

The space of order-preserving functionals and Π -completeness

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ABSTRACT

For a Tychonoff space X , let $O_f(X)$ denote the space of order-preserving functionals. An open cover γ_0 of $O_f(X)$ is constructed from a cover γ_X of X , and it is shown that disjointness of γ_X implies disjointness of γ_0 . It is further proved that the space $O_f(\beta, X)$ of order-preserving functionals on the Stone-Čech compactification βX is a perfect compactification of $O_f(X)$. As a consequence of the Π -completeness criterion, $O_f(X)$ is Π -complete if and only if X is Π -complete.

Keywords: Π -complete space, perfect compactification, order-preserving functional, finite-component cover.

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1. Introduction

The interplay between completeness properties and compactifications is an important theme in the study of Tychonoff spaces and has attracted considerable attention in the literature. In this context, various notions of completeness have been introduced to describe more refined topological properties of spaces and their compactifications. One such notion is Π -completeness, which was introduced and studied in [1]. Several related classes of spaces, referred to as compact-type spaces, were considered in [2]. The behaviour of hyperspace functors on such spaces was investigated in [3], where the criterion for Π -completeness of Tychonoff spaces established in [4] played an essential role.

The space of order-preserving functionals on a compact Hausdorff space was studied in [5]. In particular, the normality properties introduced in [6] were established for the functor O of order-preserving functionals on the category $Comp$ of compact Hausdorff spaces and continuous maps, with the exception of the preservation of preimages. Idempotent probability measures form a special class of weakly additive, order-preserving, normalized functionals and have applications in several areas; see [7], [8] and the references therein. Using the construction proposed in [9], the functor O was extended in [10] to the category $\mathcal{T}ych$ of Tychonoff spaces and continuous maps.

The aim of the present paper is to characterize the Π -completeness of the space $O_f(X)$ in terms of the Π -completeness of the underlying Tychonoff space X . The proof is based on the criterion from [4]. A key step is to show that $O_f(\beta, X)$, where βX denotes the Stone-Čech compactification of X , is a perfect compactification of $O_f(X)$. We also associate with a system γ_X of subsets of X a corresponding system γ_0 in $O_f(X)$ and show that openness and disjointness are preserved under this construction. Using these results, we prove that $O_f(X)$ is Π -complete if and only if X is Π -complete.

2. Preliminaries

In the present paper, by a space we mean a topological T_1 -space, and by a map we mean a continuous map.

A collection ω of subsets of a set X is said to be star countable (respectively, star-finite) if each element of ω intersects at most countably many (respectively, finitely many) elements of ω . A collection ω of subsets of a set X *refines* a collection Ω of subsets of X if for each element $A \in \omega$ there is an element $B \in \Omega$ such that $A \subset B$. It is also said that ω is a *refinement* of Ω . For a point $x \in X$ and a natural number n the inequality $Kp(x, \omega) \leq n$ means that no more than n elements of ω contain x ([11], p. 270), and $Kp \omega \leq n$ if $Kp(x, \omega) \leq n$ for every $x \in X$.

A finite sequence of subsets $M_0, \dots, M_s$ of a set X is called a chain [1] connecting the sets M_0 and M_s if $M_{i-1} \cap M_i \neq \emptyset$ for all $i = 1, \dots, s$. A collection ω of subsets of a set X is said to be *connected* if for any pair of sets $M, M' \subset X$ there exists a chain ω connecting the sets M and M' . The maximal connected sub-collections of ω are called *components* of ω . A star-finite open cover of a space X is said to be a *finite-component cover* if the number of elements in each component is finite.

For a collection $\omega = \{M_\alpha : \alpha \in A\}$ of subsets of a space X we put $[\omega] = [\omega]_X = \{[M_\alpha]_X : \alpha \in A\}$. For a topological space X , a subspace $W \subseteq X$, and a point $x \in X \setminus W$, a cover λ of W by sets open in X pricks out the point x in X if $x \notin \cup[\lambda]_X$ [2].

For a Tychonoff space X , let βX be its Stone-Čech compactification (i. e. the maximal compact extension).

Definition 2.1. [2] A Tychonoff space X is said to be Π -complete if for every point $x \in \beta X \setminus X$ there exists a finite component cover ω of X which pricks out the point x in βX .

Let νX be a compact extension of a Tychonoff space X . If $H \subset X$ is an open set in X , then by $\mathfrak{D}(H)$ (or by $\mathfrak{D}_{\nu X}(H)$) we denote the maximal open set (concerning inclusion) in νX satisfying $\mathfrak{D}_{\nu X}(H) \cap X = H$. It is easy to see that

$$\mathfrak{D}_{\nu X}(H) = \bigcup_{\substack{\Gamma \in \tau_{\nu X}, \\ \Gamma \cap X = H}} \Gamma$$

where $\tau_{\nu X}$ is the topology of the compact extension νX .

