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Research article

Dynamics of Compositions of Lotka-Volterra Operators Corresponding to Some Partially Oriented Graphs in a Three-Dimensional Simplex

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Abstract

The paper is devoted to the study of the dynamics of the trajectories of the inner points of the composition of two quadratic Lotka-Volterra dynamical systems operating in a three-dimensional simplex. Four compositional operators are investigated in this work. Fixed points are found for them and their characters are investigated by analyzing the Jacobian spectrum. The compositions of two discrete dynamic Lotka-Volterra systems are interesting because they can be applied in epidemiology problems.

Keywords: *Lotka–Volterra mapping, oriented graph, fixed point, repeller, attractor.*

AMS Subject Classification (2020): *Primary: 37B25; Secondary: 37C25, 37C27.*

1. Introduction

It is known that many applied problems today are solved using non-linear dynamics. In [1], the dynamics of the composition of some Lotka-Volterra operators operating in a two-dimensional simplex is investigated. Later work [2] was devoted to the classification of fixed points, according to the theory introduced in [3] for the composition of some Lotka-Volterra operators operating in two- and three-dimensional simplexes. In [4], the composition of two Lotka-Volterra operators corresponding to strong oppositely directed tournaments is proposed as a discrete model of sexually transmitted viruses. In this paper, these studies continue; that is, the dynamics of the composition of Lotka-Volterra operators operating in a three-dimensional simplex corresponding to partially oriented graphs is investigated. Here, for the first time, according to the works [6]-[8], the correspondence of partially oriented graphs for the operators under consideration is shown. In [9], compositions of operators of this type were also studied using the one-dimensional dynamics apparatus of A.N. Sharkovsky. In this paper, fixed points are found for the compositional operators considered and their characters are investigated. Composite operators of this type are relevant for research because they act as discrete models to study the dynamics of the spread of computer viruses in two operating systems.

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2. Preliminary information.

Let be us quadratic Lotka—Volterra operator

$$V_1 : x'_k = x_k \left(1 + \sum_{i=1}^{m} a_{ki} x_i\right), \ and \ \ V_2 : x'_k = x_k \left(1 + \sum_{i=1}^{m} b_{ki} x_i\right), \ k = \overline{1, m} \tag{2.1}$$

on the simplex

$$S^3 = \left\{ x = (x_1, x_2, x_3, x_4) : x_i \geq 0, \sum_{i=1}^{4} x_i = 1 \right\} \subset R^4$$

Definition 2.1. [1]. A complex operator satisfying the equalities

$$(V_1 \circ V_2)(x) = V_1 (V_2 (x)) \ or \ (V_2 \circ V_1)(x) = V_2 (V_1 (x))$$

is called a composition of operators V_1 and V_2.

Confirmation. According to [1], the composition of operators and can be expressed in the form

$$W = V_1 \circ V_2 : x'_k = x_k \left(1 + f_k \left(x_1, x_2, ..., x_{k-1}, x_{k+1}, ..., x_m\right)\right), \ k = \overline{1, m} \tag{2.2}$$

Before proceeding to the main results, we recall the information from [2]-[5] related to fixed points and their characters.

Definition 2.2. [2] A point x satisfying the equality $W(x) = x$ is called a fixed point of the operator W and is denoted as $Fix(W) = \{x \in S^{m-1} : \ W(x) = x\}$.

Definition 2.3. [3], [4]. Suppose x_0 is a fixed point for W. Then x_0 is an attracting fixed point if $|W(x_0)| < 1$.

Definition 2.4. [3], [4]. The point x_0 is a repelling fixed point if $|W(x_0)| > 1$.

Definition 2.5. [5]. Matrix of partial derivatives of operator Lotka – Volterra type is called Jacobi matrix and denoted as

$$J(W) = \begin{pmatrix} \frac{\partial x'_1}{\partial x_1} & \frac{\partial x'_1}{\partial x_2} & \cdots & \frac{\partial x'_1}{\partial x_n} \\ \frac{\partial x'_2}{\partial x_1} & \frac{\partial x'_2}{\partial x_2} & \cdots & \frac{\partial x'_2}{\partial x_n} \\ ..... & ..... & ..... & ...... \\ \frac{\partial x'_n}{\partial x_1} & \frac{\partial x'_n}{\partial x_n} & \cdots & \frac{\partial x'_n}{\partial x_n} \end{pmatrix} \tag{2.3}$$

Undirected, partially directed graphs and tournaments in S^3 are shown in Figure 1. ([6] -[8]).

