

A Comparative Study of Two Weighted Function-Space Formulations for a Degenerate–Singular Elliptic Operator

Eshmamatova D.B.* and Ochilova N.K.

ABSTRACT

We study a second-order elliptic operator that combines a leading coefficient degenerating on part of the boundary in one variable with a singular first-order coefficient of Bessel type in the other, and we compare two variational treatments of the associated semilinear Dirichlet problem with a Lipschitz nonlinearity. In the first, anisotropically weighted, setting we prove existence and uniqueness of a weak solution for every value of the degeneracy exponent, under a one-sided condition on the singular coefficient that is shown to be sharp; interior regularity holds throughout, and for strong degeneracy a conditional global estimate is obtained in an explicitly defined weighted second-order space. In the second setting an auxiliary power weight, chosen independently of the degeneracy exponent, is imposed on all derivatives. The singular term of the resulting weak form can then be computed exactly, which yields a necessary and sufficient coercivity condition that depends only on the auxiliary exponent, not on the degeneracy or on the domain. Existence, uniqueness, stability and interior regularity are established in this setting, the symmetric choice of the auxiliary weight is discussed, and the whole classical Bessel range is shown to be covered. The two thresholds coincide in the limit of a vanishing auxiliary exponent.

Keywords: degenerate elliptic operator; Bessel-type singular coefficient; weighted Sobolev space; Hardy-type inequality; coercivity threshold; Galerkin approximation; Lipschitz nonlinearity.

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1. Introduction

Diffusion in heterogeneous media, several models in population biology, and a number of problems in mathematical oncology are governed by elliptic operators whose coefficients either vanish or become unbounded on part of the domain boundary [1]. Once the leading coefficient degenerates, or a lower-order coefficient develops a singularity, the classical (unweighted) variational apparatus no longer applies directly: coercivity constants have to be re-derived, boundary conditions on the degeneration line may have to be modified, and the functional-analytic setting one chooses becomes an integral part of the mathematical problem rather than a formality preceding it. Keldysh [2] discovered that, for elliptic equations degenerating on the boundary, the lower-order coefficients decide whether a Dirichlet condition can be prescribed on the

degeneration line; for equations containing the Bessel operator $\partial_y^2 + \frac{\beta_0}{y} \partial_y$, a systematic theory was developed by Kipriyanov [3], and the symmetric weighted form $\operatorname{div}(y^\alpha \nabla u)$ of such operators has become central through the extension problem of Caffarelli and Silvestre [4]. Equations with a leading coefficient of power type y^m were studied systematically within the Central Asian school by Salakhitdinov and Mirsaburov [5], and subsequently by Mirsaburov [6, 7]; nonlinear degenerate elliptic problems in a weighted Sobolev setting close in spirit to Section 3 below have more recently been treated by Aharrouch and Bennouna [8], by El Ouaraabi, Allalou and Melliani [9], and by Alimohammady, Cattani and Koozehgar Kalleji [10], with a broader account given in the monograph of Cavalheiro [11]; regularity for degenerate elliptic boundary value problems has been revisited by Beisebay et al. [12], by Di Fazio, Fanciullo and Zamboni up to the boundary [13], and, via a Harnack inequality for a related class of degenerate sub-elliptic operators, by Do [14]. On the analytic side, the theory of Muckenhoupt weights goes back to Muckenhoupt [15], local regularity for equations with A_2 -weights is due to Fabes, Kenig and Serapioni [16], and elliptic equations governed by degenerate weights of this type were studied recently by Balci, Diening, Giova and Passarelli di Napoli [17]; the Hardy-type inequalities used below are classical [18], and recent instances of the monotone-operator machinery for nonlinear elliptic problems are given by Wang, Hou and Ge [19] and by Tiarimti Alaoui and Jourhmane [20].

Given a fixed operator of this type, the interesting question is not merely whether *some* well-posedness theory can be produced, but which weighted setting produces it, and what has to be paid for it. This is the question we take up here, for

$$Lu = \partial_x(y^m u_x) + u_{yy} + \frac{\beta_0}{y} u_y, \quad (1.1)$$

by building, and then placing next to each other, two settings for this operator: one in which the degeneracy weight y^m acts only on u_x , and a second in which an auxiliary exponent $\alpha \in (0, 1)$, unrelated to m , is imposed uniformly on both derivatives. Both settings work for every $m > 0$. In the first one, the singular term of the weak form can be evaluated exactly, which yields the one-sided condition $\beta_0 < 2/C_H(\Omega)$; when the degenerate part of the boundary contains a segment this is $\beta_0 < 1/2$, and the bound is sharp. In the second one, the derivative of the auxiliary weight shifts the singular coefficient from $-\beta_0$ to $\alpha - \beta_0$, and the same exact evaluation yields the sharp, m - and domain-independent threshold $\beta_0 < \beta_0^*(\alpha) := (1 + \alpha)/2$, which fills the whole Bessel range $\beta_0 < 1$ as $\alpha \rightarrow 1^-$ – at the price of a weaker energy norm and of a Poincaré constant that blows up as $\alpha \rightarrow 1^-$. A biological reading is available throughout: $u(x, y)$ can stand for the concentration of oxygen, a drug, or a signalling molecule diffusing through heterogeneous tissue, y for depth from a vascular interface, y^m for an anisotropic diffusivity that vanishes toward that interface, and $\frac{\beta_0}{y} u_y$ for a drift term concentrated near it [21]; we mention this interpretation without developing it further in the present note.

2. Statement of the problems

Throughout, $\Omega \subset \{(x, y) \in \mathbf{R}^2 : y > 0\}$ denotes a bounded domain, $\partial\Omega = \Gamma_0 \cup \Gamma_+$, where $\Gamma_0 = \partial\Omega \cap \{y = 0\}$ is the degenerate part of the boundary, $h := \sup\{y : (x, y) \in \Omega\}$, and $m > 0, \beta_0 \in \mathbf{R}$ are fixed parameters. No smoothness of $\partial\Omega$ is needed for the existence, uniqueness and interior regularity results of Section 3; the results up to the boundary (Theorem 3.4) and the whole of Section 4 are stated on the rectangle $\Omega = (0, \ell) \times (0, h)$, for which $\Gamma_0 = [0, \ell] \times \{0\}$.

2.1. Problem I: the anisotropically weighted nonlinear problem

$$-\frac{\partial}{\partial x} \left(y^m \frac{\partial u}{\partial x} \right) - \frac{\partial^2 u}{\partial y^2} - \frac{\beta_0}{y} \frac{\partial u}{\partial y} = f(u), \quad (x, y) \in \Omega, \quad u = 0 \text{ on } \partial\Omega, \quad (2.1)$$

where $f : \mathbf{R} \rightarrow \mathbf{R}$ satisfies

(A1) $|f(s) - f(t)| \leq L_f |s - t|$ for all $s, t \in \mathbf{R}$ (global Lipschitz continuity);

(A2) $|f(s)| \leq C_0(1 + |s|)$ for all $s \in \mathbf{R}$ (linear growth).

Because $y^m \rightarrow 0$ as $y \rightarrow 0^+$, the natural bilinear form ceases to be uniformly elliptic in the x -variable near Γ_0 ; this is what forces the choice of the anisotropic space

$$H_0^1(\Omega, y^m) := \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{H_0^1(\Omega, y^m)}}, \quad \|u\|_{H_0^1(\Omega, y^m)}^2 = \int_{\Omega} (y^m u_x^2 + u_y^2) dx dy.$$

2.2. Problem II: the auxiliary-weighted problem

On the rectangle $\Omega = (0, \ell) \times (0, h)$, consider

$$-\partial_x(y^m u_x) - u_{yy} - \frac{\beta_0}{y} u_y = f(u) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \quad (2.2)$$

governed by the identical operator that appears in (2.1), but now posed in the auxiliary weighted space $V_{\alpha, m}(\Omega)$ of Section 4, where $\alpha \in (0, 1)$ is selected independently of m and $\|u\|_{V_{\alpha, m}}^2 = \int_{\Omega} (y^{\alpha+m} u_x^2 + y^{\alpha} u_y^2 + y^{\alpha} u^2)$. As will be seen, the resulting threshold on β_0 is a function of α only, with no dependence on m . Here f is required to satisfy (H1) global Lipschitz continuity with constant L and (H2) linear growth – the same two requirements as (A1) and (A2), relabelled for Problem II.