For a topological space X and its subset A , a set $Fr_X A = [A]_X \cap [X \setminus A]_X = [A]_X \setminus Int_X A$ is called a boundary of A in X . A compactification νX of a Tychonoff space X is called *perfect with respect to an open set H in X* if the equality $[Fr_X H]_{\nu X} = Fr_{\nu X} \mathfrak{D}_{\nu X}(H)$ holds. If νX is perfect for every open set in X , then it is called a *perfect compactification* of the space X ([11], p. 232).

A compactification νX of space X is perfect if and only if for any two disjoint open sets U_1 and U_2 in X the equality $\mathfrak{D}(U_1 \cup U_2) = \mathfrak{D}(U_1) \cup \mathfrak{D}(U_2)$ holds [11]. The Stone-Čech compactification βX of a Tychonoff space X is a perfect compactification of X . The equality $\mathfrak{D}(U_1 \cup U_2) = \mathfrak{D}(U_1) \cup \mathfrak{D}(U_2)$ holds for every pair of open sets U_1 and U_2 in X if and only if X is normal, and the compactification νX coincides with the Stone-Čech compactification βX , i. e. $\nu X \cong \beta X$.

The following criterion plays a key role in investigating the class of Π -complete spaces ([4], Theorem 1.1. Π), see pages 16-17).

Theorem 2.1. [4] A Tychonoff space X is Π -complete if and only if for every $x \in \beta X \setminus X$ of an arbitrary perfect compactification bX there exists an open $bX \setminus X$ cover ω of X with $Kp \omega = 1$, pricking out x in bX (i. e. $x \notin \cup[\omega]_{bX}$).

Since the Stone-Čech compactification βX of a Tychonoff space X is a perfect compactification of X , Theorem 2.1 implies the following assertion.

Corollary 2.1. A Tychonoff space X is Π -complete if and only if for every $x \in \beta X \setminus X$ there exists a cover ω of X with $Kp \omega = 1$, pricking out x in βX .

Note that every compact Hausdorff space is a Π -complete space. The square of the Sorgenfrey line is Π -complete, but it is not a paracompact space (hence, it is not a compact Hausdorff space). The space $T(\omega_1)$ of all ordinal numbers less than the first uncountable ordinal number ω_1 is a normal space but it is not Π -complete.

Let us list some known properties of Π -complete spaces.

1. A closed subset of a Π -complete space is Π -complete ([1], p. 19).
2. If $f: X \rightarrow Y$ is a perfect map in a Π -complete space Y then X is also Π -complete ([1], p. 26).
3. A Π -complete space is complete in the Dieudonné sense ([1], p. 18.)

The author of [5] studied the functor $O: \mathfrak{Comp} \rightarrow \mathfrak{Comp}$ and showed that it satisfies the normality conditions except for the preservation of preimages. Then, using the construction suggested by A.Ch. Chigogidze [9], an extension $O_\beta: \mathfrak{Tych} \rightarrow \mathfrak{Tych}$ was obtained in [10]. Here, $\mathfrak{Comp}$ denotes the category of compact Hausdorff spaces and their continuous maps, and $\mathfrak{Tych}$ the category of Tychonoff spaces and their continuous maps.

For a compact Hausdorff space X a functional $\mu: C(X) \rightarrow \mathbb{R}$ is said [5] to be:

- 1) *normalized* if $\mu(c_X) = c$ for every constant function $c_X: X \rightarrow \mathbb{R}$, $c \in \mathbb{R}$. Here $c_X(x) = c$;
- 2) *weakly additive* if $\mu(\varphi + c) = \mu(\varphi) + c$ for every $c \in \mathbb{R}$ and each $\varphi \in C(X)$;
- 3) *order-preserving* if $\mu(\varphi) \leq \mu(\psi)$ for any pair of functions $\varphi, \psi \in C(X)$ with $\varphi \leq \psi$.

For brevity, a weakly additive, order-preserving, normalized functional is called an order-preserving functional. A set of all order-preserving functionals on $C(X)$ is denoted by $O(X)$. It is endowed with the topology τ_p of pointwise convergence. For $\mu \in O(X)$ sets

$$\langle \mu; \varphi_1, \dots, \varphi_n; \theta \rangle = \{ \nu \in O(X) : |\nu(\varphi_i) - \mu(\varphi_i)| < \theta, i = 1, \dots, n \}$$

form a base of $O(X)$ at μ . Here $\varphi, \dots, \varphi_n \in C(X)$, $\theta > 0$.

Let A be a closed subset of a compact Hausdorff space X . An order-preserving functional $\mu \in O(X)$ is *supported* on A if $\mu \in O(A) \subset O(X)$. The support of $\mu \in O(X)$ is denoted by $\text{supp } \mu$. For a positive integer n let $O_n(X)$ denote a set of all order-preserving functionals whose support consists no more than n points. Put $O_\omega(X) = \bigcup_{n=1}^{\infty} O_n(X)$. An order-preserving functional $\mu \in O_\omega(X)$ is said to be an order-preserving functional with finite support. For a point $x \in X$ the Dirac measure $\delta_x: C(X) \rightarrow \mathbb{R}$ is defined by equality $\delta_x(\varphi) = \varphi(x)$ for all $\varphi \in C(X)$. Clearly, $\text{supp } \delta_x = \{x\}$.