3. Main results.

Let us assume that the Lotka–Volterra quadratic operators corresponding to partially oriented graphs (see Figure 1) and the compositions of their compositions

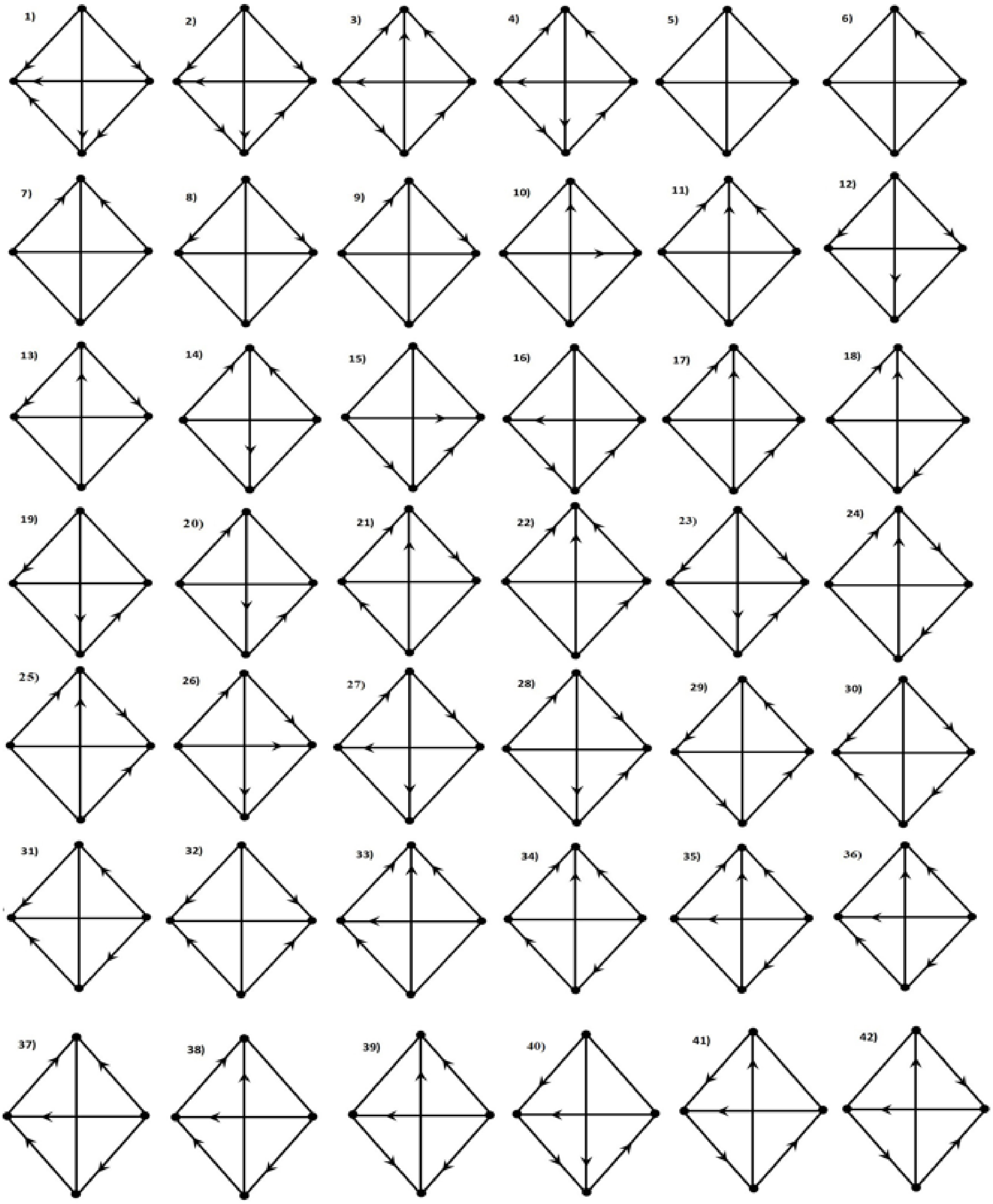

Figure 1. Tournaments defined in three-dimensional simplex.

$$V_1:\begin{cases} x'_1=x_1\left(1+a_{12}x_2+a_{13}x_3+a_{14}x_4\right); \\ x'_2=x_2\left(1-a_{12}x_1\right); \\ x'_3=x_3\left(1-a_{13}x_1\right); \\ x'_4=x_4\left(1-a_{14}x_1\right); \end{cases} \quad V_2:\begin{cases} x'_1=x_1\left(1-b_{12}x_2-b_{13}x_3-b_{14}x_4\right); \\ x'_2=x_2\left(1+b_{12}x_1\right); \\ x'_3=x_3\left(1+b_{13}x_1\right); \\ x'_4=x_4\left(1+b_{14}x_1\right). \end{cases} \tag{3.1}$$

Partially oriented graphs corresponding to these operators are imaged in Figure 2.