3. The anisotropically weighted space $H_0^1(\Omega, y^m)$

For every $m > 0$ the following two inequalities hold:

$$\int_{\Omega} \frac{u^2}{y^2} \leq C_H(\Omega) \int_{\Omega} u_y^2, \quad \int_{\Omega} u^2 \leq C_P(\Omega) \int_{\Omega} (y^m u_x^2 + u_y^2), \quad u \in H_0^1(\Omega, y^m), \quad (3.1)$$

where $C_H(\Omega)$ and $C_P(\Omega)$ denote the best constants; one always has $C_H(\Omega) \leq 4$ and $C_P(\Omega) \leq h^2/2$.

Remark 3.1 (Scope of (3.1); no restriction on m). For $u \in C_c^\infty(\Omega)$ and fixed x , the slice $\psi(y) = u(x, y)$, extended by zero, belongs to $C_c^\infty((0, \infty))$, and the classical one-dimensional Hardy inequality $\int_0^\infty \psi^2/y^2 dy \leq 4 \int_0^\infty \psi'^2 dy$ [18] gives the first inequality in (3.1) with constant 4 after integration in x ; the second one, with constant $h^2/2$, is proved in Remark 3.2. Both pass to all of $H_0^1(\Omega, y^m)$ by density (Fatou's lemma for the first). Neither step involves the weight y^m at all. In particular, the Muckenhoupt property $y^m \in A_2$, which holds precisely for $-1 < m < 1$ [15], is *not* used anywhere in this section. It would matter if one wished to identify $H_0^1(\Omega, y^m)$ with the subspace of a weighted space $W^{1,2}$ of functions with vanishing trace by means of the general theory [11, 16]; such an identification is not needed here, because $H_0^1(\Omega, y^m)$ is defined as a completion of $C_c^\infty(\Omega)$ and the vanishing of its elements on Γ_0 follows directly from Remark 3.2 for every $m > 0$. Consequently Lemmas 3.1–3.3 and Theorems 3.1–3.3 below hold for *every* $m > 0$, including $m \geq 1$. The degeneracy exponent enters only the regularity statement of Theorem 3.4, where $m > 1$ is needed for the Hardy–Rellich inequality of Lemma 3.4.

Remark 3.2 (An elementary global L^2 -bound, valid for every $m > 0$). The space $H_0^1(\Omega, y^m)$ embeds continuously into $L^2(\Omega)$ for every $m > 0$, which is what makes every element of the completion an actual function on Ω rather than a merely abstract Cauchy-sequence class. For $u \in C_c^\infty(\Omega)$, the representation $u(x, y) = \int_0^y u_y(x, t) dt$ (valid since $u \equiv 0$ near Γ_0), together with Cauchy–Schwarz, gives $|u(x, y)|^2 \leq y \int_0^h u_y(x, t)^2 dt$

for every $y \in (0, h)$; integrating over Ω gives the m -independent bound

$$\int_{\Omega} u^2 dx dy \leq \frac{h^2}{2} \int_{\Omega} u_y^2 dx dy \leq \frac{h^2}{2} \|u\|_{H_0^1(\Omega, y^m)}^2, \quad u \in C_c^\infty(\Omega).$$

Since this holds on the dense subspace $C_c^\infty(\Omega)$ and $L^2(\Omega)$ is complete, it extends by continuity to all of $H_0^1(\Omega, y^m)$: every $\|\cdot\|_{H_0^1(\Omega, y^m)}$ -Cauchy sequence is in particular $L^2(\Omega)$ -Cauchy, so the completion is canonically identified with a closed subspace of $L^2(\Omega)$, and every $u \in H_0^1(\Omega, y^m)$ is the $L^2(\Omega)$ -limit – along a subsequence, the a.e. pointwise limit – of any approximating sequence $\phi_j \in C_c^\infty(\Omega)$. This proves the second inequality of (3.1) with $C_P(\Omega) \leq h^2/2$, uses no property of A_2 -weights and no restriction on m , and is exactly what justifies statements such as “ $u(x, 0) = 0$ for a.e. x ” and the density argument in the proof of Lemma 3.3 below for every $m > 0$.

Multiplying (2.1) by $v \in C_c^\infty(\Omega)$ and integrating by parts gives the bilinear form

$$a_m(u, v) := \int_{\Omega} \left(y^m u_x v_x + u_y v_y - \frac{\beta_0}{y} u_y v \right) dx dy. \quad (3.2)$$

A function $u \in H_0^1(\Omega, y^m)$ will be called a weak solution of (2.1) whenever

$$a_m(u, v) = \int_{\Omega} f(u) v dx dy \quad \forall v \in H_0^1(\Omega, y^m). \quad (3.3)$$

The singular term is well defined: Cauchy–Schwarz combined with the first estimate of (3.1), applied to v , gives

$$\left| \int_{\Omega} \frac{\beta_0}{y} u_y v dx dy \right| \leq |\beta_0| C_H(\Omega)^{1/2} \|u\|_{H_0^1(\Omega, y^m)} \|v_y\|_{L^2(\Omega)}, \quad (3.4)$$

so a_m is a continuous bilinear form on $H_0^1(\Omega, y^m) \times H_0^1(\Omega, y^m)$.

Lemma 3.1 (Exact form of the singular term). *Let $m > 0$. For every $u \in H_0^1(\Omega, y^m)$,*

$$\int_{\Omega} \frac{u_y u}{y} dx dy = \frac{1}{2} \int_{\Omega} \frac{u^2}{y^2} dx dy, \quad \text{hence} \quad a_m(u, u) = \|u\|_{H_0^1(\Omega, y^m)}^2 - \frac{\beta_0}{2} \int_{\Omega} \frac{u^2}{y^2} dx dy. \quad (3.5)$$

Consequently, with $\beta_0^+ := \max\{\beta_0, 0\}$,

$$a_m(u, u) \geq \kappa_I \|u\|_{H_0^1(\Omega, y^m)}^2, \quad \kappa_I := 1 - \frac{\beta_0^+ C_H(\Omega)}{2}. \quad (3.6)$$

Proof. For $u \in C_c^\infty(\Omega)$ and fixed x , the slice $\psi = u(x, \cdot)$ has compact support in $\{y > 0\}$, so integration by parts produces no boundary terms and, since $(1/y)' = -1/y^2$,

$$\int \frac{\psi \psi'}{y} dy = \frac{1}{2} \int \frac{(\psi^2)'}{y} dy = \frac{1}{2} \int \frac{\psi^2}{y^2} dy.$$

Integrating in x gives (3.5) on $C_c^\infty(\Omega)$. Both sides are continuous on $H_0^1(\Omega, y^m)$ by (3.1) and (3.4), so the identity extends by density. If $\beta_0 \leq 0$, the last term in (3.5) is nonnegative and $a_m(u, u) \geq \|u\|_{H_0^1(\Omega, y^m)}^2$; if $\beta_0 > 0$, (3.1) gives $\int_{\Omega} u^2/y^2 \leq C_H(\Omega) \|u\|_{H_0^1(\Omega, y^m)}^2$. In both cases (3.6) follows. $\square$

We impose

$$\beta_0 C_H(\Omega) < 2 \quad (\text{A0}), \quad C_0 C_P(\Omega) < \kappa_I \quad (\text{A2}'). \quad (3.7)$$

Condition (A0) is equivalent to $\kappa_I > 0$; it imposes no lower bound on β_0 , and, since $C_H(\Omega) \leq 4$, it holds in particular whenever $\beta_0 < 1/2$.

