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Research article

Nonlocal fundamental splines on rectangular finite elements

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Abstract

In this work, we investigate the construction of nonlocal fundamental splines on rectangular finite elements. To achieve this, we utilize the coefficients of an optimal algebraic-trigonometric interpolation formula, derived via Sobolev's method in a Hilbert space of differentiable functions. Furthermore, we establish and prove a theorem that characterizes the essential properties of these nonlocal fundamental splines.

Keywords: *Interpolation; optimal interpolation formula; error functional; Hilbert space; approximation; fundamental spline; finite elements.*

AMS Subject Classification (2020): *Primary: 41A05, 41A15; Secondary: 65D05, 65D07.*

1. Introduction

It is known that function interpolation is used in solving many practical problems. In particular, interpolation methods are widely used in modeling complex engineering systems and reconstructing computed tomography images. In interpolation theory, it is of fundamental importance to construct optimal interpolation formulas corresponding to fundamental splines and to evaluate their errors. The problem of constructing an optimal interpolation formula was first posed and investigated in 1961 by S. L. Sobolev in the Hilbert space $W_2^{(m)}$ [1]. To this end, S. L. Sobolev utilized a discrete analogue $D_{hH}^{(m)}(h\beta)$ of the polyharmonic differential operator

$$\Delta^m = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \ldots + \frac{\partial^2}{\partial x_n^2}\right)^m .$$

The construction of the discrete operator $D_{hH}^{(m)}(h\beta)$ in the n-dimensional case is a very difficult problem. For $n = 1$, the problem of constructing optimal interpolation formulas using S. L. Sobolev's method has been extensively studied by Kh. M. Shadimetov and A. R. Hayotov [2, 3, 4, 5, 6].

In particular, a discrete analogue $\Delta_2(h\beta)$ of the differential operator $\frac{d^6}{dx^6} + 2\omega^2\frac{d^4}{dx^4} + \omega^4\frac{d^2}{dx^2}$ was constructed and its properties were investigated in [7]. Subsequently, in [8], an optimal algebraic-trigonometric interpolation formula was developed by utilizing the results obtained in [7]. In the present work, we construct nonlocal fundamental splines of two variables and investigate their key properties. To this end, we employ the coefficients of the optimal algebraic-trigonometric interpolation formula established in [8]. For the sake of completeness, a brief overview of these results is presented in the next section.

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2. An algebraic-trigonometric optimal interpolation formula in the Hilbert space $K_{2,\omega}^{(3)}$

We consider the Hilbert space defined in the form [6]

$$K_{2,\omega}^{(3)} = \{\varphi : [0,1] \to \mathbb{R} | \varphi'' \text{ is absolutely continuous and } \varphi''' \in L_2(0,1)\}.$$

This space is equipped by the norm

$$\|\varphi\|_{K_{2,\omega}^{(3)}} = \left(\int_0^1 \left(\varphi'''(x) + \omega^2 \varphi'(x) \right)^2 dx \right)^{\frac{1}{2}},$$

where $\omega \in \mathbb{R} \setminus \{0\}$.

Suppose we are given the values $\varphi(x_0), \varphi(x_1), \ldots, \varphi(x_N)$ of a function $\varphi(x) \in K_{2,\omega}^{(3)}$ at the nodes $x_k = kh$ for $k = 0, 1, \ldots, N$, where $h = \frac{1}{N}$. We consider the problem of approximating the function $\varphi(x) \in K_{2,\omega}^{(3)}$ by an elementary function $P_3(\varphi, x)$ as follows:

$$\varphi(x) \cong P_3(\varphi, x) \text{ for } x \in [0,1],$$

where

$$P_3(\varphi, x) = \sum_{\beta=0}^{N} C_\beta(x) \varphi(x_\beta) \tag{2.1}$$

is an approximating function and $C_\beta(x)$ $(\beta = 0, 1, \ldots, N)$ are its coefficients. If the approximating function $P_3(\varphi, x)$ satisfies the interpolation conditions

$$\varphi(x_\beta) = P_3(\varphi, x_\beta), \; \beta = 0, 1, \ldots, N,$$

then equality (2.1) is called an interpolation formula.

One of the main problems in interpolation theory is finding the exact upper bound for the error of the interpolation formula (2.1).