The composition of these operators is

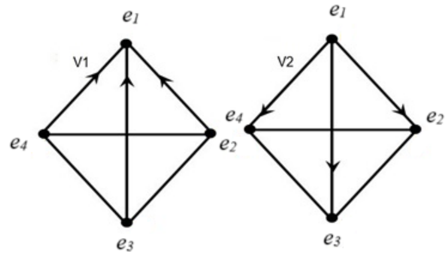

Figure 2. Partially oriented graphs corresponding to operators V_1 and V_2

$$V_1 \circ V_2 : \begin{cases} x'_1 = x_1 \left(1 - b_{12}x_2 - b_{13}x_3 - b_{14}x_4\right) \left(1 + a_{12}x_2 \left(1 + b_{12}x_1\right) + \right. \\ \left. + a_{13}x_3 \left(1 + b_{13}x_1\right) + a_{14}x_4 \left(1 + b_{14}x_1\right)\right); \\ x'_2 = x_2 \left(1 + b_{12}x_1\right) \left(1 - a_{12}x_1 \left(1 - b_{12}x_2 - b_{13}x_3 - b_{14}x_4\right)\right); \\ x'_3 = x_3 \left(1 + b_{13}x_1\right) \left(1 - a_{13}x_1 \left(1 - b_{12}x_2 - b_{13}x_3 - b_{14}x_4\right)\right); \\ x'_4 = x_4 \left(1 + b_{14}x_1\right) \left(1 - a_{14}x_1 \left(1 - b_{12}x_2 - b_{13}x_3 - b_{14}x_4\right)\right); \end{cases} \tag{3.2}$$

Lemma 3.1. *The following confirmations are satisfying for operator $V_1 \circ V_2$:*
i. Vertices of simplex I, P, H, and D i, points on the edges Γ_{IP}, Γ_{IH}, Γ_{ID} and side Γ_{PHD} are fixed points of composition operator;
ii. All vertices of operator $V_1 \circ V_2$ are attractor;
iii. Fixed points on the edges operator $V_1 \circ V_2$ are saddle point.

Proof. According to the definition of a fixed point, the solutions of the equation $Vx = x$ define the fixed points of the operator.

$$\begin{cases} x_1 = x_1 \left(1 - b_{12}x_2 - b_{13}x_3 - b_{14}x_4\right) \left(1 + a_{12}x_2 \left(1 + b_{12}x_1\right) + \right. \\ \left. + a_{13}x_3 \left(1 + b_{13}x_1\right) + a_{14}x_4 \left(1 + b_{14}x_1\right)\right); \\ x_2 = x_2 \left(1 + b_{12}x_1\right) \left(1 - a_{12}x_1 \left(1 - b_{12}x_2 - b_{13}x_3 - b_{14}x_4\right)\right); \\ x_3 = x_3 \left(1 + b_{13}x_1\right) \left(1 - a_{13}x_1 \left(1 - b_{12}x_2 - b_{13}x_3 - b_{14}x_4\right)\right); \\ x_4 = x_4 \left(1 + b_{14}x_1\right) \left(1 - a_{14}x_1 \left(1 - b_{12}x_2 - b_{13}x_3 - b_{14}x_4\right)\right); \\ x_1 + x_2 + x_3 + x_4 = 1. \end{cases} \tag{3.3}$$