Lemma 3.2 (A priori energy bound). *Let $m > 0$ and suppose that (A0), (A2) and (A2') hold. Then any weak solution $u \in H_0^1(\Omega, y^m)$ of (2.1) obeys $\|u\|_{H_0^1(\Omega, y^m)}^2 \leq C$, the constant C depending only on C_0 , $C_P(\Omega)$, $|\Omega|$ and κ_I .*

Proof. Insert $v = u$ into (3.3) and use Lemma 3.1:

$$\kappa_I \|u\|_{H_0^1(\Omega, y^m)}^2 \leq a_m(u, u) = \int_{\Omega} f(u)u \, dx \, dy.$$

The linear-growth hypothesis (A2) yields $\int_{\Omega} f(u)u \leq C_0 \int_{\Omega} (|u| + u^2)$. The quadratic part is bounded by $C_0 C_P(\Omega) \|u\|_{H_0^1(\Omega, y^m)}^2$, and the linear part by $C_0 |\Omega|^{1/2} C_P(\Omega)^{1/2} \|u\|_{H_0^1(\Omega, y^m)} \leq \varepsilon \|u\|_{H_0^1(\Omega, y^m)}^2 + C_1$ (Young's inequality), with $C_1 = C_0^2 |\Omega| C_P(\Omega) / (4\varepsilon)$. Hence

$$\kappa_I \|u\|_{H_0^1(\Omega, y^m)}^2 \leq C_1 + (C_0 C_P(\Omega) + \varepsilon) \|u\|_{H_0^1(\Omega, y^m)}^2.$$

Taking $\varepsilon = \frac{1}{2}(\kappa_I - C_0 C_P(\Omega))$, which (A2') guarantees is positive, we obtain

$$\frac{1}{2}(\kappa_I - C_0 C_P(\Omega)) \|u\|_{H_0^1(\Omega, y^m)}^2 \leq C_1,$$

which is the asserted bound. $\square$

The space $H_0^1(\Omega, y^m)$ is separable (it is isometric to a subspace of $L^2(\Omega)^2$ via $u \mapsto (y^{m/2}u_x, u_y)$), so there are linearly independent $\psi_1, \psi_2, \dots \in C_c^\infty(\Omega)$ whose span is dense. Set $V_N = \text{span}\{\psi_1, \dots, \psi_N\}$ and look for $u_N \in V_N$ with $a_m(u_N, \psi_j) = \int_{\Omega} f(u_N)\psi_j$, $j = 1, \dots, N$. Define $\Phi_N : V_N \rightarrow V_N$ by $(\Phi_N(w), v)_{H_0^1(\Omega, y^m)} = a_m(w, v) - \int_{\Omega} f(w)v$ for $v \in V_N$; it is continuous by (A1), and the computation in the proof of Lemma 3.2 gives $(\Phi_N(w), w) \geq \frac{1}{2}(\kappa_I - C_0 C_P(\Omega))\|w\|^2 - C_1 > 0$ on a sphere $\|w\| = R$ with R independent of N . By the acute-angle lemma (a standard corollary of Brouwer's fixed-point theorem), Φ_N has a zero u_N with $\|u_N\|_{H_0^1(\Omega, y^m)} \leq R$ for every N .

Lemma 3.3 (Compactness, valid for every $m > 0$). *Let $m > 0$, and let $\{u_N\}$ be a bounded sequence in $H_0^1(\Omega, y^m)$. Then there exist a subsequence, still denoted by $\{u_N\}$, and a function $u \in H_0^1(\Omega, y^m)$ such that*

$$u_N \rightharpoonup u \quad \text{weakly in } H_0^1(\Omega, y^m), \quad u_N \rightarrow u \quad \text{strongly in } L^2(\Omega),$$

and, after passage to a further subsequence, $u_N \rightarrow u$ almost everywhere in Ω .

Proof. Weak compactness of bounded sets in the Hilbert space $H_0^1(\Omega, y^m)$ gives the weak limit u . Given $\delta > 0$, split Ω into the near-degenerate strip $\Omega_\delta = \Omega \cap \{0 < y < \delta\}$ and the remainder $\Omega^\delta = \Omega \cap \{y \geq \delta\}$. Fix $\zeta \in C_c^\infty(\mathbf{R})$ with $\zeta = 0$ on $(-\infty, \delta/2]$, $\zeta = 1$ on $[\delta, \infty)$. Since $y^m \geq (\delta/2)^m$ on the support of ζ , the functions ζu_N belong to the unweighted space $H_0^1(\Omega)$ (they are limits of $\zeta \phi_j \in C_c^\infty(\Omega)$) and are bounded there; the Rellich–Kondrachov theorem for H_0^1 on a bounded open set, which requires no regularity of $\partial\Omega$, supplies a subsequence converging strongly in $L^2(\Omega)$, hence a subsequence of $\{u_N\}$ converging strongly in $L^2(\Omega^\delta)$. On Ω_δ , since $C_c^\infty(\Omega)$ is dense in $H_0^1(\Omega, y^m)$ by definition, the pointwise representation

$$u_N(x, y) = \int_0^y u_{N,y}(x, t) \, dt,$$

valid for $u_N \in C_c^\infty(\Omega)$ by the fundamental theorem of calculus, extends by density to every $u_N \in H_0^1(\Omega, y^m)$: choosing $\phi_j \in C_c^\infty(\Omega)$ with $\phi_j \rightarrow u_N$ in $\|\cdot\|_{H_0^1(\Omega, y^m)}$, we have in particular $\phi_{j,y} \rightarrow u_{N,y}$ in $L^2(\Omega)$, so along a subsequence $\phi_{j,y}(x, \cdot) \rightarrow u_{N,y}(x, \cdot)$ in $L^2((0, \delta))$ for a.e. x , which lets the right-hand side of the displayed

identity (written for ϕ_j) pass to the limit for a.e. such x ; since $\phi_j \rightarrow u_N$ in $L^2(\Omega)$ as well (Remark 3.2), a further subsequence gives $\phi_j(x, y) \rightarrow u_N(x, y)$ for a.e. (x, y) , identifying the limit of the left-hand side with $u_N(x, y)$ and so transferring the identity to u_N . Together with the resulting pointwise bound, valid for a.e. x and $0 < y < \delta$,

$$|u_N(x, y)|^2 = \left| \int_0^y u_{N,y}(x, t) dt \right|^2 \leq y \int_0^\delta |u_{N,y}(x, t)|^2 dt,$$

integration over Ω_δ gives $\int_{\Omega_\delta} u_N^2 \leq \frac{\delta^2}{2} \int_{\Omega} u_{N,y}^2$, with a right-hand side bounded uniformly in N ; the same bound holds for u . The L^2 -mass carried on Ω_δ can therefore be forced below any prescribed level by taking δ small, uniformly in N . Letting $\delta_k \downarrow 0$ and extracting a diagonal subsequence produces the claimed strong $L^2(\Omega)$ -convergent subsequence, whose limit is u by uniqueness of weak limits, and a further subsequence of which converges pointwise a.e. No property of A_2 -weights was used, so the argument is valid for every $m > 0$. $\square$

Theorem 3.1 (Existence of a weak solution). *Let $m > 0$ and assume (A0), (A1), (A2) and (A2'). Then problem (2.1) admits at least one weak solution $u \in H_0^1(\Omega, y^m)$.*

Proof. The Galerkin approximations u_N constructed after Lemma 3.2 are bounded uniformly in N in $H_0^1(\Omega, y^m)$; by Lemma 3.3, along a subsequence, $u_N \rightharpoonup u$ weakly in $H_0^1(\Omega, y^m)$, strongly in $L^2(\Omega)$ and a.e. The Lipschitz property (A1) then forces $f(u_N) \rightarrow f(u)$ strongly in $L^2(\Omega)$, since

$$\|f(u_N) - f(u)\|_{L^2(\Omega)} \leq L_f \|u_N - u\|_{L^2(\Omega)} \rightarrow 0.$$