Following [12] an order-preserving functional $\mu \in O_\omega(X)$ with finite support $\text{supp } \mu = \{x_1, \dots, x_n\}$ can be represented as a relation $\mu = \mu(\delta_{x_1}, \dots, \delta_{x_n})$ between the Dirac measures δ_{x_i} , $i = 1, \dots, n$. This relation has a view $\mu(\varphi) = \mu(\varphi(x_1), \dots, \varphi(x_n))$, $\varphi \in C(X)$.

We consider the following types of “restrictions”:

$$\begin{aligned} \mu_{x_i}(\varphi) &= \mu(\epsilon, \dots, \epsilon, \varphi(x_i), \epsilon, \dots, \epsilon), \\ \mu^{x_i}(\varphi) &= \mu(\varphi(x_1), \dots, \varphi(x_{i-1}), \epsilon, \varphi(x_{i+1}), \dots, \varphi(x_n)), \end{aligned} \tag{2.1}$$

for each $\varphi \in C(X)$. Here $\epsilon = -\infty, 0$ or $+\infty$ (for more details, see below).

For α , $-\infty < \alpha < +\infty$, and a subset $A \subset O(X)$ we put $\alpha A = \{\alpha\mu: \mu \in A\}$. If $\alpha = 0$ then $\alpha A = \{0\}$, and if $\alpha = 1$ then $\alpha A = A$. Note that for numbers α, β , subsets $A, B \subset O(X)$ we have $\alpha A + \beta B = \{\alpha\mu + \beta\nu: \mu \in A, \nu \in B\} \subset O(X)$ if and only if $\alpha, \beta \geq 0$ and $\alpha + \beta = 1$. Moreover, the set $O(X)$ of all order-preserving functionals is closed under affine combination. In other words, for $\{\mu_i: i = 1, \dots, n\} \subset O(X)$ and $\{\alpha_i: i = 1, \dots, n\} \subset [0, 1]$ with $\sum_{i=1}^n \alpha_i = 1$ we have $\sum_{i=1}^n \alpha_i \mu_i \in O(X)$. Obviously, $\alpha_i \mu_i \in \alpha_i O(X)$, and

$\sum_{i=1}^n \alpha_i O(X) = O(X)$. If $0 \leq \alpha_i < 1$ then $\alpha_i O(X) \neq O(X)$. In the case $0 < \alpha_i < 1$ we will say that $\alpha_i O(X)$ is parallel to $O(X)$. For an open neighborhood $\langle \mu_i; \varphi_{i1}, \dots, \varphi_{in_i}; \theta_i \rangle$ the set $\alpha_i \langle \mu_i; \varphi_{i1}, \dots, \varphi_{in_i}; \theta_i \rangle$ is an open in $\alpha_i O(X)$ neighborhood of $\alpha_i \mu_i$, $i = 1, \dots, n$. Consequently, the sum $\sum_{i=1}^n \alpha_i \langle \mu_i; \varphi_{i1}, \dots, \varphi_{in_i}; \theta_i \rangle$ is open in $\sum_{i=1}^n \alpha_i O(X) = O(X)$.

Thus, affine combination is one of the rules for forming an order-preserving functional, using a given (at most countable) set of order-preserving functionals. In this case, any order-preserving functional μ with finite support has the form $\mu = \sum_{i=1}^n \alpha_i \delta_{x_i}$ for $\alpha_i \geq 0$, $i = 1, \dots, n$, and $\sum_{i=1}^n \alpha_i = 1$. Consequently, for an arbitrary $\varphi \in C(X)$ we have

$$\begin{aligned}\mu_{x_i}(\varphi) &= \alpha_i \varphi(x_i), \\ \mu^{x_i}(\varphi) &= \alpha_1 \varphi(x_1) + \dots + \alpha_{i-1} \varphi(x_{i-1}) + \alpha_{i+1} \varphi(x_{i+1}) + \dots + \alpha_n \varphi(x_n).\end{aligned}$$

Since $\varphi \in C(X)$ is arbitrary, in this case equalities (2.1) take the form

$$\begin{aligned}\mu_{x_i} &= \mu(0, \dots, 0, \delta_{x_i}, 0, \dots, 0) = \alpha_i \delta_{x_i}, \\ \mu^{x_i} &= \mu(\delta_{x_1}, \dots, \delta_{x_{i-1}}, 0, \delta_{x_{i+1}}, \dots, \delta_{x_n}) = \\ &= \alpha_1 \delta_{x_1} + \dots + \alpha_{i-1} \delta_{x_{i-1}} + \alpha_{i+1} \delta_{x_{i+1}} + \dots + \alpha_n \delta_{x_n}.\end{aligned}$$

Let us consider two other rules. For α , $-\infty \leq \alpha \leq +\infty$, and a subset $A \subset O(X)$ we put $\alpha + A = \{\alpha + \mu : \mu \in A\}$. Clearly, if $\alpha = 0$ then $\alpha + A = A$ and it lies in $O(X)$. In the case $\alpha \neq 0$ and $-\infty < \alpha < +\infty$ the set $\alpha + A$ does not lie in $O(X)$ but lies in the “parallel” set $\alpha + O(X)$ in $W(X)$ of all order-preserving, weakly additive (not necessarily normalized) functionals on $C(X)$. In the cases $\alpha = \pm\infty$ we get singletons $\alpha + A = \{-\infty\}$ or $\alpha + A = \{+\infty\}$ which can also be considered as “parallel” sets.