The solution of the system of equations define the fixed points of the given operator. The solutions of equation (2.3) are vertices of simplex $I\,(1;0;0;0)$, $P\,(0;1;0;0)$, $H\,(0;0;1;0)$, $D\,(0;0;0;1)$ and points

$$O_1\left(\frac{(b_{12}-2)\sqrt{a_{12}} + \sqrt{a_{12}b_{12}^2 + 4b_{12}}}{2b_{12}\sqrt{a_{12}}}; \frac{(b_{12}+2)\sqrt{a_{12}} - \sqrt{a_{12}b_{12}^2 + 4b_{12}}}{2b_{12}\sqrt{a_{12}}}; 0; 0\right),$$

$$O_2\left(\frac{(b_{13}-2)\sqrt{a_{13}} + \sqrt{a_{13}b_{13}^2 + 4b_{13}}}{2b_{13}\sqrt{a_{13}}}; 0; \frac{(b_{13}+2)\sqrt{a_{13}} - \sqrt{a_{13}b_{13}^2 + 4b_{13}}}{2b_{13}\sqrt{a_{13}}}; 0\right),$$

$$O_3\left(\frac{(b_{14}-2)\sqrt{a_{14}} + \sqrt{a_{14}b_{14}^2 + 4b_{14}}}{2b_{14}\sqrt{a_{14}}}; 0; 0; \frac{(b_{14}+2)\sqrt{a_{14}} - \sqrt{a_{14}b_{14}^2 + 4b_{14}}}{2b_{14}\sqrt{a_{14}}}\right),$$

$$O_4\left(0; x_2; x_3; 1 - x_2 - x_3\right),$$

they represent the points belongs to edges $\Gamma_{IP},\ \Gamma_{IH},\ \Gamma_{ID}$ and sides Γ_{PHD} of the simplex. The elements of the Jacobi matrix of this operator are:

$\frac{\partial x'_1}{\partial x_1} = (1 - b_{12}x_2 - b_{13}x_3 - b_{14}x_4)\,(1 + a_{12}x_2\,(1 + b_{12}x_1) + a_{13}x_3\,(1 + b_{13}x_1) + a_{14}x_4\,(1 + b_{14}x_1)) +$
$+x_1\,(1 - b_{12}x_2 - b_{13}x_3 - b_{14}x_4)\,(a_{12}b_{12}x_2 + a_{13}b_{13}x_3 + a_{14}b_{14}x_4)\,;$
$\frac{\partial x'_1}{\partial x_2} = -\,\mathrm{b}_{12}x_1\,(1 + a_{12}x_2\,(1 + b_{12}x_1) + a_{13}x_3\,(1 + b_{13}x_1) + a_{14}x_4\,(1 + b_{14}x_1)) +$
$+a_{12}x_1\,(1 - b_{12}x_2 - b_{13}x_3 - b_{14}x_4)\,(1 + b_{12}x_1)\,;$
$\frac{\partial x'_2}{\partial x_1} = b_{12}x_2\,(1 - a_{12}x_1\,(1 - b_{12}x_2 - b_{13}x_3 - b_{14}x_4)) - a_{12}x_2\,(1 + b_{12}x_1)\,(1 - b_{12}x_2 - b_{13}x_3 - b_{14}x_4)\,;$
$\frac{\partial x'_2}{\partial x_2} = (1 + b_{12}x_1)\,(1 - a_{12}x_1\,(1 - b_{12}x_2 - b_{13}x_3 - b_{14}x_4)) + a_{12}b_{12}x_1x_2\,(1 + b_{12}x_1)\,;$
$\frac{\partial x'_2}{\partial x_3} = a_{12}b_{13}x_1x_2\,(1 + b_{12}x_1)\,;$
$\frac{\partial x'_2}{\partial x_4} = a_{12}b_{14}x_1x_2\,(1 + b_{12}x_1)\,;$
$\frac{\partial x'_3}{\partial x_1} = b_{13}x_3\,(1 - a_{13}x_1\,(1 - b_{12}x_2 - b_{13}x_3 - b_{14}x_4)) - a_{13}x_3\,(1 + b_{13}x_1)\,(1 - b_{12}x_2 - b_{13}x_3 - b_{14}x_4)\,;$
$\frac{\partial x'_3}{\partial x_2} = a_{13}b_{12}x_1x_3\,(1 + b_{13}x_1)\,;$
$\frac{\partial x'_3}{\partial x_3} = (1 + b_{13}x_1)\,(1 - a_{13}x_1\,(1 - b_{12}x_2 - b_{13}x_3 - b_{14}x_4)) + a_{13}b_{13}x_1x_3\,(1 + b_{13}x_1)\,;$
$\frac{\partial x'_3}{\partial x_4} = a_{13}b_{14}x_1x_3\,(1 + b_{13}x_1)\,;$
$\frac{\partial x'_4}{\partial x_1} = b_{14}x_4\,(1 - a_{14}x_1\,(1 - b_{12}x_2 - b_{13}x_3 - b_{14}x_4)) - a_{14}x_4\,(1 + b_{14}x_1)\,(1 - b_{12}x_2 - b_{13}x_3 - b_{14}x_4)\,;$
$\frac{\partial x'_4}{\partial x_2} = a_{14}b_{12}x_1x_4\,(1 + b_{14}x_1)\,;$
$\frac{\partial x'_4}{\partial x_3} = a_{14}b_{13}x_1x_4\,(1 + b_{14}x_1)\,;$
$\frac{\partial x'_4}{\partial x_4} = (1 + b_{14}x_1)\,(1 - a_{14}x_1\,(1 - b_{12}x_2 - b_{13}x_3 - b_{14}x_4)) + a_{14}b_{14}x_1x_4\,(1 + b_{14}x_1)\,.$