Letting $N \rightarrow \infty$ in the Galerkin identities $a_m(u_N, \psi_j) = \int_{\Omega} f(u_N) \psi_j$ – first against a fixed ψ_j , using that $a_m(\cdot, \psi_j)$ is a continuous linear functional for the left-hand side and the strong L^2 -convergence of $f(u_N)$ for the right-hand side, then extending the resulting identity from $\bigcup_N V_N$ to all of $H_0^1(\Omega, y^m)$ by density – shows that u satisfies (3.3). Thus $u \in H_0^1(\Omega, y^m)$ is a weak solution of (2.1). $\square$

Theorem 3.2 (Uniqueness). *Let $m > 0$, assume (A0) and (A1), and suppose $L_f C_P(\Omega) < \kappa_I$. Then the weak solution of (2.1) is unique.*

Proof. Let u_1 and u_2 be two weak solutions and set $w = u_1 - u_2$. Subtracting the two weak formulations and testing against $v = w$ gives $a_m(w, w) = \int_{\Omega} (f(u_1) - f(u_2))w$. By Lemma 3.1 the left-hand side is at least $\kappa_I \|w\|_{H_0^1(\Omega, y^m)}^2$, while (A1) together with (3.1) bounds the right-hand side from above by $L_f C_P(\Omega) \|w\|_{H_0^1(\Omega, y^m)}^2$. Hence

$$(\kappa_I - L_f C_P(\Omega)) \|w\|_{H_0^1(\Omega, y^m)}^2 \leq 0,$$

and w vanishes identically. $\square$

Remark 3.3 (Sharpness of (A0)). Suppose that Ω contains a rectangle $(a, b) \times (0, \delta_0)$ with $[a, b] \times \{0\} \subset \Gamma_0$ (as is the case for the rectangle of Problem II). Then $C_H(\Omega) = 4$, so that (A0) reads $\beta_0 < 1/2$, and for every $m > 0$ the form a_m fails to be coercive on $H_0^1(\Omega, y^m)$ whenever $\beta_0 \geq 1/2$. Indeed, the functions $u_k = \xi(x)\eta(y)y^{1/2+1/k}$ of Theorem 4.1(ii) below, taken with $\alpha = 0$ and supported in $(a, b) \times (0, \delta_0)$, belong to $H_0^1(\Omega, y^m)$ (by the same truncation argument) and satisfy, by (3.5) and the asymptotics computed there, $\int u_k^2/y^2 / \int u_{k,y}^2 \rightarrow 4$ and

$$\frac{a_m(u_k, u_k)}{\|\xi\|_{L^2}^2 \int_0^h \psi'_k(y)^2 dy} \rightarrow 1 - 2\beta_0.$$

Thus the condition $\beta_0 < 1/2$ is necessary and sufficient for coercivity of a_m , and no lower bound on β_0 is required.

Theorem 3.3 (Interior regularity and smoothness, no restriction on m). *Let $m > 0$ and let $u \in H_0^1(\Omega, y^m)$ be a weak solution of (2.1) with f satisfying (A1)–(A2). Then $u \in H_{\text{loc}}^2(\Omega)$, the equation (2.1) holds a.e. in Ω , and for every $\Omega' \Subset \Omega$ one has $\|u\|_{H^2(\Omega')} \leq C_{\Omega'}(\|f(u)\|_{L^2(\Omega)} + \|u\|_{H_0^1(\Omega, y^m)})$. If, in addition, $f \in C^\infty(\mathbf{R})$, then $u \in C^\infty(\Omega)$.*

Proof. Since $u \in L^2(\Omega)$, (A2) gives $f(u) \in L^2(\Omega)$. Since $\Omega \subset \{y > 0\}$ and $\Omega' \Subset \Omega$, we have $y \geq \delta > 0$ on a neighbourhood $\Omega'' \Subset \Omega$ of $\overline{\Omega'}$; there every coefficient appearing in (2.1) is smooth and bounded, the principal ones bounded away from zero, and $u \in H^1(\Omega'')$. The first claim follows from the standard interior regularity theory for uniformly elliptic equations [22]. If $f \in C^\infty(\mathbf{R})$, then $u \in H_{\text{loc}}^2(\Omega) \subset C(\Omega)$ in two dimensions, so $f(u) \in H_{\text{loc}}^1(\Omega)$ and interior regularity gives $u \in H_{\text{loc}}^3(\Omega)$; inductively, $u \in H_{\text{loc}}^k(\Omega)$ for every k , whence $u \in C^\infty(\Omega)$ by Sobolev embedding. Smoothness up to Γ_0 is not claimed. $\square$

Lemma 3.4 (Weighted Hardy–Rellich inequality on an interval). *Let $m > 1$ and $\delta > 0$. Suppose $\psi \in H_{\text{loc}}^1((0, \delta])$ (that is, $\psi \in H^1((\eta, \delta))$ for every $0 < \eta < \delta$) and*

$$B := \int_0^\delta y^m \psi'(y)^2 dy < \infty.$$

Then $A_0 := \int_0^\delta y^{m-2} \psi(y)^2 dy$ is finite and

$$\int_0^\delta y^{m-2} \psi(y)^2 dy \leq \frac{4}{(m-1)^2} \int_0^\delta y^m \psi'(y)^2 dy + \frac{2\delta^{m-1}}{m-1} \psi(\delta)^2. \quad (3.8)$$

Proof. Fix $0 < \eta < \delta$ and set $A_0(\eta) := \int_\eta^\delta y^{m-2} \psi^2 dy < \infty$. Since ψ is absolutely continuous on $[\eta, \delta]$ and $\frac{d}{dy} \left[\frac{y^{m-1}}{m-1} \right] = y^{m-2}$, integration by parts gives

$$A_0(\eta) = \frac{\delta^{m-1}}{m-1} \psi(\delta)^2 - \frac{\eta^{m-1}}{m-1} \psi(\eta)^2 - \frac{2}{m-1} \int_\eta^\delta y^{m-1} \psi \psi' dy \leq K + c A_0(\eta)^{1/2} B^{1/2},$$

where $K := \frac{\delta^{m-1}}{m-1} \psi(\delta)^2$ and $c := \frac{2}{m-1}$; here the term at η was dropped because it is nonpositive ($m > 1$), and Cauchy–Schwarz was applied to the cross term. Writing $t = A_0(\eta)^{1/2}$, the inequality $t^2 - cB^{1/2}t - K \leq 0$ gives $t \leq \frac{1}{2}(cB^{1/2} + \sqrt{c^2B + 4K})$, and squaring together with $(p+q)^2 \leq 2p^2 + 2q^2$ yields $A_0(\eta) \leq c^2B + 2K$. The right-hand side does not depend on η ; letting $\eta \downarrow 0$ (monotone convergence) gives $A_0 < \infty$ and (3.8). $\square$

For the regularity statement up to Γ_0 we need a precisely defined weighted second-order space. For $m > 1$ set

$$\begin{aligned} \mathcal{H}_m^2(\Omega) &:= \{u \in H_0^1(\Omega, y^m) \cap H_{\text{loc}}^2(\Omega) : \|u\|_{\mathcal{H}_m^2(\Omega)} < \infty\}, \\ \|u\|_{\mathcal{H}_m^2(\Omega)}^2 &:= \|u\|_{H_0^1(\Omega, y^m)}^2 + \int_\Omega (y^{3m} u_{xx}^2 + y^m u_{yy}^2 + y^{m-2} u_y^2) dx dy. \end{aligned} \quad (3.9)$$

The weights are dictated by the operator: y^m on u_{yy} is the weight of the hypothesis below; y^{m-2} on u_y is the weight produced from it by Lemma 3.4; and since y^m does not depend on x , the equation (2.1), which holds a.e. by Theorem 3.3, can be solved for u_{xx} ,

$$y^m u_{xx} = -u_{yy} - \frac{\beta_0}{y} u_y - f(u) \quad \text{a.e. in } \Omega, \quad (3.10)$$

so that multiplication by $y^{m/2}$ gives exactly the weight y^{3m} on u_{xx}^2 . The mixed derivative u_{xy} is not controlled.