Let

$$\begin{aligned}\max\{\alpha + A, \beta + B\} &= \{\max\{\alpha + \mu, \beta + \nu\} : \mu \in A, \nu \in B\}, \\ \min\{\alpha + A, \beta + B\} &= \{\min\{\alpha + \mu, \beta + \nu\} : \mu \in A, \nu \in B\}\end{aligned}$$

for some α, β , $-\infty \leq \alpha, \beta \leq +\infty$ and $A, B \subset O(X)$. Evidently, $\max\{\alpha + A, \beta + B\} \subset O(X)$ if and only if $-\infty \leq \alpha, \beta \leq 0$ and $\max\{\alpha, \beta\} = 0$. And also, $\min\{\alpha + A, \beta + B\} \subset O(X)$ if and only if $0 \leq \alpha, \beta \leq +\infty$ and $\min\{\alpha, \beta\} = 0$. In these two cases, any order-preserving functional μ with finite support has one of the following forms:

$$\begin{aligned}\mu &= \max_{i \in \{1, \dots, n\}} \{\alpha_i + \delta_{x_i}\} \text{ for } -\infty \leq \alpha_i \leq 0, \quad i = 1, \dots, n, \text{ and } \max_{i \in \{1, \dots, n\}} \{\alpha_i\} = 0; \\ \mu &= \min_{i \in \{1, \dots, n\}} \{\alpha_i + \delta_{x_i}\} \text{ for } 0 \leq \alpha_i \leq +\infty, \quad i = 1, \dots, n, \text{ and } \min_{i \in \{1, \dots, n\}} \{\alpha_i\} = 0.\end{aligned}$$

For an arbitrary $\varphi \in C(X)$ we have

$$\begin{aligned}\mu_{x_i}(\varphi) &= \alpha_i + \varphi(x_i), \\ \mu^{x_i}(\varphi) &= \max\{\alpha_1 + \varphi(x_1), \dots, \alpha_{i-1} + \varphi(x_{i-1}), \alpha_{i+1} + \varphi(x_{i+1}), \dots, \alpha_n + \varphi(x_n)\}\end{aligned}$$

in the maximum case, and

$$\begin{aligned}\mu_{x_i}(\varphi) &= \alpha_i + \varphi(x_i), \\ \mu^{x_i}(\varphi) &= \min\{\alpha_1 + \varphi(x_1), \dots, \alpha_{i-1} + \varphi(x_{i-1}), \alpha_{i+1} + \varphi(x_{i+1}), \dots, \alpha_n + \varphi(x_n)\}\end{aligned}$$

in the minimum case.

By arbitrariness of $\varphi \in C(X)$ we get

$$\begin{aligned}\mu_{x_i} &= \max\{-\infty, \dots, -\infty, \alpha_i + \delta_{x_i}, -\infty, \dots, -\infty\} = \alpha_i + \delta_{x_i}, \\ \mu^{x_i} &= \max\{\alpha_1 + \delta_{x_1}, \dots, \alpha_{i-1} + \delta_{x_{i-1}}, -\infty, \alpha_{i+1} + \delta_{x_{i+1}}, \dots, \alpha_n + \delta_{x_n}\} = \\ &= \max\{\alpha_1 + \delta_{x_1}, \dots, \alpha_{i-1} + \delta_{x_{i-1}}, \alpha_{i+1} + \delta_{x_{i+1}}, \dots, \alpha_n + \delta_{x_n}\}\end{aligned}$$

and

$$\begin{aligned}\mu_{x_i} &= \min\{+\infty, \dots, +\infty, \alpha_i + \delta_{x_i}, +\infty, \dots, +\infty\} = \alpha_i + \delta_{x_i}, \\ \mu^{x_i} &= \min\{\alpha_1 + \delta_{x_1}, \dots, \alpha_{i-1} + \delta_{x_{i-1}}, +\infty, \alpha_{i+1} + \delta_{x_{i+1}}, \dots, \alpha_n + \delta_{x_n}\} = \\ &= \min\{\alpha_1 + \delta_{x_1}, \dots, \alpha_{i-1} + \delta_{x_{i-1}}, \alpha_{i+1} + \delta_{x_{i+1}}, \dots, \alpha_n + \delta_{x_n}\}.\end{aligned}$$

Now we note the following remarkable property.