This difference

$$(\ell, \varphi) = \varphi(x) - P_3(\varphi, x),$$

is called the error of the interpolation formula (2.1). This difference represents a linear functional ℓ at a fixed point $x = z \in [0,1]$, and this functional is called the error functional of the interpolation formula (2.1). It is easy to see that the error functional ℓ is defined in the form

$$\ell(x, z) = \delta(x - z) - \sum_{\beta=0}^{N} C_\beta(z) \delta(x - x_\beta), \tag{2.2}$$

where $\delta(x)$ is Dirac's delta-function. Since the functional ℓ is defined in $K_{2,\omega}^{(3)}$, it satisfies the following conditions

$$(\ell(x,z), 1) = 0, \tag{2.3}$$

$$(\ell(x,z), \sin(\omega x)) = 0, \tag{2.4}$$

$$(\ell(x,z), \cos(\omega x)) = 0. \tag{2.5}$$

The functional ℓ is bounded and linear in the Hilbert space $K_{2,\omega}^{(3)}$. Therefore, we consider the problem of finding the maximum of this functional in the Hilbert space $K_{2,\omega}^{(3)}$.

Problem 1. Find the following quantity

$$\|\ell\| = \sup_{\substack{\varphi\in K_{2,\omega}^{(3)} \\ \|\varphi\|\neq 0}} \frac{|(\ell,\varphi)|}{\|\varphi\|_{K_{2,\omega}^{(3)}}}. \tag{2.6}$$

From (2.6), we obtain the estimate

$$|(\ell,\ \varphi)| \le \|\varphi\|_{K_{2,\omega}^{(3)}}\ \|\ell\|_{K_{2,\omega}^{(3)*}}$$

for the error of the interpolation formula (2.1). Where $K_{2,\omega}^{(3)*}$ is the conjugate space to the space $K_{2,\omega}^{(3)}$.

It should be noted that equations (2.3), (2.4) and (2.5) follow from relation (2.6).

To calculate the norm of the error functional (2.2), we use the concept of an extremal function for the interpolation formula (2.1).

Definition 2.1. (Sobolev [1]) A function $\psi_\ell \in K_{2,\omega}^{(3)}$ satisfying the equation

$$(\ell,\psi_\ell) = \|\ell\|_{K_{2,\omega}^{(3)*}}\|\psi_\ell\|_{K_{2,\omega}^{(3)}}$$

is called an extremal function for the interpolation formula (2.1).

According to the Riesz representation theorem for continuous linear functionals in a Hilbert space, there exists a unique function $\psi_\ell \in K_{2,\omega}^{(3)}$ such that for all $\varphi \in K_{2,\omega}^{(3)}$, the following equalities hold [9]:

$$(\ell,\varphi) = \langle\psi_\ell,\varphi\rangle_{K_{2,\omega}^{(3)}} \tag{2.7}$$

and

$$|\ell|K2,\omega^{(3)*} = |\psi_\ell|K2,\omega^{(3)}. \tag{2.8}$$

Here $\langle\psi_\ell,\varphi\rangle_{K_{2,\omega}^{(3)}}$ is the inner product of the functions ψ_ℓ and φ and is determined by the expression

$$\langle\psi_\ell,\varphi\rangle_{K_{2,\omega}^{(3)}} = \int_0^1 \left(\psi_\ell'''(x) + \omega^2\psi_\ell'(x)\right)\left(\varphi'''(x) + \omega^2\varphi'(x)\right)dx.$$

Thus, by solving the functional equation (2.7), we obtain the expression

$$\psi_\ell(x) = -\ell(x) * G_3(x) + d_1\sin(\omega x) + d_2\cos(\omega x) + p_0 \tag{2.9}$$

for the extremal function ψ_ℓ, corresponding to the error functional (2.2) of the interpolation formula (2.1). Here

$$G_3(x) = \frac{\operatorname{sign}(x)}{4\omega^5}\left(\omega x\cos(\omega x) - 3\sin(\omega x) + 2\omega x\right), \tag{2.10}$$

where d_1, d_2 and p_0 are constants.

For solving the functional equation (2.7), we used the method of finding the extremal function, described in Section 2 of the work [10].

As a result, using (2.7) and (2.8), taking into account equalities (2.3), (2.4) and (2.5), we obtain

$$\|\ell\|^2_{K_{2,\omega}^{(3)*}} = (\ell(x,z),\psi_\ell(x)) = \tag{2.11}$$

$$= -\sum_{\beta=0}^{N}\sum_{\gamma=0}^{N} C_\beta(z)C_\gamma(z)G_3(h\beta - h\gamma) + 2\sum_{\beta=0}^{N} C_\beta(z)G_3(z - h\beta)$$

for the solution of Problem 1.