The elements of Jacobi matrix for simplex vertices are listed in Table 1.

Table 1.

	e_1	e_2	e_3	e_4
$\frac{\partial x'_1}{\partial x_1}$	1	$(1-b_{12})(1+a_{12})$	$(1-b_{13})(1+a_{13})$	$(1-b_{14})(1+a_{14})$
$\frac{\partial x'_1}{\partial x_2}$	$-b_{12}+a_{12}(1+b_{12})$	0	0	0
$\frac{\partial x'_1}{\partial x_3}$	$-b_{13}+a_{13}(1+b_{13})$	0	0	0
$\frac{\partial x'_1}{\partial x_4}$	$-b_{14}+a_{14}(1+b_{14})$	0	0	0
$\frac{\partial x'_2}{\partial x_1}$	0	$b_{12}-a_{12}(1-b_{12})$	0	0
$\frac{\partial x'_2}{\partial x_2}$	$(1+b_{12})(1-a_{12})$	1	1	1
$\frac{\partial x'_2}{\partial x_3}$	0	0	0	0
$\frac{\partial x'_2}{\partial x_4}$	0	0	0	0
$\frac{\partial x'_3}{\partial x_1}$	0	0	$b_{13}-a_{13}(1-b_{13})$	0
$\frac{\partial x'_3}{\partial x_2}$	0	0	0	0
$\frac{\partial x'_3}{\partial x_3}$	$(1+b_{13})(1-a_{13})$	1	1	1
$\frac{\partial x'_3}{\partial x_4}$	0	0	0	0
$\frac{\partial x'_4}{\partial x_1}$	0	0	0	$b_{14}-a_{14}(1-b_{14})$
$\frac{\partial x'_4}{\partial x_2}$	0	0	0	0
$\frac{\partial x'_4}{\partial x_3}$	0	0	0	0
$\frac{\partial x'_4}{\partial x_4}$	$(1+b_{14})(1-a_{14})$	1	1	1

Table 2 below lists the eigenvalues of fixed points I, P, H and D of the composition operator $V_1 \circ V_2$.

Table 2.

	e_1	e_2	e_3	e_4
λ_1	1	1	1	1
λ_2	$1-a_{12}b_{12}-a_{12}+b_{12}$	1	1	1
λ_3	$1-a_{13}b_{13}-a_{13}+b_{13}$	1	1	1
λ_4	$1-a_{14}b_{14}-a_{14}+b_{14}$	$1-a_{12}b_{12}+a_{12}-b_{12}$	$1-a_{13}b_{13}+a_{13}-b_{13}$	$1-a_{14}b_{14}+a_{14}-b_{14}$

According to Table 2, the absolute value of eigenvalues of fixed points I, P, H and D of the composition operator $V_1 \circ V_2$ is $|\lambda_i| < 1,\ i = \overline{1,4}$. Then, I, P, H and D points are attractor. The eigenvalues of point O_1 are:

$\lambda_1 = 1;$

$\lambda_2 = 1 - \frac{1}{4b_{13}\sqrt{a_{13}}}\left(b_{13}\sqrt{a_{13}} - \sqrt{b_{13}\left(a_{13}b_{13}+4\right)}\right)\left(-2b_{13} + a_{13}b_{13} + 3\sqrt{a_{13}b_{13}\left(a_{13}b_{13}+4\right)} - 2a_{13}\right)$