Remark 2.1. For every pair of $\alpha, \beta, -\infty \leq \alpha, \beta \leq 0$, with $\max\{\alpha, \beta\} = 0$, we have

$$\max\{\alpha + O(X), \beta + O(X)\} = O(X).$$

Similarly, for every pair of $\alpha, \beta, 0 \leq \alpha, \beta \leq +\infty$, with $\min\{\alpha, \beta\} = 0$, we have

$$\min\{\alpha + O(X), \beta + O(X)\} = O(X).$$

These properties can be demonstrated by direct verification. So, for $\mu, \nu \in O(X)$, and $\alpha, \beta, -\infty \leq \alpha, \beta \leq 0$, with $\max\{\alpha, \beta\} = 0$, the maximum $\max\{\alpha + \mu, \beta + \nu\}$ is an order-preserving, weakly additive, normalized functional on $C(X)$. Analogously, for $\mu, \nu \in O(X)$, and $\alpha, \beta, 0 \leq \alpha, \beta \leq +\infty$, with $\min\{\alpha, \beta\} = 0$ the minimum $\min\{\alpha + \mu, \beta + \nu\}$ is an order-preserving, weakly additive, normalized functional on $C(X)$.

Proposition 2.1. Let $\langle \mu; \varphi_1, \dots, \varphi_n; \theta \rangle$ and $\langle \nu; \psi_1, \dots, \psi_k; \delta \rangle$ be open neighborhoods of order-preserving functionals μ and ν , respectively. Then the sets

$$\begin{aligned}&\max\{\alpha + \langle \mu; \varphi_1, \dots, \varphi_n; \theta \rangle, \beta + \langle \nu; \psi_1, \dots, \psi_k; \delta \rangle\} \\ &(\text{for each pair of } \alpha, \beta, -\infty \leq \alpha, \beta \leq 0 \text{ with } \max\{\alpha, \beta\} = 0)\end{aligned}$$

and

$$\begin{aligned}&\min\{\alpha + \langle \mu; \varphi_1, \dots, \varphi_n; \theta \rangle, \beta + \langle \nu; \psi_1, \dots, \psi_k; \delta \rangle\} \\ &(\text{for each pair of } \alpha, \beta, 0 \leq \alpha, \beta \leq +\infty \text{ with } \min\{\alpha, \beta\} = 0)\end{aligned}$$

are open in $O(X)$.

Proof. It is enough to establish the openness of one of the sets. The openness of the other one will be established similarly. We will prove the openness of the first set. Let $\Omega = \max\{\alpha + \langle \mu; \varphi_1, \dots, \varphi_n; \theta \rangle, \beta + \langle \nu; \psi_1, \dots, \psi_k; \delta \rangle\}$.

Three cases are possible.

1) $-\infty \leq \alpha < \beta = 0$. Then $\Omega \cap O(X) = \langle \nu; \psi_1, \dots, \psi_k; \delta \rangle$; in this case the set $\alpha + \langle \mu; \varphi_1, \dots, \varphi_n; \theta \rangle$ lies in the “parallel” space $\alpha + O(X)$ as an open subset.

2) $-\infty \leq \beta < \alpha = 0$. Then $\Omega \cap O(X) = \langle \mu; \varphi_1, \dots, \varphi_n; \theta \rangle$, and $\beta + \langle \nu; \psi_1, \dots, \psi_k; \delta \rangle$ is open in the “parallel” space $\beta + O(X)$.

So, in the above two cases, in general, the set $\alpha + \langle \mu; \varphi_1, \dots, \varphi_n; \theta \rangle$ is open in the space $\alpha + O(X)$, and $\beta + \langle \nu; \psi_1, \dots, \psi_k; \delta \rangle$ in $\beta + O(X)$. Consequently, the set Ω is open in $\max\{\alpha + O(X), \beta + O(X)\} = O(X)$.

3) $\alpha = \beta = 0$. Then $\Omega \cap O(X) = \Omega$ is open in $\max\{O(X), O(X)\} = O(X)$. Proposition 2.1 is proved. $\square$

Remark 2.2. Consider any finite set of elements $\alpha_i \in [-\infty, 0]$ with $\max\{\alpha_i: i = 1, \dots, n\} = 0$ ($\alpha_i \in [0, +\infty]$ with $\min\{\alpha_i: i = 1, \dots, n\} = 0$). Then the first (second) equality in Remark 2.1 can be rewritten as

$$\max\{\alpha_i + O(X): i = 1, \dots, n\} = O(X) \quad (\min\{\alpha_i + O(X): i = 1, \dots, n\} = O(X)).$$

Consequently, Proposition 2.1 can be reformulated for any finite maximum (minimum).

Note that there are many such rules for forming an order-preserving functional using a given set of order-preserving functionals on $C(X)$ even in the case where X consists of two points. An example showing the existence of such rules was constructed by A. Zaitov (see [13]). Let ${}^Z f$ be such a rule. Then, to a given set of order-preserving functionals $\{\mu_t: t \in \mathfrak{T}\}$, it assigns an order-preserving functional μ_f , i. e. ${}^Z f((\mu_t)_{t \in \mathfrak{T}}) = \mu_f$. If ${}^Z f$ is an affine combination, then the set $\mathfrak{T}$ has at most countably many indices; if ${}^Z f$ is the rule of taking the maximum or minimum, then $|\mathfrak{T}| \leq |O(X)|$.