It is known that the expression (2.11) is a quadratic function with respect to the coefficients C_β and always takes non-negative values.

An interpolation formula of the form (2.1) with coefficients $C_\beta = \mathring{C}_\beta(z)$, $(\beta = 0, 1, ..., N)$, giving the smallest value to expression (2.11) under conditions (2.3), (2.4) and (2.5), is called the optimal interpolation formula. The coefficients of the optimal interpolation formula are called optimal coefficients.

Thus, to construct the optimal interpolation formula of the form (2.1), we need to solve the following problem.

Problem 2. Find the coefficients $\mathring{C}_\beta(z)$ $(\beta = 0, 1, ..., N)$ that give the minimum to the quantity (2.11) under the conditions (2.3), (2.4) and (2.5).

In the next section we consider solving of the Problem 2.

3. Solving of the problem 2

It should be noted that the quantity $\|\ell\|_{K_{2,\omega}^{(3)*}}$ reaches its minimum value at the coefficients $\mathring{C}_\beta(z)$ $(\beta = 0, 1, ..., N)$. To substantiate this, we use Lagrange's method of undetermined multipliers for finding of conditional extremum of the multivariable function. Therefore, we consider the function

$$\Lambda\left(C_0, C_1, ..., C_N, d_1, d_2, p_0\right) =$$

$$= \|\ell\|^2_{K_{2,\omega}^{(3)*}} + 2\left(d_1\left(\ell, \sin(\omega z)\right) + d_2\left(\ell, \cos(\omega z)\right) + p_0\left(\ell, 1\right)\right).$$

To find the stationary points of this function, we take the first-order partial derivatives with respect to all its variables and equate them to zero. As a result, we obtain the following system of linear equations:

$$\sum_{\gamma=0}^{N} C_\gamma(z)G_3(h\beta - h\gamma) + d_1\sin(\omega h\beta) + d_2\cos(\omega h\beta) + p_0 = G_3(z - h\beta), \tag{3.1}$$

$$\beta = 0, 1, ..., N,$$

$$\sum_{\gamma=0}^{N} C_\gamma(z) = 1, \tag{3.2}$$

$$\sum_{\gamma=0}^{N} C_\gamma(z)\sin(\omega h\gamma) = \sin(\omega z), \tag{3.3}$$

$$\sum_{\gamma=0}^{N} C_\gamma(z)\cos(\omega h\gamma) = \cos(\omega z), \tag{3.4}$$

here $G_3(z)$ is defined by (2.10).

It should be noted that the system (3.1)-(3.4) has a unique solution provided $N + 1 \geq m$ and this solution gives the minimum to the quantity (2.11) under the conditions (2.3), (2.4) and (2.5). This statement can proved similarly as existence and uniqueness of the solution of the system (3.1)-(3.2) in the work [10]. Here we omit the proof of this statement.

To find the analytical solution the system of linear equations (3.1)-(3.3), we use the discrete analog $D_3(h\beta)$ of the differential operator $\frac{d^6}{dx^6}+2\omega^2\frac{d^4}{dx^4}+\omega^4\frac{d^2}{dx^2}$ and the properties of this discrete analog. For this purpose, we present the following theorems.

Theorem 3.1. *A discrete analogue of the differential operator* $\frac{d^6}{dx^6}+2\omega^2\frac{d^4}{dx^4}+\omega^4\frac{d^2}{dx^2}$ *has the following form [7]:*

$$D_3(h\beta) = p\begin{cases} \sum\limits_{k=1}^{2} B_k {q_k}^{|\beta|-1}, & |\beta|\geq 2,\\ 1+\sum\limits_{k=1}^{2} B_k, & |\beta|=1,\\ C+\sum\limits_{k=1}^{2} \frac{B_k}{q_k}, & \beta=0, \end{cases}$$

where p, B_k, C *and* q_k *are all known quantities.*

Theorem 3.2. *The discrete analogue* $D_3\left(h\beta\right)$ *of the differential operator* $\frac{d^6}{dx^6}+2\omega^2\frac{d^4}{dx^4}+\omega^4\frac{d^2}{dx^2}$ *satisfies the following equalities [7]*

$$\begin{aligned} &D_3\left(h\beta\right)*\sin\left(\omega h\beta\right)=0,\\ &D_3\left(h\beta\right)*\cos\left(\omega h\beta\right)=0,\\ &D_3\left(h\beta\right)*\left(\omega h\beta\right)\sin\left(\omega h\beta\right)=0,\\ &D_3\left(h\beta\right)*\left(\omega h\beta\right)\cos\left(\omega h\beta\right)=0,\\ &D_3\left(h\beta\right)*1=0,\\ &D_3\left(h\beta\right)*\left(\omega h\beta\right)=0. \end{aligned}$$