Theorem 3.4 (Conditional global weighted regularity for $m > 1$). *Let $\Omega = (0, \ell) \times (0, h)$, $m > 1$, let f satisfy (A1)–(A2), and let $u \in H_0^1(\Omega, y^m)$ be a weak solution of (2.1) such that $y^{m/2}u_{yy} \in L^2(\Omega)$. Then $u \in \mathcal{H}_m^2(\Omega)$, and*

$$\int_{\Omega} y^{m-2} u_y^2 dx dy \leq \frac{4}{(m-1)^2} \int_{\Omega} y^m u_{yy}^2 dx dy + C_1 (\|f(u)\|_{L^2(\Omega)} + \|u\|_{H_0^1(\Omega, y^m)})^2, \quad (3.11)$$

$$\int_{\Omega} y^{3m} u_{xx}^2 dx dy \leq 3 \int_{\Omega} y^m u_{yy}^2 dx dy + 3\beta_0^2 \int_{\Omega} y^{m-2} u_y^2 dx dy + 3h^m \|f(u)\|_{L^2(\Omega)}^2, \quad (3.12)$$

so that $\|u\|_{\mathcal{H}_m^2(\Omega)} \leq C (\|y^{m/2}u_{yy}\|_{L^2(\Omega)} + \|f(u)\|_{L^2(\Omega)} + \|u\|_{H_0^1(\Omega, y^m)})$, with C_1, C depending only on m, β_0, ℓ and h .

Proof. Step 1: the non-degenerate part. Fix $\zeta \in C^\infty(\mathbf{R})$ with $\zeta = 0$ on $(-\infty, h/4]$ and $\zeta = 1$ on $[h/2, \infty)$, and set $R' = (0, \ell) \times (h/8, h)$ and $w = \zeta u$. Since $y^m \geq (h/8)^m$ on R' , $w \in H_0^1(R')$ (it is the limit of $\zeta \phi_j \in C_c^\infty(R')$). Testing (3.3) with $\zeta \varphi$, $\varphi \in C_c^\infty(R')$, and rearranging gives

$$\int_{R'} (y^m w_x \varphi_x + w_y \varphi_y) = \int_{R'} g \varphi, \quad g := \zeta f(u) + \frac{\beta_0}{y} \zeta u_y - 2\zeta' u_y - \zeta'' u \in L^2(R'),$$

with $\|g\|_{L^2(R')} \leq C (\|f(u)\|_{L^2(\Omega)} + \|u\|_{H_0^1(\Omega, y^m)})$. The rectangle R' is convex and the coefficients $y^m, 1$ are Lipschitz on $\overline{R'}$ and bounded below by positive constants, so the H^2 -regularity theorem for the Dirichlet problem on convex domains [23, Theorem 3.2.1.2] gives $w \in H^2(R')$ with $\|w\|_{H^2(R')} \leq C \|g\|_{L^2(R')}$. (This replaces, on the rectangle, the boundary regularity for C^2 domains, which is not available at the corners.) As $w = u$ on $\Omega^{h/2} := (0, \ell) \times (h/2, h)$,

$$\|u\|_{H^2(\Omega^{h/2})} \leq C (\|f(u)\|_{L^2(\Omega)} + \|u\|_{H_0^1(\Omega, y^m)}),$$

and the trace theorem applied to $u_y \in H^1(\Omega^{h/2})$ on the segment $\{y = h/2\}$ bounds $\int_0^\ell u_y(x, h/2)^2 dx$ by the square of the same right-hand side.

Step 2: the strip near Γ_0 . Since $u \in H_{\text{loc}}^2(\Omega)$ (Theorem 3.3) and $y^{m/2}u_{yy} \in L^2(\Omega)$, for a.e. $x \in (0, \ell)$ the slice $\psi = u_y(x, \cdot)$ belongs to $H_{\text{loc}}^1((0, h/2])$, satisfies $\int_0^{h/2} y^m \psi'^2 dy < \infty$, and $\psi(h/2)$ coincides with the trace of Step 1. Lemma 3.4 with $\delta = h/2$, integrated over $x \in (0, \ell)$, gives

$$\int_{(0, \ell) \times (0, h/2)} y^{m-2} u_y^2 \leq \frac{4}{(m-1)^2} \int_{\Omega} y^m u_{yy}^2 + \frac{2(h/2)^{m-1}}{m-1} \int_0^\ell u_y(x, h/2)^2 dx.$$

Step 3: conclusion. On $\Omega^{h/2}$ the weight y^{m-2} is bounded by $\max\{h^{m-2}, (h/2)^{m-2}\}$, so $\int_{\Omega^{h/2}} y^{m-2} u_y^2 \leq C \|u\|_{H_0^1(\Omega, y^m)}^2$. Adding this to Step 2 and using Step 1 for the trace term gives (3.11). Finally, (3.10) gives $y^{3m/2} u_{xx} = -y^{m/2} u_{yy} - \beta_0 y^{m/2-1} u_y - y^{m/2} f(u)$, and $(p+q+r)^2 \leq 3(p^2+q^2+r^2)$ together with $y^m \leq h^m$ yields (3.12). Hence $u \in \mathcal{H}_m^2(\Omega)$ with the stated bound. $\square$

Remark 3.4. Theorem 3.4 is conditional in an essential way: it assumes $y^{m/2}u_{yy} \in L^2(\Omega)$ rather than deriving it from (2.1). The restriction to the rectangle is used twice: in Step 1, where convexity replaces boundary regularity at the corners, and in Step 2, where every vertical slice has the full height h . On a general domain the same argument applies on any rectangle $(a, b) \times (0, h_0) \subset \Omega$ adjacent to Γ_0 , but near the points where Γ_0 meets Γ_+ the slice heights degenerate and the argument gives no uniform bound. Whether global regularity in $\mathcal{H}_m^2(\Omega)$ can be obtained for $m > 1$ without the hypothesis on u_{yy} is left open; related smoothness questions for degenerate elliptic boundary value problems, by different methods, are treated in [12, 13].

4. The auxiliary-weighted space $V_{\alpha,m}(\Omega)$

4.1. Problem II: the auxiliary-weighted formulation

Throughout this section $\Omega = (0, \ell) \times (0, h)$. For $0 < \alpha < 1$ and $m > 0$, define

$$V_{\alpha,m}(\Omega) = \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{V_{\alpha,m}}}, \quad \|u\|_{V_{\alpha,m}}^2 = \int_{\Omega} \left(y^{\alpha+m} u_x^2 + y^\alpha u_y^2 + y^\alpha u^2 \right) dx dy, \quad (4.1)$$

and write $|u|_1^2 := \int_{\Omega} (y^{\alpha+m} u_x^2 + y^\alpha u_y^2) dx dy$ for the energy seminorm.

Remark 4.1 (Motivating the choice of $V_{\alpha,m}$). One might first attempt a space weighted uniformly by y^α , independently of m , namely

$$\int_{\Omega} y^\alpha \left(u^2 + u_x^2 + u_y^2 \right) dx dy.$$

This choice, however, is not compatible with the anisotropic principal part of (2.2). Indeed, the coercivity argument controls $\int_{\Omega} y^{\alpha+m} u_x^2 dx dy$, whereas, for $m > 0$, the weight $y^{\alpha+m}$ is smaller than y^α near $y = 0$. Thus control of the former does not provide control of $\int_{\Omega} y^\alpha u_x^2 dx dy$. The space in (4.1) avoids this mismatch by weighting u_x with exactly $y^{\alpha+m}$, which is the coefficient appearing in the principal part of the weighted weak form below. Equivalently, $\|u\|_{V_{\alpha,m}}^2 = \int_{\Omega} y^\alpha (y^m u_x^2 + u_y^2 + u^2) dx dy$. Thus $V_{\alpha,m}$ preserves the anisotropic structure of $H_0^1(\Omega, y^m)$ while introducing the auxiliary factor y^α .