For an arbitrary order-preserving, weakly additive, normalized functional μ , it is easy to see that

$$\min\{\varphi(x): x \in \text{supp } \mu\} \leq \mu(\varphi) \leq \max\{\varphi(x): x \in \text{supp } \mu\}$$

for each $\varphi \in C(X)$.

The following results were obtained by A. Zaitov.

Theorem 2.2. For a compact Hausdorff space X , every order-preserving functional $\mu \in O(X)$ whose support has no isolated points and each function $\varphi \in C(X)$ there is a point $x_0 \in \text{supp } \mu$ such that

$$\mu(\varphi) = \varphi(x_0).$$

Here the absence of isolated points in the support $\text{supp } \mu$ (which is a compact Hausdorff space) is essential. Indeed, if $x' \in \text{supp } \mu$ is an isolated point, then, taking any $x'' \in \text{supp } \mu$, consider the order-preserving functional $\mu = \frac{1}{2}\delta_{x'} + \frac{1}{2}\delta_{x''}$. Let $\varphi \in C(X)$ such that $\varphi(x') = 1$ and $\varphi(x) = 0$ for all $x \in X$. Then $\mu(\varphi) = \frac{1}{2}\varphi(x') = \frac{1}{2}$, and there are no points $x \in \text{supp } \mu$ satisfying equality $\mu(\varphi) = \varphi(x)$.

Theorem 2.3. For a compact Hausdorff space X , any order-preserving functional $\mu \in O(X)$, and every point $x_0 \in \text{supp } \mu$, there is a function $\varphi \in C(X)$ satisfying the equation

$$\mu(\varphi) = \varphi(x_0).$$

3. Π -completeness of the space $O_f(X)$ of order-preserving functionals

For a compact Hausdorff space X , define

$$\begin{aligned} O_f(X) = \Big\{ \mu \in O_\omega(X): \text{ there exists a unique point } x \in \text{supp } \mu = \{x_1, \dots, x_n\} \\ \text{ such that } \left(\mu_x(1_X) \geq \frac{n}{n+1} \text{ and } \mu^x(1_X) \leq \frac{1}{n+1} \right) \text{ or} \\ \left(\mu_x(1_X) = 1 \text{ and } |\mu^x(0_X)| \geq \frac{n}{n+1} \right) \Big\}. \end{aligned} \quad (3.1)$$

This construction was suggested in [14]. The topological and categorical properties of the space $P_f(X)$ of probability measures were studied in [15], while the corresponding properties of the space $I_f(X)$ of idempotent probability measures were studied in [16]. Note that $P_f(X)$ and $I_f(X)$ are subsets of $O_f(X)$.

For a compact Hausdorff space X , the set $O_f(X)$ is closed in $O(X)$.

For a point $x \in X$, define

$$O_f^x(X) = \left\{ \mu = \mu(\delta_{x_1}, \dots, \delta_{x_n}) \in O_f(X) : x \in \text{supp } \mu, \right. \\ \left. \mu_x(1_X) \geq \frac{n}{n+1} \text{ or } |\mu^x(0_X)| \geq \frac{n}{n+1} \right\}.$$

It is easy to see that

$$O_f^x(X) \cap O_f^y(X) = \emptyset$$

for every pair of distinct points $x, y \in X$, and

$$O_f(X) = \bigcup_{x \in X} O_f^x(X).$$

For a subset $M \subset X$, define

$$\langle M \rangle = \left\{ \mu \in O_f(X) : \text{there exists } x \in M \text{ such that } \mu \in O_f^x(X) \right\}.$$

Proposition 3.1. *For an open set $U \subset X$, the set $\langle U \rangle$ is open in $O_f(X)$.*

Proof. Clearly, $\delta_x \in \langle U \rangle$ for every $x \in U$. Indeed, $\delta_x \in O_f^x(X)$ for each $x \in X$, and hence, in particular, for each $x \in U$.

Fix x_0 and $\mu_0 \in O_f^{x_0}(X)$ with $|\text{supp } \mu_0| \geq 2$. Consider a function $\varphi \in C(X)$, $\varphi: X \rightarrow [0, 1]$, such that $\varphi(x_0) = 1$ and $\varphi^{-1}\left(\left(\frac{1}{2}, 1\right]\right) = U$. Then $\frac{2}{3} \leq \mu_{0,x_0}(\varphi) \leq \mu_0(\varphi) \leq 1$, i. e., $\frac{2}{3} \leq \mu_0(\varphi) \leq 1$.

Let $\nu \in \langle \mu_0; \varphi; \frac{1}{6} \rangle \cap O_f(X)$. Suppose that $\text{supp } \nu = \{z\}$ and $z \notin U$. Then $\nu = \delta_z$ and $\nu(\varphi) = \varphi(z) \leq \frac{1}{2}$. Combining $\frac{2}{3} \leq \mu_0(\varphi) \leq 1$ and $-\frac{1}{2} \leq -\nu(\varphi) \leq 0$, we obtain $\frac{1}{6} \leq \mu_0(\varphi) - \nu(\varphi) \leq 1$, which yields a contradiction. Hence, $z \in U$, i. e., $\nu \in \langle U \rangle$.

