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Research article

Classification of Character of Rest Points of the Lotka-Volterra Operator Acting in S^4

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Abstract

In the paper, we consider the problem of studying the trajectories of points under the action of the Lotka-Volterra operator. In short, the goal is to study the dynamics of trajectories of interior points by finding fixed points and studying the Jacobian spectrum at these points of the operator in question. It turned out that in a number of applied problems, there are Lotka-Volterra mappings of exactly this type, and the points of the simplex in this case are considered as the states of the system under study. In this case, the simplex-preserving mapping defines the discrete law of evolution of the given system. Starting from a certain starting point, we can consider the sequence that determines the evolution of this point. The work explicitly shows sets of limit points for positive and negative trajectories, which in turn describe in applied problems the beginning and the end of the evolutionary process, respectively.

Keywords: *skew-symmetric matrix; boundary point; interior point; fixed point; partially oriented graph; eigenvalue; saddle point; attractor; repeller.*

AMS Subject Classification (2020): *Primary: 37B25; Secondary: 37C25; 37C27.*

1. Introduction

It is known that nonlinear dynamics has been formed relatively over the last $30-35$ years, using systems of nonlinear mathematical models [1-4]. In this it is necessary to note differential equations and discrete mappings, among which an important role is played by discrete Lotka-Volterra mappings.

It is known [5],[6] that discrete Lotka-Volterra mappings are uniquely determined by specifying the skew-symmetric matrix $A=(a_{ij})$, $a_{ij}=-a_{ji}$, $i,j=\overline{1,m}$ and act on the simplex

$$S^{m-1}=\left\{x\in R^m:\ \sum_{i=1}^{m}x_i=1,\ x_i\geq 0\right\}$$

according to the formulas

$$V: x'_k=x_k\left(1+\sum_{i=1}^{m}a_{ki}x_i\right), k=1,...,m \tag{1.1}$$

on condition $|a_{ki}|\leq 1$. We define it as $V:S^{m-1}\to S^{m-1}$ [5-7]. By specifying the mapping in the form (1.1), it is not difficult to notice that it is uniquely determined by the skew-symmetric matrix of $A=(a_{ki})$. In [5-7] the concepts of a skew-symmetric matrix of general position are introduced.

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Definition 1.1. A skew-symmetric matrix $A = (a_{ki})$ is called in general position if the major minors of an even order are nonzero.

If the definition condition is not met, then such a skew-symmetric matrix is called degenerate.

Also, from the works [5-7] we know that if a skew-symmetric matrix is in general position, then in this case a complete oriented graph corresponds to this matrix, i.e. a tournament. In this paper, we consider the degenerate case of Lotka-Volterra mappings. Mappings of this type are defined by a degenerate skew-symmetric matrix. This type of mapping, acting in two-dimensional and three-dimensional simplexes, began to be considered in [8],[9] and they show that partially oriented graphs correspond to matrices of this type.

Interest in the study of degenerate Lotka-Volterra mappings arises from the fact that they can be used in modeling epidemiological situations, up to recurrent ([8],[10]) and non-recurrent diseases [9], as well as in modeling environmental problems ([11]). Consequently, the work is devoted to the continuation of the study of mappings of this type.

Partially oriented graphs in the cases of $m = 3, 4$ are given in the monograph [12]. This work is devoted to finding rest points and investigating their character of the degenerate Lotka-Volterra mapping operating in a four-dimensional simplex, i.e. S^4. This study gives us a picture of the flow of trajectories of the internal points of the simplex.

Now let us introduce the basic definitions that we will refer to throughout our work.

Definition 1.2. [1] Let $x \in S^{m-1}$. If condition $Vx = x$ is satisfied, then point x is called fixed.

The Jacobi matrix for some $f_k\,(x_1, x_2, ..., x_m)$ has the following form:

$$J(x) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_m} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_m} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_m} \end{pmatrix} \tag{1.2}$$

Let $x^* = (x_1, x_2, ..., x_m) \in S^{m-1}$ be a fixed point of a system of type (1.1). If

$$|J\,(x^*) - \lambda E| = 0, \tag{1.3}$$

then the numbers $\lambda_i,\ i = \overline{1, m}$, which are solutions to the equation (1.3), are called the eigenvalues of the fixed point x^* [9].