To find the analytical expression of the coefficients of the optimal interpolation formula (2.1), we use the known equality

$$C_\beta(z) = D_3(h\beta) * u(z,h\beta), \quad \text{for } \beta\in\mathbb{Z}. \tag{3.5}$$

Where $u(z,h\beta)$ is defined as follows:

$$u\left(z,h\beta\right)=\begin{cases} -\frac{1}{4\omega^5}\left(\omega h\beta\cos\left(\omega h\beta-\omega z\right)+2\omega h\beta\right)\\ +d_1^-\sin\left(\omega h\beta\right)+d_2^-\cos\left(\omega h\beta\right)+p_0^-, \beta=-1,-2,...,\\ G_3\left(z-h\beta\right), \beta=0,1,...,N,\\ \frac{1}{4\omega^5}\left(\omega h\beta\cos\left(\omega h\beta-\omega z\right)+2\omega h\beta\right)\\ +d_1^+\sin\left(\omega h\beta\right)+d_2^+\cos\left(\omega h\beta\right)+p_0^+, \beta=N+1,N+2,... \end{cases} \tag{3.6}$$

Here, all the elements on the right side of the equality are known quantities [8].

Finally, using expression (3.5), taking into account (3.6) and using Theorem 1 and Theorem 2, we obtain the following theorem for the analytical expression of the coefficients of the optimal interpolation formula (2.1).

Theorem 3.3. *The coefficients of the optimal interpolation formula* (2.1) *are determined by the following equations:*

$$\mathring{C}_0(z) = p\sum_{k=1}^{2} B_k\left(M_k+q_k^N N_k\right) \tag{3.7}$$

$$+p\left(\frac{h}{4\omega^4}(\cos(\omega h+\omega z)+2)-d_1^-\sin(\omega h)+d_2^-\cos(\omega h)+p_0^-\right)$$

$$+p\left(CG_3(z)+G_3(z-h)+\sum_{\gamma=0}^{N}\sum_{k=1}^{2}B_kq_k^{\gamma-1}G_3(z-h\gamma)\right),$$

$$\mathring{C}_\beta(z)=p\sum_{k=1}^{2}B_k\left(q_k^{\beta}M_k+q_k^{N-\beta}N_k\right) \tag{3.8}$$

$$+p\left(G_3(z-h(\beta-1))+CG_3(z-h\beta)+G_3(z-h(\beta+1))\right)$$

$$+p\sum_{\gamma=0}^{N}\sum_{k=1}^{2}B_kq_k^{|\beta-\gamma|-1}G_3(z-h\gamma),\ \beta=1,2,...,N-1,$$

$$\mathring{C}_N(z)=p\sum_{k=1}^{2}B_k\left(q_k^{N}M_k+N_k\right) \tag{3.9}$$

$$+p\left(\frac{1+h}{4\omega^4}(\cos(\omega+\omega h-\omega z)+2)+d_1^+\sin(\omega h+\omega)+d_2^+\cos(\omega h+\omega)+p_0^+\right)$$

$$+p\left(CG_3(z-1)+G_3(z-1+h)+\sum_{\gamma=0}^{N}\sum_{k=1}^{2}B_kq_k^{N-\gamma-1}G_3(z-h\gamma)\right),$$

here, all the elements on the right side of the equalities are known quantities [8].

Without affecting the generality, we take $z=x$. Now, to geometrically represent the coefficients $\mathring{C}_i(x)$, we present their graphs. The graphs of the coefficients $\mathring{C}_i(x)$ $(i=0,1,2,3,4,5)$ for the case $N=5$ are shown in Figure 1.

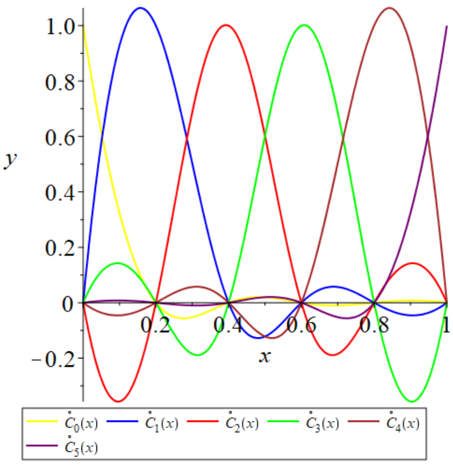

Figure 1. This figure shows the graphs of $\mathring{C}_0(x)$, $\mathring{C}_1(x)$, $\mathring{C}_2(x)$, $\mathring{C}_3(x)$, $\mathring{C}_4(x)$ and $\mathring{C}_5(x)$ when $N=5$ and $\omega=1$ in expressions (3.7)-(3.9), respectively.