$\lambda_3 = 1 - \frac{1}{4b_{14}^2\sqrt{a_{14}^3}}\left(\left(b_{14}-2\right)\sqrt{a_{14}} + \sqrt{b_{14}\left(a_{14}b_{14}+4\right)}\right)\cdot$

$\cdot\left(\sqrt{b_{14}\left(a_{14}b_{14}+4\right)}\sqrt{a_{14}}\left(a_{14}b_{12} - a_{12}b_{14} + a_{12}b_{14}\right) + a_{14}^2b_{12}b_{14} + a_{12}a_{14}b_{14}^2 - a_{12}a_{14}b_{12}b_{14} - 2a_{12}b_{12}b_{14}\right);$

$\lambda_4 = 2 + \frac{1}{4b_{14}^2\sqrt{a_{14}^3}}\left(\left(b_{14}-2\right)\sqrt{a_{14}} + \sqrt{b_{14}\left(a_{14}b_{14}+4\right)}\right).$

The eigenvalues of point O_2:

$\lambda_1 = 1;$

$\lambda_2 = 1 - \frac{1}{4b_{13}\sqrt{a_{13}}}\left(b_{13}\sqrt{a_{13}} - \sqrt{b_{13}\left(a_{13}b_{13}+4\right)}\right)\left(-2b_{13} + a_{13}b_{13} + 3\sqrt{a_{13}b_{13}\left(a_{13}b_{13}+4\right)} - 2a_{13}\right);$

$\lambda_3 = 1 - \frac{1}{4b_{13}^2\sqrt{a_{13}^3}}\left(\left(b_{13}-2\right)\sqrt{a_{13}} + \sqrt{b_{13}\left(a_{13}b_{13}+4\right)}\right)\cdot$

$\cdot\left(\sqrt{b_{13}\left(a_{13}b_{13}+4\right)}\sqrt{a_{13}}\left(a_{13}b_{12} - a_{12}b_{13} + a_{12}b_{12}\right) + a_{13}^2b_{12}b_{13} + a_{12}a_{13}b_{13}^2 - a_{12}a_{13}b_{12}b_{13} - 2a_{12}b_{12}b_{13}\right);$

$\lambda_4 = 2 + \frac{\left(b_{13}\sqrt{a_{13}} - 2\sqrt{a_{13}} + \sqrt{b_{13}\left(a_{13}b_{13}+4\right)}\right)\left(a_{13}b_{13} - 2b_{13} + \sqrt{a_{13}b_{13}\left(a_{13}b_{13}+4\right)}\right)}{4\sqrt{a_{13}}b_{13}};$

The eigenvalues of point O_3:

$\lambda_1 = 1;$

$\lambda_2 = 1 - \frac{1}{4b_{14}\sqrt{a_{14}}}\left(b_{14}\sqrt{a_{14}} - \sqrt{b_{14}\left(a_{14}b_{14}+4\right)}\right)\left(-2b_{14} + a_{14}b_{14} + 3\sqrt{a_{14}b_{14}\left(a_{14}b_{14}+4\right)} - 2a_{14}\right);$

$\lambda_3 = 1 - \frac{1}{4b_{14}^2\sqrt{a_{14}^3}}\left(\left(b_{14}-2\right)\sqrt{a_{14}} + \sqrt{b_{14}\left(a_{14}b_{14}+4\right)}\right)\cdot$

$\cdot\left(\sqrt{b_{14}\left(a_{14}b_{14}+4\right)}\sqrt{a_{14}}\left(a_{14}b_{12} - a_{12}b_{14} + a_{12}b_{14}\right) + a_{14}^2b_{12}b_{14} + a_{12}a_{14}b_{14}^2 - a_{12}a_{14}b_{12}b_{14} - 2a_{12}b_{12}b_{14}\right);$

$\lambda_4 = 2 + \frac{1}{4b_{14}^2\sqrt{a_{14}^3}}\left(\left(b_{14}-2\right)\sqrt{a_{14}} + \sqrt{b_{14}\left(a_{14}b_{14}+4\right)}\right)\cdot$

$\cdot\left(\sqrt{b_{14}\left(a_{14}b_{14}+4\right)}\sqrt{a_{14}}\left(a_{14}b_{12} - a_{12}b_{14} + a_{12}b_{14}\right) + a_{14}^2b_{12}b_{14} + a_{12}a_{14}b_{14}^2 - a_{12}a_{14}b_{12}b_{14} - 2a_{12}b_{12}b_{14}\right).$

Lemma is proved.