Definition 1.3. If at a fixed point $x^* = (x_1, x_2, ..., x_m) \in S^{m-1}$ all eigenvalues $\lambda_i,\ i = \overline{1, m}$ are less than 1 in absolute value, then this point is called attracting.

Definition 1.4. If all eigenvalues $\lambda_i,\ i = \overline{1, m}$ at a fixed point $x^* = (x_1, x_2, ..., x_m) \in S^{m-1}$ are greater than 1, then this fixed point is called repulsive.

In all other cases, the fixed point is called a saddle point.

2. Results

Let the degenerate skew-symmetric matrix A have the form

$$A = \begin{pmatrix} 0 & 0 & -a & b & 0 \\ 0 & 0 & c & -e & 0 \\ a & -c & 0 & 0 & -d \\ -b & e & 0 & 0 & f \\ 0 & 0 & d & -f & 0 \end{pmatrix}, and\ 0 < a,\ b,\ c,\ d,\ e,\ f \leq 1.$$

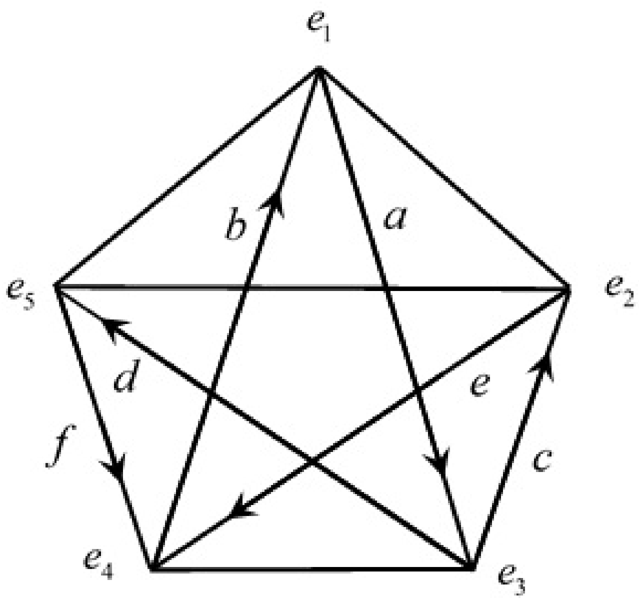

Figure 1. Partially directed graph corresponding to matrix A.

Then the matrix A defines a partially directed graph (see Figure 1).

In this case, operator $V : S^4 \to S^4$, defined by matrix A, will have the following form:

$$V : \begin{cases} x_1' = x_1(1 - ax_3 + bx_4), \\ x_2' = x_2(1 + cx_3 - ex_4), \\ x_3' = x_3(1 + ax_1 - cx_2 - dx_5), \\ x_4' = x_4(1 - bx_1 + ex_2 + fx_5), \\ x_5' = x_5(1 + dx_3 - fx_4). \end{cases} \tag{2.1}$$

We are faced with the task of studying the dynamics of the interior points of operator (2.1). To do this, we find fixed points and study their characters.

First, we introduce the following notation:

Let $I = \{1, 2, ..., m\}$, points $e_i = (\delta_{i1}, ..., \delta_{im})$ be the vertices of the simplex, where δ_{ij} is the Kronecker symbol, $(i, j \in I)$. $\Gamma_\alpha-$ convex hull of vertices $\{e_i\}_{i\in\alpha}$ for $\alpha \subset I$. The interior of the face is $\Gamma_\alpha - ri\Gamma_\alpha$ and the relative boundary is $\partial\Gamma_\alpha$. Let $|\alpha|$ be the number of elements of $\alpha \subset I$. First, let us find the fixed points of operator (2.1). To do this, we solve the system of equations $Vx = x$. As a result, they belong to S^4, i.e. that is, the edges and faces of the simplex

$$\Gamma_{34} = \left\{(0,\ 0,\ \gamma,\ 1-\gamma,\ 0) \in S^4,\ 0 \le \gamma \le 1\right\},$$

$$\Gamma_{125} = \left\{(\alpha,\ \beta,\ 0,\ 0,\ 1-\alpha-\beta) \in S^4,\ 0 \le \alpha,\ \beta \le 1,\ \ \alpha+\beta \le 1\right\}.$$

