-Sp(ZA)}\, dA. \tag{3.6}$$

Introduce the change of variables $Y = Z^{1/2}AZ^{1/2}$. Then $A = Z^{-1/2}YZ^{-1/2}, dA = (\det(Z))^{-\frac{m+1}{2}}\, dY$, and $Sp(ZA) = Sp\left(Z\,Z^{-1/2}YZ^{-1/2}\right) = Sp(Y)$. Hence, $e^{-Sp(ZA)} = e^{-Sp(Y)}$. Moreover, the condition $A > B$ transforms into $Y > Z^{1/2}BZ^{1/2}$. Consequently,

$$J(B) = (\det(Z))^{-\frac{m+1}{2}} \int_{Y>Z^{1/2}BZ^{1/2}} e^{-Sp(Y)}\, dY. \tag{3.7}$$

Substituting (3.7) into (3.5), we get

$$\int_{B>0} (\det(B))^{-\frac{m+1}{2}} f(B) \left[(\det(Z))^{-\frac{m+1}{2}} \int_{Y>Z^{1/2}BZ^{1/2}} e^{-Sp(Y)}\, dY\right] dB. \tag{3.8}$$

Now introduce the transformation $W = B^{-1/2}YB^{-1/2}$. Then $Y = B^{1/2}WB^{1/2}, dY = (\det(B))^{\frac{m+1}{2}}\, dW$, and $Sp(Y) = Sp(B^{1/2}WB^{1/2}) = Sp(WB)$. Thus, $e^{-Sp(Y)} = e^{-Sp(WB)}$. The boundary condition $Y > Z^{1/2}BZ^{1/2}$ is transformed by multiplying both sides by $B^{-1/2}(\cdot)B^{-1/2}$:

$$W > (B^{-1/2}Z^{1/2}B^{1/2})(B^{1/2}Z^{1/2}B^{-1/2}).$$

Let $U = B^{-1/2}Z^{1/2}B^{1/2}$, $U' = B^{1/2}Z^{1/2}B^{-1/2}$, and hence $W > UU'$. Denote $V = UU'$. It is known that the matrices UU' and $U'U$ have identical eigenvalues. Moreover, $U'U = (B^{1/2}Z^{1/2}B^{-1/2})(B^{-1/2}Z^{1/2}B^{1/2})$. Therefore, the eigenvalues of V coincide with those of Z. Hence, the matrices V and Z are spectrally equivalent. Consequently, the condition $W > V$ can be written as $W > Z$. Substituting all the obtained transformations into (3.8), we obtain

$$(\det(Z))^{-\frac{m+1}{2}} \int_{B>0} f(B) \left[\int_{W>Z} e^{-Sp(WB)}\, dW\right] dB. \tag{3.9}$$

Applying Fubini's theorem once again, we interchange the order of integration:

$$(\det(Z))^{-\frac{m+1}{2}} \int_{W>Z} \left[\int_{B>0} e^{-Sp(WB)} f(B)\, dB\right] dW. \tag{3.10}$$

By Definition 2.7, the inner integral in (3.10) is the matrix Laplace transform with respect to the parameter W:

$$\mathcal{L}_W\{f(B)\} = \int_{B>0} e^{-Sp(WB)} f(B)\, dB.$$

Thus,

$$(\det(Z))^{-\frac{m+1}{2}} \int_{W>Z} \mathcal{L}_W\{f(B)\}\, dW.$$

The theorem has been proven.

Theorem 3.2. *Let $f(A) \in S_m \subset \mathbb{R}[m \times m]$ be a matrix original function. Assume that there exists a matrix $Z \in \mathfrak{R}_{II}(m)$, $\det(Z) \neq 0$, such that the Laplace transform relation $\mathcal{L}_Z\{f(A)\} = F(Z)$ holds. Suppose further that the integral*

$$\int_{0<W<Z} \mathcal{L}_W\{f(B)\}\, dW$$

is convergent. Then the following equality holds:

$$\mathcal{L}_Z\left\{\int_{B>A}(\det(B))^{-\frac{m+1}{2}}f(B)\,dB\right\}=(\det(Z))^{-\frac{m+1}{2}}\int_{0<W<Z}\mathcal{L}_W\{f(B)\}\,dW. \tag{3.11}$$