The introduction of the auxiliary factor y^α must be taken into account when deriving the corresponding weak formulation. Multiplying (2.2) by $y^\alpha v$, with $v \in C_c^\infty(\Omega)$, and integrating by parts (no boundary terms appear, since v has compact support in Ω) gives

$$\begin{aligned} - \int_{\Omega} y^\alpha \partial_x (y^m u_x) v dx dy &= \int_{\Omega} y^{\alpha+m} u_x v_x dx dy, \\ - \int_{\Omega} y^\alpha u_{yy} v dx dy &= \int_{\Omega} y^\alpha u_y v_y dx dy + \alpha \int_{\Omega} y^{\alpha-1} u_y v dx dy, \end{aligned}$$

while

$$- \int_{\Omega} \frac{\beta_0}{y} u_y y^\alpha v dx dy = -\beta_0 \int_{\Omega} y^{\alpha-1} u_y v dx dy.$$

Consequently, the two singular first-order contributions combine as $(\alpha - \beta_0) \int_{\Omega} y^{\alpha-1} u_y v dx dy$; for $\alpha = 0$ this reduces to the term $-\beta_0 \int_{\Omega} y^{-1} u_y v$ of (3.2). Therefore, the auxiliary weight does not merely multiply the original weak formulation: its derivative contributes an additional singular term. The corresponding weighted bilinear form is

$$a_{w,m}(u, v) = \int_{\Omega} \left(y^{\alpha+m} u_x v_x + y^\alpha u_y v_y + (\alpha - \beta_0) y^{\alpha-1} u_y v \right) dx dy, \quad (4.2)$$

and it is the form associated with the operator

$$L_m u := -\partial_x (y^{\alpha+m} u_x) - \partial_y (y^\alpha u_y) + (\alpha - \beta_0) y^{\alpha-1} u_y = y^\alpha \left(-\partial_x (y^m u_x) - u_{yy} - \frac{\beta_0}{y} u_y \right), \quad (4.3)$$

that is, y^α times the operator of (2.2). A function $u \in V_{\alpha,m}(\Omega)$ is called a weak solution of (2.2) if

$$a_{w,m}(u, v) = \int_{\Omega} y^\alpha f(u) v dx dy, \quad \forall v \in V_{\alpha,m}(\Omega). \quad (4.4)$$

The singular term in (4.2) is controlled by a weighted Hardy inequality, which we now state and prove explicitly, since it is used repeatedly below.

Lemma 4.1 (Weighted Hardy inequality). *Let $0 < \alpha < 1$. For $u \in V_{\alpha,m}(\Omega)$,*

$$\int_{\Omega} y^{\alpha-2} u^2 dx dy \leq C_H \int_{\Omega} y^{\alpha} u_y^2 dx dy, \quad C_H = C_H(\alpha) = \frac{4}{(1-\alpha)^2}. \quad (4.5)$$

Proof. It suffices to prove (4.5) for $u \in C_c^\infty(\Omega)$ and then extend it by density. Fix such a u and a value of x , and set $\psi(y) := u(x, y)$; since u has compact support in Ω , ψ has compact support in $(0, h)$, so $y^{\alpha-1}\psi(y)^2$ vanishes identically near both $y = 0$ and $y = h$. Since $\alpha \neq 1$ and $\frac{d}{dy}[y^{\alpha-1}] = (\alpha-1)y^{\alpha-2}$, integrating by parts over $(0, h)$ therefore produces no boundary term:

$$\int_0^h y^{\alpha-2} \psi^2 dy = \frac{1}{\alpha-1} \int_0^h (y^{\alpha-1})' \psi^2 dy = \frac{2}{1-\alpha} \int_0^h y^{\alpha-1} \psi \psi' dy. \quad (4.6)$$

By Cauchy–Schwarz, applied to $y^{(\alpha-2)/2}\psi$ and $y^{\alpha/2}\psi'$,

$$\int_0^h y^{\alpha-1} \psi \psi' dy \leq \left(\int_0^h y^{\alpha-2} \psi^2 dy \right)^{1/2} \left(\int_0^h y^{\alpha} \psi'(y)^2 dy \right)^{1/2}.$$

Writing $A := \int_0^h y^{\alpha-2} \psi^2 dy$ and $B := \int_0^h y^{\alpha} \psi'(y)^2 dy$ – both finite, since $\psi \in C_c^\infty((0, h))$ – the two displays combine to $A \leq \frac{2}{1-\alpha} A^{1/2} B^{1/2}$. If $A = 0$ the claim is trivial; otherwise dividing by $A^{1/2} > 0$ gives $A \leq \frac{4}{(1-\alpha)^2} B$, which is (4.5) restricted to the slice through x . Integrating over x gives (4.5) for $u \in C_c^\infty(\Omega)$. For general $u \in V_{\alpha,m}(\Omega)$, choose $u_j \in C_c^\infty(\Omega)$ with $u_j \rightarrow u$ in $\|\cdot\|_{V_{\alpha,m}}$ -norm; the same a.e.-convergent-subsequence argument used in the proof of Lemma 3.3 gives $u_j \rightarrow u$ almost everywhere in Ω along a subsequence, and Fatou's lemma, together with $u_{j,y} \rightarrow u_y$ in $L^2(\Omega; y^\alpha)$, gives

$$\int_{\Omega} y^{\alpha-2} u^2 \leq \liminf_j \int_{\Omega} y^{\alpha-2} u_j^2 \leq C_H \liminf_j \int_{\Omega} y^{\alpha} u_{j,y}^2 = C_H \int_{\Omega} y^{\alpha} u_y^2,$$

which is (4.5) for u . □

Lemma 4.2 (Weighted Poincaré inequality). *Let $0 < \alpha < 1$ and $m > 0$. Then*

$$\int_{\Omega} y^{\alpha} u^2 dx dy \leq C_P \int_{\Omega} \left(y^{\alpha+m} u_x^2 + y^{\alpha} u_y^2 \right) dx dy, \quad u \in V_{\alpha,m}(\Omega), \quad C_P = \frac{h^2}{2(1-\alpha)}. \quad (4.7)$$

Proof. For $u \in C_c^\infty(\Omega)$, Cauchy–Schwarz gives

$$|u(x, y)|^2 = \left| \int_0^y u_y(x, t) dt \right|^2 \leq \left(\int_0^y t^{-\alpha} dt \right) \left(\int_0^y t^{\alpha} u_y(x, t)^2 dt \right) = \frac{y^{1-\alpha}}{1-\alpha} \int_0^y t^{\alpha} u_y(x, t)^2 dt,$$

so that $y^{\alpha} u(x, y)^2 \leq \frac{y}{1-\alpha} \int_0^h t^{\alpha} u_y(x, t)^2 dt$; integrating first over $y \in (0, h)$ and then over x gives $\int_{\Omega} y^{\alpha} u^2 \leq \frac{h^2}{2(1-\alpha)} \int_{\Omega} y^{\alpha} u_y^2$. Adding the nonnegative term $\int_{\Omega} y^{\alpha+m} u_x^2$ to the right-hand side gives (4.7) on $C_c^\infty(\Omega)$; as in Remark 3.2, the bound extends to all of $V_{\alpha,m}(\Omega)$ by density and completeness of $L^2(\Omega; y^\alpha)$. □

The decisive point is that the singular term of $a_{w,m}(u, u)$ need not be estimated: it can be computed.