Since the proof of the results derived from this lemma can be applied to the dynamics of compositions of Lotka-Volterra type quadratic operators corresponding to partially oriented graphs and fully oriented tournaments given in Fig. 1, we present the rest of the lemmas without proof. Let us first consider the composition of Lotka–Volterra type quadratic operators corresponding to fully oriented and unoriented graphs. Suppose the following operators are given:

I)

$$1):\begin{cases} x'_1 = x_1\left(1 - a_{12}x_2 - a_{13}x_3 - a_{14}x_4\right); \\ x'_2 = x_2\left(1 + a_{12}x_1 - a_{23}x_3 - a_{24}x_4\right); \\ x'_3 = x_3\left(1 + a_{13}x_1 + a_{23}x_2 - a_{34}x_4\right); \\ x'_4 = x_4\left(1 + a_{14}x_1 + a_{24}x_2 + a_{34}x_3\right); \end{cases} \quad 5):\begin{cases} x'_1 = x_1; \\ x'_2 = x_2; \\ x'_3 = x_3; \\ x'_4 = x_4. \end{cases} \tag{3.4}$$

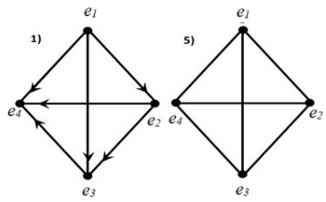

Figure 3. Complete oriented tournament and unoriented graph.

The form of Composition of these operators is

$$1)\circ 5):\begin{cases} x'_1 = x_1\left(1 - a_{12}x_2 - a_{13}x_3 - a_{14}x_4\right); \\ x'_2 = x_2\left(1 + a_{12}x_1 - a_{23}x_3 - a_{24}x_4\right); \\ x'_3 = x_3\left(1 + a_{13}x_1 + a_{23}x_2 - a_{34}x_4\right); \\ x'_4 = x_4\left(1 + a_{14}x_1 + a_{24}x_2 + a_{34}x_3\right); \end{cases} \tag{3.5}$$

Lemma 3.2. *The subsequent confirmations are pleasing to the operator* $1)\circ 5)$*:*
i. There are no fixed points other than simplex vertices I, P, H, and D;
ii.Vertice I of operator $1)\circ 5)$ *is repeller, Vertices P and H are saddle point and vertice D is attractor;*

II)

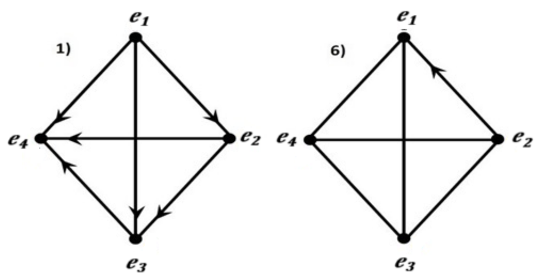

Figure 4. Complete oriented tournament and graph which one edge is oriented.

$$1):\begin{cases} x'_1 = x_1\left(1 - a_{12}x_2 - a_{13}x_3 - a_{14}x_4\right); \\ x'_2 = x_2\left(1 + a_{12}x_1 - a_{23}x_3 - a_{24}x_4\right); \\ x'_3 = x_3\left(1 + a_{13}x_1 + a_{23}x_2 - a_{34}x_4\right); \\ x'_4 = x_4\left(1 + a_{14}x_1 + a_{24}x_2 + a_{34}x_3\right); \end{cases} \quad 6):\begin{cases} x'_1 = x_1\left(1 + b_{12}x_2\right); \\ x'_2 = x_2\left(1 - b_{12}x_1\right); \\ x'_3 = x_3; \\ x'_4 = x_4. \end{cases} \tag{3.6}$$

The form of Composition of these operators is

$$1)\circ 6):\begin{cases} x'_1 = x_1\left(1 + b_{12}x_2\right)\left(1 - a_{12}x_2\left(1 - b_{12}x_1\right) - a_{13}x_3 - a_{14}x_4\right); \\ x'_2 = x_2\left(1 - b_{12}x_1\right)\left(1 + a_{12}x_1\left(1 + b_{12}x_2\right) - a_{23}x_3 - a_{24}x_4\right); \\ x'_3 = x_3\left(1 + a_{13}x_1\left(1 + b_{12}x_2\right) + a_{23}x_2\left(1 - b_{12}x_1\right) - a_{34}x_4\right); \\ x'_4 = x_4\left(1 + a_{14}x_1\left(1 + b_{12}x_2\right) + a_{24}x_2\left(1 - b_{12}x_1\right) + a_{34}x_3\right); \end{cases} \tag{3.7}$$