Consider now the case $|\text{supp } \nu| \geq 2$. Let $\nu \in O_f^y(X)$. Then $\nu(\varphi) \geq \nu_y(\varphi) \geq \frac{2}{3}\varphi(y)$. Assuming $y \notin U$, we get $-\frac{1}{3} \leq -\nu(\varphi) \leq 0$, and consequently, $\frac{1}{3} \leq \mu_0(\varphi) - \nu(\varphi) \leq 1$. This again yields a contradiction. Hence, $y \in U$, and therefore $\nu \in \langle U \rangle$.

Thus, $\langle \mu_0; \varphi; \frac{1}{6} \rangle \cap O_f(X) \subset \langle U \rangle$. This proves Proposition 3.1. $\square$

Proposition 3.1 immediately implies the following statement for closed subsets.

Corollary 3.1. *For a closed set $F \subset X$, the set $\langle F \rangle$ is closed in $O_f(X)$.*

It is easy to see that

$$[\langle M \rangle]_{O_f(X)} = \langle [M]_X \rangle \quad (3.2)$$

for an arbitrary $M \subset X$.

For a Tychonoff space X , by the definition given in [10], the set $O_\beta(X)$ has the form

$$O_\beta(X) = \{\mu \in O(\beta X) : \text{supp } \mu \subset X\}.$$

Since the support $\text{supp } \mu$ of every order-preserving functional $\mu \in O_\beta(X)$ is compact, μ is called an order-preserving functional with compact support. The set $O_\beta(X)$ is considered as a subspace of $O(\beta X)$.

For a Tychonoff space X , the set $O_f(X)$ is defined by the same rule as in (3.1).

Lemma 3.1. *Let U and V be nonempty subsets of a Tychonoff space X . Then*

$$\langle U \rangle \cap \langle V \rangle \neq \emptyset$$

if and only if

$$U \cap V \neq \emptyset.$$

Corollary 3.2. *Let U and V be nonempty subsets of a Tychonoff space X . Then*

$$\langle U \rangle \subset \langle V \rangle$$

if and only if

$$U \subset V.$$

Lemma 3.2. *For an arbitrary open cover $\gamma_X = \{U_\alpha : \alpha \in \mathfrak{A}\}$ of a Tychonoff space X , the system*

$$\gamma_0 = \{\langle U \rangle : U \in \gamma\}$$

is an open cover of $O_f(X)$.

Proof. It suffices to verify that

$$\bigcup \gamma_0 \supset O_f(X).$$

Let μ be an arbitrary element of $O_f(X)$. There exists a point $x \in X$ such that $x \in \text{supp } \mu$ and $\mu \in O_f^x(X)$. Consequently, $\mu \in \langle U_x \rangle$ for each open neighborhood U_x of x . Since γ_X is an open cover of X , there exists a set $U \in \gamma$ containing x . Hence, $\mu \in \langle U \rangle$. Thus, every element of $O_f(X)$ belongs to some element of γ_0 , which completes the proof. $\square$

Proposition 3.2. *Let X be a Tychonoff space and $\gamma_X = \{U_\alpha : \alpha \in \mathfrak{A}\}$ an open cover of X . If $Kp \gamma_X = 1$, then $Kp \gamma_0 = 1$ for the cover*

$$\gamma_0 = \{\langle U \rangle : U \in \gamma\}$$

of $O_f(X)$.

Proof. Let $\langle U \rangle$ and $\langle V \rangle$ be elements of γ_0 . Since $Kp \gamma_X = 1$, Lemma 3.1 implies that $\langle U \rangle \cap \langle V \rangle \neq \emptyset$ if and only if $U = V$. Hence, $Kp \gamma_0 = 1$. Lemma 3.2 completes the proof. $\square$

The following statement will be used in the proof of the next result.

Lemma 3.3. [3] *Let γX be a compact extension of a space X , and let V and W be disjoint open sets in γX . Let $V \cap X = V_\gamma$ and $W \cap X = W_\gamma$. Then*

$$[X \setminus V_\gamma]_{\gamma X} \cap [X \setminus W_\gamma]_{\gamma X} = [X \setminus (V_\gamma \cup W_\gamma)]_{\gamma X}.$$

Note that $O_f(X)$ is dense in $O_f(\beta X)$ and $O_f(\beta X)$ is a compact Hausdorff space. Therefore, $O_f(\beta X)$ is a compact extension of $O_f(X)$. The following theorem gives a more precise description of this compactification.