These graphs also confirm that formula (2.1) is interpolational.

Remark 3.1. The optimal coefficients $\mathring{C}_i(x)$ $(i=0,1,...,N)$ are nonlocal algebraic-trigonometric fundamental splines in the Hilbert space $K_{2,\omega}^{(3)}$.

The validity of Remark 3.1 follows from the definition of the fundamental spline [11] and Figure 1.

In the next section, we consider the problem of constructing nonlocal fundamental splines of two variables.

4. Constructing nonlocal fundamental splines

Let us be given a domain $\Omega = [0,1] \times [0,1] \subset \mathbb{R}^2$ and a mesh

$$\Delta_{h_1,h_2} :=$$

$$= \left\{ (x_i,\ y_j) : x_i = ih_1,\ y_j = jh_2,\ i = 0,\ 1,\ ...,\ m,\ j = 0,\ 1,\ ...,\ n,\ h_1 m = 1,\ h_2 n = 1 \right\}$$

in this domain. Through points x_i and y_j of mesh Δ_{h_1,h_2}, we draw lines parallel to the ordinate and abscissa axes, respectively. As a result, the domain Ω is divided into rectangular parts (i.e., finite elements) of the same shapes $[x_i,\ x_{i+1}] \times [y_j,\ y_{j+1}]$ $(i = 0,\ 1,\ ...,\ m-1,\ j = 0,\ 1,\ ...,\ n-1)$.

To construct fundamental splines of two variables on finite elements $[x_i,\ x_{i+1}] \times [y_j,\ y_{j+1}]$ $(i = 0,\ 1,\ ...,\ m-1,\ j = 0,\ 1,\ ...,\ n-1)$ in the domain Ω, we use the concept of the tensor product of functions of one variable.

Definition 4.1. Let f and g be functions of one variable. We define a two-variable function $f \otimes g$ using the expression

$$f \otimes g = f \otimes g(x,\ y) = f(x)g(y). \tag{4.1}$$

The function $f \otimes g$ of two variables is called the tensor product of the functions f and g of one variable. Where the symbol $\otimes$ represents the tensor product [12].

In general, there are two methods for constructing fundamental splines of two variables. These are the construction method using the tensor product and the geometric method [12]. To simplify the calculations, we construct two-variable fundamental splines using the tensor product below. For this, we assign a function $\varphi_{i,j}(x,\ y)$, defined in the form

$$\varphi_{i,j}(x,\ y) = \varphi_i \otimes \varphi_j(x,\ y) = \varphi_i(x)\varphi_j(y)\ (i = 0,\ 1,\ ...,\ m;\ j = 0,\ 1,\ ...,\ n)$$

and satisfying the relation

$$\varphi_{i,j}(x_p,\ y_q) = \begin{cases} 1, & p = i \text{ and } q = j, \\ 0, & \text{otherwise} \end{cases}$$

i.e.,

$$\varphi_{i,j}(x_p,\ y_q) = \delta_{i,p}\delta_{j,q}\ (p = 0,\ 1,\ ...,\ m,\ q = 0,\ 1,\ ...,\ n), \tag{4.2}$$

to each node $(x_i,\ y_j)$. Where $\varphi_i \otimes \varphi_j\ (x,\ y)$ is defined by (4.1), $\varphi_i(x)$ and $\varphi_j(y)$ are determined using expressions (3.7)-(3.9), $\delta_{i,p}$ and $\delta_{j,q}$ are Kronecker symbols.

It should be noted that the functions $\varphi_{i,j}(x,\ y)$ $(i = 0,\ 1,\ ...,\ m,\ j = 0,\ 1,\ ...,\ n)$ form a finite-dimensional linear space in the domain Ω, and therefore these functions are uniquely defined by their values at the nodes $(x_i,\ y_j)$.

Through these linear transformations

$$x = -\frac{1}{a-b}s + \frac{a}{a-b},\ y = -\frac{1}{a-b}t + \frac{a}{a-b},$$

it is possible to convert the domain Ω into any arbitrary domain $[a,b] \times [a,\ b]$. Here a and b are some real numbers, and s and t are new variables.