Let us be given the Lotka-Volterra operator of the form (2.1). First, we write down the general Jacobian matrix for $Vx = \left(x_1',\ x_2',\ x_3',\ x_4',\ x_5'\right) \in S^4$. To do this, we use formula (1.2) for operator (2.1):

$$J(Vx) = \begin{pmatrix} 1 - ax_3 + bx_4 & 0 & -ax_1 & bx_1 & 0 \\ 0 & 1 + cx_3 - ex_4 & cx_2 & -ex_2 & 0 \\ ax_3 & -cx_3 & 1 + ax_1 - cx_2 - dx_5 & 0 & -dx_3 \\ -bx_4 & ex_4 & 0 & 1 - bx_1 + ex_2 + fx_5 & -fx_4 \\ 0 & 0 & dx_5 & -fx_5 & 1 + dx_3 - fx_4 \end{pmatrix}.$$

Theorem 2.1. *All vertices of simplex S^4 are fixed saddle points.*

Proof. We know that the vertices of the simplex S^4 lie at points
$e_1 = (1, 0, 0, 0, 0)$, $e_2 = (0, 1, 0, 0, 0)$, $e_3 = (0, 0, 1, 0, 0)$, $e_4 = (0, 0, 0, 1, 0)$, $e_5 = (0, 0, 0, 0, 1)$.
All these points satisfy the condition $Vx = x$.

1) Find the eigenvalues using the Jacobian matrix for point $e_1 = (1, 0, 0, 0, 0)$ and formula (1.3), i.e. solve an equation of the form

$$(1-\lambda)^3(1+a-\lambda)(1-b-\lambda)=0.$$

Its solutions have the form $\lambda_1 = \lambda_2 = \lambda_3 = 1$, $\lambda_4 = 1 + a$, $\lambda_5 = 1 - b$. This means that $|\lambda_4| > 1$, $|\lambda_5| < 1$. Vertex e_1 is a saddle point.

2) For vertex $e_2 = (0, 1, 0, 0, 0)$ we obtain equation

$$(1-\lambda)^3(1+e-\lambda)(1-c-\lambda)=0.$$

Its solutions $\lambda_1 = \lambda_2 = \lambda_3 = 1$, $\lambda_4 = 1 + e$, $\lambda_5 = 1 - c$, i.e. $|\lambda_4| > 1$, $|\lambda_5| < 1$. Vertex e_2 is also a saddle. Carrying out the same actions for e_3, e_4 and e_5, we establish that among their eigenvalues there are modulo greater than 1 and less than 1. This means that the vertices e_3, e_4, e_5 are also saddles. The theorem has been proven.

Now let us turn our attention to other fixed points of the simplex. All points of edge $\Gamma_{34} \subset S^4$ satisfy condition $Vx = x$. We solve equations $-ax_3 + bx_4 = 0$, $cx_3 - ex_4 = 0$, $dx_3 - fx_4 = 0$ taking into account $x_3 + x_4 = 1$.

The solutions to these equations consist of neutral fixed points. Defining them as

$$E_1=\left(0,\ 0,\ \frac{b}{a+b},\ \frac{a}{a+b},\ 0\right),\ E_2=\left(0,\ 0,\ \frac{e}{c+e},\ \frac{c}{c+e},\ 0\right),\ E_3=\left(0,\ 0,\ \frac{f}{d+f},\ \frac{d}{d+f},\ 0\right),$$

respectively.

Theorem 2.2. *If condition $de > cf$ is satisfied, then all fixed points belonging to the interval $[E_1,\ E_2] \subset \Gamma_{34}$ are repellers, if condition $de > cf$ is satisfied, then $[E_1,\ E_3] \subset \Gamma_{34}$ the interval also consists entirely of repellers.*

Proof. Consider the set consisting of these fixed points of the edge Γ_{34}. According to (1.3), we find the eigenvalues:

$$(1-\lambda)^2(1-a\gamma+b(1-\gamma)-\lambda)(1+c\gamma-e(1-\gamma)-\lambda)(1+d\gamma-f(1-\gamma)-\lambda)=0.$$

Its solutions have the form $\lambda_1 = \lambda_2 = 1$, $\lambda_3 = 1 + (-a\gamma + b(1-\gamma))$, $\lambda_4 = 1 + c\gamma - e(1-\gamma)$,$\lambda_5 = 1 + d\gamma - f(1-\gamma)$. In simplex S^4 we consider the system of inequalities

$$\begin{cases} -a\gamma+b(1-\gamma)>0, \\ c\gamma-e(1-\gamma)>0, \\ d\gamma-f(1-\gamma)>0. \end{cases} \tag{2.2}$$

And according to the results found λ, we divide the set of points Γ_{34} into classes. If condition $de > cf$ is satisfied, then the solutions to the system of inequalities (2.2) consist of interval $[E_1,\ E_2] \subset \Gamma_{34}$ (see Figure 2). The eigenvalues of fixed points belonging to both sets $[E_1,\ E_2]$ and $[E_1,\ E_3]$ have the form
$\lambda_1 = \lambda_2 = 1, \lambda_3, \lambda_4, \lambda_5 > 1$,. This means that the intervals consist entirely of repellers.