Theorem 3.1. *For a Tychonoff space X the space $O_f(\beta X)$ of all order-preserving functionals on $C(\beta X) \cong C_b(X)$ is a perfect compactification of the space $O_f(X)$.*

Proof. It is enough to consider open subsets of $O_f(X)$ of the form $\langle U \rangle$, where $U \subset X$ is open. Let U_1 and U_2 be disjoint open sets in X . Since βX is a perfect compactification of X , we have

$$\mathfrak{D}_{\beta X}(U_1 \cup U_2) = \mathfrak{D}_{\beta X}(U_1) \cup \mathfrak{D}_{\beta X}(U_2)$$

(see Corollary 2.1). Consider the open sets

$$\langle U_i \rangle \quad \text{in} \quad O_f(X), \quad i = 1, 2.$$

By Lemma 3.2, $\langle U_1 \rangle \cap \langle U_2 \rangle = \emptyset$. We will show that

$$\mathfrak{D}_{O_f(\beta X)}(\langle U_1 \rangle \cup \langle U_2 \rangle) = \mathfrak{D}_{O_f(\beta X)}(\langle U_1 \rangle) \cup \mathfrak{D}_{O_f(\beta X)}(\langle U_2 \rangle).$$

The inclusion $\supset$ follows directly from the definition of sets of the form $\mathfrak{D}(H)$. Therefore, it remains to prove the reverse inclusion. Let $\mu' \in O_f(\beta X)$ be an order-preserving functional such that

$$\mu' \notin \mathfrak{D}_{O_f(\beta X)}(\langle U_1 \rangle) \cup \mathfrak{D}_{O_f(\beta X)}(\langle U_2 \rangle).$$

Then $\mu' \in O_f(\beta X) \setminus \mathfrak{D}_{O_f(\beta X)}(\langle U_i \rangle)$, $i = 1, 2$. By [11] (see p. 234), we have

$$O_f(\beta X) \setminus \mathfrak{D}_{O_f(\beta X)}(\langle U_i \rangle) = [O_f(X) \setminus \langle U_i \rangle]_{O_f(\beta X)}, \quad i = 1, 2.$$

Hence, $\mu' \in [O_f(X) \setminus \langle U_i \rangle]_{O_f(\beta X)}$, $i = 1, 2$. Since $\langle U_1 \rangle \cap \langle U_2 \rangle = \emptyset$, by Lemma 3.3 we have

$$[O_f(X) \setminus \langle U_1 \rangle]_{O_f(\beta X)} \cap [O_f(X) \setminus \langle U_2 \rangle]_{O_f(\beta X)} = [O_f(X) \setminus (\langle U_1 \rangle \cup \langle U_2 \rangle)]_{O_f(\beta X)}.$$

Therefore,

$$\mu' \in [O_f(X) \setminus (\langle U_1 \rangle \cup \langle U_2 \rangle)]_{O_f(\beta X)},$$

or, equivalently,

$$\mu' \in O_f(\beta X) \setminus \mathfrak{D}_{O_f(\beta X)}(\langle U_1 \rangle \cup \langle U_2 \rangle).$$

Consequently,

$$\mu' \notin \mathfrak{D}_{O_f(\beta X)}(\langle U_1 \rangle \cup \langle U_2 \rangle).$$

Thus, we have established the reverse inclusion

$$\mathfrak{D}_{O_f(\beta X)}(\langle U_1 \rangle \cup \langle U_2 \rangle) \subset \mathfrak{D}_{O_f(\beta X)}(\langle U_1 \rangle) \cup \mathfrak{D}_{O_f(\beta X)}(\langle U_2 \rangle).$$

This proves Theorem 3.1. □

We are now ready to state the main result of the paper.

Theorem 3.2. *Let X be a Tychonoff space. Then $O_f(X)$ is Π -complete if and only if X is Π -complete.*

Proof. Suppose first that $O_f(X)$ is Π -complete. Then Property 1 (see p. 24) implies the Π -completeness of the closed subset $X \subset O_f(X)$.

Now suppose that X is a Π -complete space and let $\mu \in O_f(\beta X) \setminus O_f(X)$. By Theorem 3.1, $O_f(\beta X)$ is a perfect compactification of $O_f(X)$. Consequently, it is enough to show that there exists an open cover Ω of $O_f(X)$ with $Kp \Omega = 1$ that pricks out μ in $O_f(\beta X)$.

We have $\text{supp } \mu \not\subset X$. By Corollary 2.1, for every point $x \in \text{supp } \mu \setminus X \subset \beta X \setminus X$, there exists an open cover ω_x with $Kp \omega_x = 1$ that pricks out x in βX , i. e. $x \notin \cup[\omega_x]_{\beta X}$. Fix a point $x_0 \in \text{supp } \mu \setminus X$. Then $\text{supp } \mu \not\subset [U]_{\beta X}$ for every $U \in \omega_{x_0}$. Hence, $\mu \notin \langle [U]_{\beta X} \rangle$ for any $U \in \omega_{x_0}$. Equality 3.2 then gives

$$\mu \notin \cup[\Omega]_{O_f(\beta X)} = \cup\{\langle [U]_{\beta X} \rangle_{O_f(\beta X)} : U \in \omega_{x_0}\}.$$

Finally, Theorem 2.1 together with Proposition 3.2 completes the proof. □

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