Proof. To prove relation (3.11), we write its left-hand side:

$$\mathcal{L}_Z\left\{\int_{B>A}(\det(B))^{-\frac{m+1}{2}}f(B)\,dB\right\}=\int_{A>0}e^{-Sp(ZA)}\left[\int_{B>A}(\det(B))^{-\frac{m+1}{2}}f(B)\,dB\right]dA.$$

The integration domain is determined by the conditions $A=A'>0, B=B'>0, B-A>0$. Therefore, unlike Theorem 3.1, we rewrite the integration domain in the form $B>A, A>0$, and change the order of integration:

$$\int_{B>0}\left[\int_{0<A<B}e^{-Sp(ZA)}(\det(B))^{-\frac{m+1}{2}}f(B)\,dA\right]dB.$$

Applying the substitutions $Y=Z^{1/2}AZ^{1/2}, W=B^{-1/2}YB^{-1/2}$, the integration domain transforms from $0<A<B$ into $0<W<Z$. Consequently, we obtain

$$(\det(Z))^{-\frac{m+1}{2}}\int_{0<W<Z}\left[\int_{B>0}e^{-Sp(WB)}f(B)\,dB\right]dW.$$

By Definition 2.7, the inner integral is precisely the matrix Laplace transform $L_W\{f(B)\}$. Therefore,

$$(\det(Z))^{-\frac{m+1}{2}}\int_{0<W<Z}\mathcal{L}_W\{f(B)\}\,dW.$$

The remaining arguments proceed in complete parallel with the proof of Theorem 3.1. Hence, the theorem is proved.

Theorem 3.3. *For an arbitrary Laplace transform relation* $\mathcal{L}_Z\{f(A)\}=F(Z)$ *defined in classical domain of the second type, there exists a parametric matrix* $V\in\mathfrak{R}_{II}(m)$*, such that:*

$$\mathcal{L}_Z\left\{e^{-Sp(VA)}f(A)\right\}=\mathcal{L}_{Z+V}\{f(A)\}, \tag{3.12}$$

the relationship will be appropriate.

Proof. According to *Definition 2.7* of the Laplace transform defined in classical domains of the second type (2.1):

$$\mathcal{L}_Z\{f(A)\}=\int\limits_{A>0}e^{-Sp(ZA)}f(A)\,dA$$

the relationship is valid. Now, $g(A)=e^{-Sp(VA)}f(A)$ let's look at the substitution in this relationship:

$$\mathcal{L}_Z\left\{e^{-Sp(VA)}f(A)\right\}=\int\limits_{A>0}e^{-Sp(ZA)}e^{-Sp(VA)}f(A)\,dA. \tag{3.13}$$

We have the right to write the exponentials in the integral (3.10) together, since the matrix trace has the linearity property [9,20,30]: $Sp(ZA)+Sp(VA)=Sp((Z+V)A)$.
Therefore, relation (3.10) becomes:

$$\mathcal{L}_Z\left\{e^{-Sp(VA)}f(A)\right\}=\int\limits_{A>0}e^{-Sp((Z+V)A)}f(A)\,dA=\mathcal{L}_{Z+V}\{f(A)\}.$$

The theorem has been proven.

4. Conclusion

The main result of this paper, in contrast to the results obtained in the authors previous works [14, 31], is the study of the problem of changing the order of integration in the Laplace transform defined over the classical domain of the second type for matrix-valued original functions defined on the class of symmetric matrices. In the paper, sufficient conditions ensuring the possibility of interchanging the order of integration in the corresponding Laplace transform integral are established. The obtained results make it possible to generalize several fundamental properties of the Laplace transform to the case of matrix arguments associated with Cartan classical domains and provide a rigorous analytical basis for such transformations.

The main results of the work are presented in *Theorems 3.1, 3.2, and 3.3*. These theorems reveal new important and elegant structural properties of the Laplace transform. In particular, in *Theorem 3.3*, when a parameter taken from the classical domain of the second type is introduced into the integral kernel, it appears in the basis of the Laplace transform in the form of a sum. This naturally forms a very elegant operator action and reveals an analytical relationship between the Laplace transform and the geometric structure of the classical domain under consideration. As a consequence, the obtained representation clarifies the analytical structure of the Laplace transform and demonstrates the harmony between matrix domains and integral transforms.

The results obtained in this paper may serve as a theoretical basis for further development of the theory of Laplace transforms with matrix arguments and their properties on classical domains. In future research, particular attention will be devoted to studying various applications of the obtained theorems and to investigating their use in the theory of integral transforms and functions defined on classical domains.

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