Under condition $de < cf$, the solutions to the system of inequalities (2.2) consist of interval $[E_1,\ E_3] \subset \Gamma_{34}$ (see Figure 3).

Theorem 2.3. *When condition $de > cf$ is satisfied, points belonging to interval $[E_3, E_1] \subset \Gamma_{34}$, and when condition $de < cf$ is satisfied, points belonging to interval $[E_2, E_1] \subset \Gamma_{34}$ are attractors.*

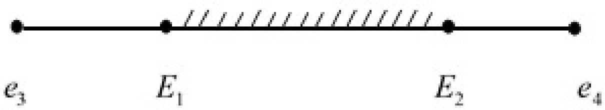

Figure 2. $[E_1, E_2]$ is part of the edge Γ_{34}.

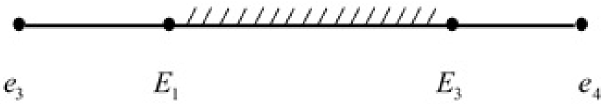

Figure 3. $[E_1, E_3]$ is part of the edge Γ_{34}.

Proof. In the proof of the above theorem, we will find the eigenvalues for the fixed points belonging to Γ_{34}. Consider the system of inequalities

$$\begin{cases} -a\gamma + b(1-\gamma) < 0, \\ c\gamma - e(1-\gamma) < 0, \\ d\gamma - f(1-\gamma) < 0 \end{cases} \tag{2.3}$$

in simplex S^4 . If $de > cf$ is true for the system of inequalities, then the solution to the system of inequalities (2.3) consists of the interval $[E_3, E_1] \subset \Gamma_{34}$. If $de < cf$ holds, then the solution to the system of inequalities (2.3) consists of $[E_2, E_1] \subset \Gamma_{34}$ the interval. The coordinates of points, E_1, E_2 and E_3 are indicated above. The eigenvalues of fixed points belonging to sets $[E_2, E_1]$, $[E_3, E_1]$, consisting of solutions to the system of inequalities (2.3), have the form $\lambda_1 = \lambda_2 = 1,\ |\lambda_3|,\ \ |\lambda_4|,\ |\lambda_5| < 1$. This means that they are attractors.The theorem has been proven.

Theorem 2.4. *If $-a\gamma + b(1-\gamma)$, $c\gamma - e(1-\gamma)$, $d\gamma - f(1-\gamma)$ have different signs, then the fixed points belonging to Γ_{34} consist of saddle points.*

Proof. We are interested in cases when these polynomials have different signs. Cases with the same symptoms were considered as follows. The table below shows cases when the signs of these polynomials are different.

	$-a\gamma + b(1-\gamma)$	$c\gamma - e(1-\gamma)$	$d\gamma - f(1-\gamma)$
1	−	+	+
2	−	−	+
3	−	+	+
4	+	−	−
5	+	−	+
6	+	+	−

Table 1. Combinations of signs of polynomials

If you create a system of inequalities using the signs of polynomials from the table, then it will consist of six inequalities. Each solution to this system of inequalities consists of an interval belonging to edge Γ_{34}. Analyzing the eigenvalues for points belonging to these intervals, we come to the conclusion that they are saddle points. The theorem has been proven.