Lemma 3.3. *The following confirmations are satisfying for operator* $1)\circ 6)$*:*
i.There is fixed point other than vertices simplex I, P, H va D;
ii. Vertices I, P and H of operator $1)\circ 6)$ *are attractor;*
iii. Fixed point of operator on the edge Γ_{IP} *is saddle point.*

III)

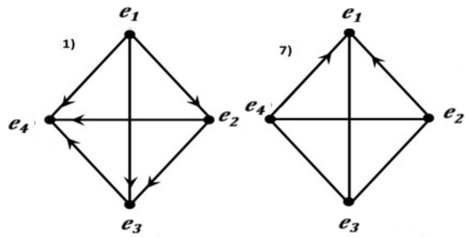

Figure 5. Complete oriented tournament and graph which two edge is oriented.

$$1):\begin{cases} x'_1 = x_1\left(1 - a_{12}x_2 - a_{13}x_3 - a_{14}x_4\right); \\ x'_2 = x_2\left(1 + a_{12}x_1 - a_{23}x_3 - a_{24}x_4\right); \\ x'_3 = x_3\left(1 + a_{13}x_1 + a_{23}x_2 - a_{34}x_4\right); \\ x'_4 = x_4\left(1 + a_{14}x_1 + a_{24}x_2 + a_{34}x_3\right); \end{cases} \quad 7):\begin{cases} x'_1 = x_1\left(1 + b_{12}x_2 + b_{14}x_4\right); \\ x'_2 = x_2\left(1 - b_{12}x_1\right); \\ x'_3 = x_3; \\ x'_4 = x_4\left(1 - b_{14}x_1\right). \end{cases} \tag{3.8}$$

The form of Composition of these operators is

$$1)\circ 7):\begin{cases} x'_1 = x_1\left(1 + b_{12}x_2 + b_{14}x_4\right)\left(1 - a_{12}x_2\left(1 - b_{12}x_1\right) - a_{13}x_3 - a_{14}x_4\left(1 - b_{14}x_1\right)\right); \\ x'_2 = x_2\left(1 - b_{12}x_1\right)\left(1 + a_{12}x_1\left(1 + b_{12}x_2 + b_{14}x_4\right) - a_{23}x_3 - a_{24}x_4\left(1 - b_{14}x_1\right)\right); \\ x'_3 = x_3\left(1 + a_{13}x_1\left(1 + b_{12}x_2 + b_{14}x_4\right) + a_{23}x_2\left(1 - b_{12}x_1\right) - a_{34}x_4\left(1 - b_{14}x_1\right)\right); \\ x'_4 = x_4\left(1 - b_{14}x_1\right)\left(1 + a_{14}x_1\left(1 + b_{12}x_2 + b_{14}x_4\right) + a_{24}x_2\left(1 - b_{12}x_1\right) + a_{34}x_3\right); \end{cases} \tag{3.9}$$

Lemma 3.4. *The subsequent confirmations are pleasing to the operator* $1)\circ 7)$*:*
i. There are fixed point other than vertices simplex I, P, H and D on the edges Γ_{IP} *and* Γ_{ID}*;*
ii. Vertices I, P and H of operator $1)\circ 7)$ *are saddle point and vertex D is attractor;*
iii. Fixed points of the operator $1)\circ 7)$ *on the edges* Γ_{IP} *and* Γ_{ID} *are saddle points.*

4. Conclusion

In the paper, according to [6]-[8], oriented and partially oriented graphs are given for $m = 4$ (they turned out to be 42, see Figure 1). All graphs are described by discrete Lotka-Volterra dynamical systems operating in a three-dimensional simplex. In [9], [10], the dynamics of the Lotka-Volterra operators are studied, and the works [1], [2], [4] are devoted to the study of the dynamics of the trajectories of the inner points of the composition of Lotka-Volterra operators operating in a two-dimensional simplex. In contrast to these works, this article examines four compositional Lotka- Volterra operators corresponding to some of the graphs in Figure 1 (see Figures 2-5), for $m = 4$. Fixed points are found for all considered compositional operators and their characters are investigated (Lemma 2.1-2.4).

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Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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