Lemma 4.3 (Exact form of the singular term). *Let $0 < \alpha < 1$ and $m > 0$. For every $u \in V_{\alpha,m}(\Omega)$,*

$$\int_{\Omega} y^{\alpha-1} u_y u dx dy = \frac{1-\alpha}{2} \int_{\Omega} y^{\alpha-2} u^2 dx dy, \quad (4.8)$$

and consequently

$$a_{w,m}(u, u) = \int_{\Omega} y^{\alpha+m} u_x^2 dx dy + \int_{\Omega} y^{\alpha} u_y^2 dx dy + \frac{(\alpha-\beta_0)(1-\alpha)}{2} \int_{\Omega} y^{\alpha-2} u^2 dx dy. \quad (4.9)$$

Proof. For $u \in C_c^\infty(\Omega)$, (4.8) is (4.6) integrated in x . Both sides of (4.8) are continuous on $V_{\alpha,m}(\Omega)$: the left-hand side because $|\int_\Omega y^{\alpha-1} u_y v| \leq \|y^{\alpha/2} u_y\|_{L^2} \|y^{\alpha/2-1} v\|_{L^2} \leq C_H^{1/2} \|u\|_{V_{\alpha,m}} \|v\|_{V_{\alpha,m}}$, and the right-hand side by (4.5). Hence (4.8) extends by density, and (4.9) follows from (4.2). $\square$

Lemma 4.4 (Boundedness of $a_{w,m}$). *The bilinear form $a_{w,m}$ defined in (4.2) is continuous on $V_{\alpha,m}(\Omega) \times V_{\alpha,m}(\Omega)$:*

$$|a_{w,m}(u, v)| \leq C_a \|u\|_{V_{\alpha,m}} \|v\|_{V_{\alpha,m}}, \quad C_a = 2 + |\alpha - \beta_0| C_H^{1/2}, \quad u, v \in V_{\alpha,m}(\Omega).$$

Proof. The first two terms in (4.2) are bounded directly by the Cauchy–Schwarz inequality, each by $\|u\|_{V_{\alpha,m}} \|v\|_{V_{\alpha,m}}$. For the singular term, the weighted Hardy inequality (4.5) gives

$$\left| \int_\Omega y^{\alpha-1} u_y v \, dx \, dy \right| \leq \left(\int_\Omega y^\alpha u_y^2 \, dx \, dy \right)^{1/2} \left(\int_\Omega y^{\alpha-2} v^2 \, dx \, dy \right)^{1/2} \leq C_H^{1/2} \|u\|_{V_{\alpha,m}} \|v\|_{V_{\alpha,m}},$$

which proves the claim. $\square$

Lemma 4.5 (Coercivity below the weighted threshold). *Let $m > 0$, $0 < \alpha < 1$, and*

$$\beta_0 < \beta_0^*(\alpha) := \frac{1 + \alpha}{2} = \alpha + C_H(\alpha)^{-1/2}. \quad (4.10)$$

Then, with $(\beta_0 - \alpha)^+ := \max\{\beta_0 - \alpha, 0\}$,

$$a_{w,m}(u, u) \geq \kappa |u|_1^2, \quad \kappa := 1 - \frac{2(\beta_0 - \alpha)^+}{1 - \alpha} \in (0, 1], \quad (4.11)$$

and

$$a_{w,m}(u, u) \geq c_* \|u\|_{V_{\alpha,m}}^2, \quad c_* := \frac{\kappa}{1 + C_P}, \quad u \in V_{\alpha,m}(\Omega).$$

Proof. If $\beta_0 \leq \alpha$, the last term in (4.9) is nonnegative, and $a_{w,m}(u, u) \geq |u|_1^2$, i.e. $\kappa = 1$. If $\alpha < \beta_0 < \beta_0^*(\alpha)$, the weighted Hardy inequality (4.5) gives

$$\frac{(\alpha - \beta_0)(1 - \alpha)}{2} \int_\Omega y^{\alpha-2} u^2 \geq -\frac{(\beta_0 - \alpha)(1 - \alpha)}{2} \cdot \frac{4}{(1 - \alpha)^2} \int_\Omega y^\alpha u_y^2 = -\frac{2(\beta_0 - \alpha)}{1 - \alpha} \int_\Omega y^\alpha u_y^2,$$

so that (4.9) yields (4.11); condition (4.10) is exactly $\kappa > 0$. Finally, (4.7) gives $\|u\|_{V_{\alpha,m}}^2 \leq (1 + C_P) |u|_1^2$, and the second inequality follows. $\square$

Remark 4.2. In an estimate that bounds the singular term by Cauchy–Schwarz alone, $|\int_\Omega y^{\alpha-1} u_y u| \leq C_H^{1/2} \int_\Omega y^\alpha u_y^2$, the sign of $\alpha - \beta_0$ is lost, and one obtains only the two-sided sufficient condition $|\alpha - \beta_0| < (1 - \alpha)/2$. The exact identity (4.8) shows that the singular term is nonnegative for $\beta_0 \leq \alpha$, so that the lower restriction $\beta_0 > \alpha - (1 - \alpha)/2$ is superfluous; by Theorem 4.1 the remaining upper restriction cannot be improved.

Lemma 4.6 (Compact embedding). *Let $m > 0$ and $0 < \alpha < 1$. Then the embedding $V_{\alpha,m}(\Omega) \hookrightarrow L^2(\Omega; y^\alpha)$ is compact.*

Proof. Let $\{u_N\}$ be a bounded sequence in $V_{\alpha,m}(\Omega)$. Since $V_{\alpha,m}(\Omega)$ is a Hilbert space, after passing to a subsequence there exists $u \in V_{\alpha,m}(\Omega)$ such that $u_N \rightharpoonup u$ weakly in $V_{\alpha,m}(\Omega)$. Fix $\delta \in (0, h)$ and set $\Omega_\delta = \Omega \cap \{0 < y < \delta\}$, $\Omega^\delta = \Omega \cap \{y \geq \delta\}$. On Ω^δ the weights y^α and $y^{\alpha+m}$ are bounded above and below by positive constants. Hence $\{u_N\}$ is bounded in the ordinary Sobolev space $H^1(\Omega^\delta)$ of the rectangle Ω^δ , and by the Rellich–Kondrachov theorem, after passing to a further subsequence, $u_N \rightarrow u$ strongly in $L^2(\Omega^\delta)$. It

remains to control the contribution from Ω_δ . For $u \in C_c^\infty(\Omega)$, since u vanishes near $\Gamma_0 = \{y = 0\}$, we have $u(x, y) = \int_0^y u_y(x, t) dt$ for every x and $0 < y < \delta$. By density of $C_c^\infty(\Omega)$ in $V_{\alpha, m}(\Omega)$ – by the same subsequence argument detailed in the proof of Lemma 3.3 (approximate in $\|\cdot\|_{V_{\alpha, m}}$ -norm, extract a subsequence converging a.e. together with $y^{\alpha/2}$ times the y -derivative converging in $L^2(\Omega)$, and pass to the limit for a.e. x in the displayed identity) – this representation extends to every $u \in V_{\alpha, m}(\Omega)$, and in particular to each u_N . Thus, for almost every x ,

$$|u_N(x, y)|^2 \leq \left(\int_0^y t^\alpha |u_{N, y}(x, t)|^2 dt \right) \left(\int_0^y t^{-\alpha} dt \right) \leq \frac{y^{1-\alpha}}{1-\alpha} \int_0^\delta t^\alpha |u_{N, y}(x, t)|^2 dt.$$

Multiplying by y^α and integrating over Ω_δ gives

$$\int_{\Omega_\delta} y^\alpha |u_N|^2 dx dy \leq \frac{\delta^2}{2(1-\alpha)} \int_{\Omega} y^\alpha |u_{N, y}|^2 dx dy \leq C\delta^2,$$

with C independent of N and δ . The same estimate holds for the weak limit u . Hence $\|u_N - u\|_{L^2(\Omega_\delta; y^\alpha)}^2 \leq C_1\delta^2$, with C_1 independent of N . Combining this estimate with the strong convergence on Ω^δ and then letting $\delta \rightarrow 0$ (with a diagonal subsequence) yields $u_N \rightarrow u$ strongly in $L^2(\Omega; y^\alpha)$. $\square$