All points belonging to face $\Gamma_{125} \subset S^4$ satisfy condition $Vx = x$. Let us study the relative boundary $\partial\Gamma_{125} \subset \Gamma_{125}$ and the relative interior $ri\Gamma_{125} \subset \Gamma_{125}$ separately. We know there is equality

$$\partial\Gamma_{125} = \partial\Gamma_{12} \cup \partial\Gamma_{25} \cup \partial\Gamma_{15}.$$

Theorem 2.5. *The set of fixed points belonging to the edge Γ_{25} consists of saddle points.*

Proof. Let $\alpha = 0$, then $\Gamma_{25} = \left\{(0,\ \beta,\ 0,\ 0,\ 1-\beta) \in S^4,\ 0 \le \beta \le 1\right\}$. We find the eigenvalues for the points from Γ_{25}. From formula (1.3) we obtain

$$(1-\lambda)^3\,(1 - c\beta - d\,(1-\beta) - \lambda)\,(1 + e\beta + f\,(1-\beta) - \lambda) = 0.$$

Its solutions have the form $\lambda_1 = \lambda_2 = \lambda_3 = 1,\quad \lambda_4 = 1 - c\beta - d\,(1-\beta),\ \lambda_5 = 1 + e\beta + f\,(1-\beta)$ and $|\lambda_4| < 1,\ |\lambda_5| > 1$. So, the set of fixed points of the edge Γ_{25} consists of saddle points. The theorem has been proven. Consider the equations $ax_1 - cx_2 - dx_5 = 0, -bx_1 + ex_2 + fx_5 = 0$. The solutions to these equations on section Γ_{15} consist of neutral fixed points. Let us denote them as $G_1\left(\frac{d}{a+d},\ 0,\ 0,\ 0,\ \frac{a}{a+d}\right)$, $G_2\left(\frac{f}{b+f},\ 0,\ 0,\ 0,\ \frac{b}{b+f}\right)$, respectively.

Theorem 2.6. *If condition $bd < af$ is satisfied, then the points belonging to the interval $[e_5, G_1] \cup [G_2,\ e_1] \subset \Gamma_{15}$ are saddles, and the points belonging to the interval $[G_1,\ G_2] \subset \Gamma_{15}$ are repellers.*

Proof. We know that if $\beta = 0$ then $\Gamma_{15} = \left\{(\alpha,\ 0\ ,\ 0,\ 0,\ 1-\alpha) \in S^4,\ 0 \le \alpha \le 1\right\}$. We find the eigenvalues of points belonging to edge Γ_{25}. From formula (1.3) we obtain

$$(1-\lambda)^3\,(1 + a\alpha - d\,(1-\alpha) - \lambda)\,(1 - b\alpha + f\,(1-\alpha) - \lambda) = 0.$$

Solutions to this equation:

$$\lambda_1 = \lambda_2 = \lambda_3 = 1,\ \lambda_4 = 1 + (a+d)\,\alpha - d, \lambda_5 = 1 + f - (b+f)\,\alpha.$$

Then, if $\alpha < \frac{d}{a+d}$, then we have $(a+d)\,\alpha - d < 0$, hence $|\lambda_4| < 1$. But if $bd < af$, then $f - (b+f)\,\alpha > 0$. Means $|\lambda_5| > 1$. So, the points of interval $[e_5, G_1] \subset \Gamma_{15}$ are saddle points. Similarly, if $bd < af$ and $\alpha > \frac{f}{b+f}$ are satisfied, then we have $|\lambda_4| > 1$ and $|\lambda_5| < 1$. Then all points of interval $[G_2,\ e_1] \subset \Gamma_{15}$ are saddle points. Values α belonging to the range $\frac{d}{a+d} < \alpha < \frac{f}{b+f}$ have $(a+d)\,\alpha - d > 0$ and $f - (b+f)\,\alpha > 0$. Then $|\lambda_4| > 1,\ |\lambda_5| > 1$. Consequently, the set of all points in interval $[G_1, G_2] \subset \Gamma_{15}$ are repulsive.The theorem has been proven.

Let us move on to considering the solution of equations $ax_1 - cx_2 - dx_5 = 0, -bx_1 + ex_2 + fx_5 = 0$ for points belonging to edge Γ_{12}. The solutions to these equations consist of neutral points in Γ_{12}, having the form $F_1\left(\frac{c}{a+c},\ \frac{a}{a+c},\ 0,\ 0,\ 0\right), F_2\left(\frac{e}{e+b},\ \frac{b}{e+b},\ 0,\ 0,\ 0\right)$.

Theorem 2.7. *If condition $bc < ae$ is satisfied, then the points belonging to the interval $[e_2, F_1] \cup [F_2,\ e_1] \subset \Gamma_{15}$ are saddle, and the points belonging to the interval $[F_1,\ F_2] \subset \Gamma_{15}$ are repulsive.*

Proof. The theorem is proved similarly to Theorem 2.6.