Remark 4.1. Functions $\varphi_{i,j}(x,\ y)$ $(i = 0,\ 1,\ ...,\ m,\ j = 0,\ 1,\ ...,\ n)$ defined in the domain Ω are fundamental splines of two variables.

The validity of Remark 4.1 follows from the definition of the fundamental spline [11] and from relation (4.2).

To geometrically represent the basis functions $\varphi_{i,j}(x, y)$ $(i = 0, 1, ..., m, j = 0, 1, ..., n)$, we present their graphs for the case $m = n = 10$ and $\omega_1 = \omega_2 = 1$ in Figure 2. Here, ω_1 and ω_2 are the parameters in the expression of the functions $\varphi_i(x)$ and $\varphi_j(y)$, respectively.

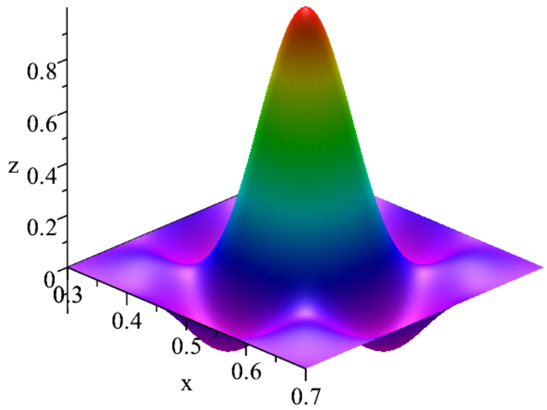

Figure 2. This figure shows the graph of the function $\varphi_{i,j}(x, y)$ $(i = 1, 2, ..., m-1, j = 1, 2, ..., n-1)$.

Theorem 4.1. *(**Theorem on the main property**) For nonlocal fundamental splines $\varphi_{i,j}(x, y)$ in the domain Ω, the following relations hold:*

$$\sum_{i=0}^{m}\sum_{j=0}^{n}\varphi_{i,j}(x, y) = 1, \tag{4.3}$$

$$\sum_{i=0}^{m}\sum_{j=0}^{n}\varphi_{i,j}(x, y)\sin(\omega_1 x_i)\sin(\omega_2 y_j) = \sin(\omega_1 x)\sin(\omega_2 y), \tag{4.4}$$

$$\sum_{i=0}^{m}\sum_{j=0}^{n}\varphi_{i,j}(x, y)\cos(\omega_1 x_i)\cos(\omega_2 y_j) = \cos(\omega_1 x)\cos(\omega_2 y), \tag{4.5}$$

$$\sum_{i=0}^{m}\sum_{j=0}^{n}\varphi_{i,j}(x, y)\sin(\omega_1 x_i)\cos(\omega_2 y_j) = \sin(\omega_1 x)\cos(\omega_2 y), \tag{4.6}$$

$$\sum_{i=0}^{m}\sum_{j=0}^{n}\varphi_{i,j}(x, y)\cos(\omega_1 x_i)\sin(\omega_2 y_j) = \cos(\omega_1 x)\sin(\omega_2 y), \tag{4.7}$$

where ω_1 and ω_2 are elements of the set $\mathbb{R} \setminus \{0\}$.

The proof of Theorem 4.1 follows from Definition 4.1 and from equalities (2.3), (2.4) and (2.5).

From relations (4.3)-(4.7), it can be stated that the nonlocal fundamental splines $\varphi_{i,j}(x,y)$ $(i=0,1,...,m,\ j=0,1,...,n)$ accurately reconstruct any linear combination of the functions 1, $\sin(\omega_1 x)\sin(\omega_2 y)$, $\cos(\omega_1 x)\cos(\omega_2 y)$, $\sin(\omega_1 x)\cos(\omega_2 y)$ and $\cos(\omega_1 x)\sin(\omega_2 y)$.

It should be noted that nonlocal fundamental splines $\varphi_{i,j}(x,y)$ $(i=0,1,...,m,\ j=0,1,...,n)$ can be applied to finite element methods and signal processing.

5. Conclusion

In this article, a bounded region on a plane is divided into rectangular finite elements, and nonlocal fundamental splines are constructed on these finite elements. For this, the coefficients of the algebraic-trigonometric optimal interpolation formula constructed in the Hilbert space were used. Also, the theorem expressing the main property of these fundamental splines was proven.

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