Above we studied the dynamics of points belonging to the boundary part of the face Γ_{125}. Now consider the set of internal points of the face

$$ri\Gamma_{125} = \left\{(\alpha,\ \beta\ ,\ 0,\ 0,\ 1-\alpha-\beta) \in S^4,\ 0 < \alpha,\ \beta < 1,\ \alpha+\beta < 1\right\}.$$

We find the eigenvalues for these points using the Jacobi matrix and formula (1.3):

$$(1-\lambda)^3\,(1 + a\alpha - c\beta - d\,(1-\alpha-\beta) - \lambda)\,(1 - b\alpha + c\beta + f\,(1-\alpha-\beta) - \lambda) = 0.$$

As a result, we get $\lambda_1 = \lambda_2 = \lambda_3 = 1$, $\lambda_4 = 1 + a\alpha - c\beta - d(1-\alpha-\beta)$, $\lambda_5 = 1 - b\alpha + c\beta + f(1-\alpha-\beta)$. Let us denote the set of solutions to the system of inequalities by T,

$$\begin{cases} a\alpha - c\beta - d(1-\alpha-\beta) < 0, \\ -b\alpha + c\beta + f(1-\alpha-\beta) < 0, \end{cases}$$

and the set of solutions to the system of inequalities through I ,

$$\begin{cases} a\alpha - c\beta - d(1-\alpha-\beta) > 0, \\ -b\alpha + c\beta + f(1-\alpha-\beta) > 0. \end{cases}$$

And the set of solutions to the system of inequalities

$$\begin{cases} a\alpha - c\beta - d(1-\alpha-\beta) > 0, \\ -b\alpha + c\beta + f(1-\alpha-\beta) < 0, \end{cases} \qquad \begin{cases} a\alpha - c\beta - d(1-\alpha-\beta) < 0, \\ -b\alpha + c\beta + f(1-\alpha-\beta) > 0. \end{cases}$$

denote by E . Based on the above mentioned, we get the following Proposition.

Proposition 1. When conditions $af < bd$, $ae < bc$ are met, the following confirmations are appropriate:

1. $T \subset \Gamma_{125}$ – all points belonging to the set are attractors;
2. All points belonging to the set $I \subset \Gamma_{125}$ are repellers;
3. All points belonging to the set $E \subset \Gamma_{125}$ are saddle points (see Figure 4).

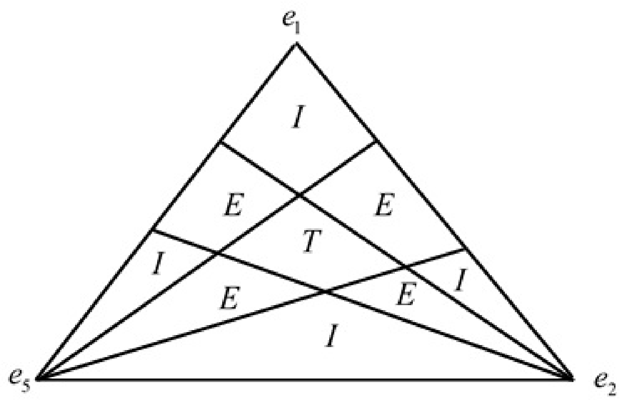

Figure 4. Classification of points belonging to the set in $ri\Gamma_{125}$.

The dynamics of internal points is studied similarly to the dynamics of boundary points discussed above.

3. Conclusion

The works devoted to discrete Lotka-Volterra mappings with non-degenerate skew-symmetric matrices are quite well known [5-7], [10]. From these works it is known that mappings of this type can be associated with elements of graph theory, in particular with the theory of tournaments. But, as it turned out, in the case when the skew-symmetric matrix is singular, then a mapping of this type can be associated with a partially directed graph. The study of this type of Lotka-Volterra mappings is relevant, since they can be considered as discrete models of epidemiology, ecology, and population genetics [8], [10]. In this regard, the work is devoted to the study of the dynamics of internal points of the degenerate Lotka-Volterra mapping acting in a four-dimensional simplex. Fixed points have been found for the operator under consideration; it turns out that there are infinitely many of them. The characters of fixed points on which the flow of trajectories of internal points of a given operator depends are studied. The following work will be devoted to finding the Fatou and Julia sets for mappings of this kind [1].

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