Theorem 4.1 (The weighted threshold and its sharpness). *Let $m > 0$, $0 < \alpha < 1$ and $\beta_0^*(\alpha) = (1 + \alpha)/2$.*

- (i) *If $\beta_0 < \beta_0^*(\alpha)$, then $a_{w, m}$ is coercive on $V_{\alpha, m}(\Omega)$ by Lemma 4.5; there is no lower restriction on β_0 , and for $\beta_0 \leq \alpha$ one has $\kappa = 1$.*
- (ii) *If $\beta_0 \geq \beta_0^*(\alpha)$, then $\inf_{u \neq 0} a_{w, m}(u, u) / \|u\|_{V_{\alpha, m}}^2 \leq 0$, and the infimum is negative if $\beta_0 > \beta_0^*(\alpha)$. Thus $\beta_0 < \beta_0^*(\alpha)$ is necessary and sufficient for coercivity, for every $m > 0$.*

Proof. Part (i) is Lemma 4.5. For (ii) we exhibit an explicit near-extremal sequence; the construction is supported in a rectangle adjacent to Γ_0 and therefore applies verbatim on any domain containing such a rectangle. Set

$$p := \frac{1-\alpha}{2}, \quad p_k := p + \frac{1}{k} \quad (k = 1, 2, \dots).$$

Fix once and for all $0 < 2\delta < h$ and a cutoff $\eta \in C^\infty([0, h])$ with $\eta \equiv 1$ on $[0, \delta]$, $\eta \equiv 0$ on $[2\delta, h]$, $0 \leq \eta \leq 1$; fix also a nonzero $\xi \in C_c^\infty(0, \ell)$. Define

$$\psi_k(y) := \eta(y) y^{p_k}, \quad u_k(x, y) := \xi(x) \psi_k(y).$$

Since $p_k > 0$ and η vanishes near $y = h$, $\psi_k(0) = \psi_k(h) = 0$; since ξ has compact support in $(0, \ell)$, u_k vanishes near $x = 0, \ell$ as well.

u_k belongs to $V_{\alpha, m}(\Omega)$. Fix $\chi \in C^\infty(\mathbf{R})$ with $\chi \equiv 0$ on $(-\infty, 1]$, $\chi \equiv 1$ on $[2, \infty)$, $0 \leq \chi \leq 1$, and for $j = 1, 2, \dots$ set

$$u_k^{(j)}(x, y) := \xi(x) \eta(y) y^{p_k} \chi(jy) \in C_c^\infty(\Omega)$$

(smooth, since it vanishes identically for $y < 1/j$, and compactly supported since ξ and η already localize it away from $x = 0, \ell$ and $y = h$). The difference $u_k - u_k^{(j)} = \xi(x) \eta(y) y^{p_k} (1 - \chi(jy))$ is supported in $\{y < 2/j\}$.

Its $L^2(y^\alpha)$ -norm satisfies $\|u_k - u_k^{(j)}\|_{L^2(y^\alpha)}^2 \leq \|\xi\|_{L^2}^2 \int_0^{2/j} y^{\alpha+2p_k} dy = O(j^{-2-2/k})$, and the x -derivative term satisfies $\int_{\Omega} y^{\alpha+m} |\partial_x(u_k - u_k^{(j)})|^2 \leq \|\xi'\|_{L^2}^2 \int_0^{2/j} y^{\alpha+m+2p_k} dy \rightarrow 0$. The y -derivative $\partial_y(u_k - u_k^{(j)})$ consists of terms of size $y^{p_k-1}(1 - \chi(jy))$ and $jy^{p_k}\chi'(jy)$, both supported in $\{y < 2/j\}$; since $j < 2/y$ there, $jy^{p_k}|\chi'(jy)| \leq 2\|\chi'\|_\infty y^{p_k-1}$, so both terms are bounded by Cy^{p_k-1} , and, using $\alpha + 2p_k - 2 = \frac{2}{k} - 1$,

$$\int_{\Omega} y^\alpha |\partial_y(u_k - u_k^{(j)})|^2 dx dy \leq C \int_0^{2/j} y^{\frac{2}{k}-1} dy = O(j^{-2/k}) \rightarrow 0, \quad j \rightarrow \infty.$$

Hence $u_k^{(j)} \rightarrow u_k$ in $\|\cdot\|_{V_{\alpha,m}}$, and $u_k \in V_{\alpha,m}(\Omega)$ for each fixed k .

The Rayleigh quotient. Because $\alpha - 1 + 2p_k = \frac{2}{k} > 0$ and $\psi_k(h) = 0$, integrating $\int_0^h y^{\alpha-1} \psi'_k \psi_k dy = \frac{1}{2} \int_0^h y^{\alpha-1} (\psi_k^2)' dy$ by parts produces no boundary contribution at either endpoint, giving (in accordance with (4.8))

$$\int_0^h y^{\alpha-1} \psi'_k \psi_k dy = \frac{1-\alpha}{2} \int_0^h y^{\alpha-2} \psi_k^2 dy =: \frac{1-\alpha}{2} A_0(k).$$

Writing also $B(k) := \int_0^h y^\alpha \psi'_k(y)^2 dy$ and $I_1(k) := \int_0^h y^{\alpha+m} \psi_k(y)^2 dy$, a direct computation from (4.2) gives

$$a_{w,m}(u_k, u_k) = \|\xi'\|_{L^2}^2 I_1(k) + \|\xi\|_{L^2}^2 \left[B(k) + (\alpha - \beta_0) \frac{1-\alpha}{2} A_0(k) \right],$$

so that

$$\frac{a_{w,m}(u_k, u_k)}{\|\xi\|_{L^2}^2 B(k)} = \frac{\|\xi'\|_{L^2}^2}{\|\xi\|_{L^2}^2} \cdot \frac{I_1(k)}{B(k)} + 1 + (\alpha - \beta_0) \frac{1-\alpha}{2} \frac{A_0(k)}{B(k)}.$$

On $[0, \delta]$, where $\eta \equiv 1$, both $A_0(k)$ and $B(k)$ reduce to multiples of $\int_0^\delta y^{-1+2/k} dy = \frac{k}{2} \delta^{2/k}$, with the factor 1 for $A_0(k)$ and p_k^2 for $B(k)$ (since $\psi'_k = p_k y^{p_k-1}$ there); the contribution of $[\delta, h]$, where η, η' and p_k stay bounded, is $O(1)$ as $k \rightarrow \infty$. Hence

$$A_0(k) = \frac{k}{2} \delta^{2/k} + O(1), \quad B(k) = p_k^2 \cdot \frac{k}{2} \delta^{2/k} + O(1), \quad \frac{A_0(k)}{B(k)} \rightarrow \frac{1}{p^2} = \frac{4}{(1-\alpha)^2} = C_H(\alpha).$$

Likewise $I_1(k) = \int_0^h y^{m+1+2/k} \eta(y)^2 dy \rightarrow \int_0^h y^{m+1} \eta^2 dy < \infty$ stays bounded while $B(k) \rightarrow \infty$, so $I_1(k)/B(k) \rightarrow 0$. Therefore

$$\frac{a_{w,m}(u_k, u_k)}{\|\xi\|_{L^2}^2 B(k)} \rightarrow 1 + \frac{2(\alpha - \beta_0)}{1 - \alpha} = \frac{2(\beta_0^*(\alpha) - \beta_0)}{1 - \alpha}.$$

This limit is negative if $\beta_0 > \beta_0^*(\alpha)$, so $a_{w,m}(u_k, u_k) < 0$ for large k . If $\beta_0 = \beta_0^*(\alpha)$ the limit is 0; since $\|u_k\|_{V_{\alpha,m}}^2 \geq \int_\Omega y^\alpha u_{k,y}^2 = \|\xi\|_{L^2}^2 B(k)$, we get $\limsup_k a_{w,m}(u_k, u_k) / \|u_k\|_{V_{\alpha,m}}^2 \leq 0$, so no positive coercivity constant exists. The argument is independent of $m > 0$. $\square$

Remark 4.3 (The symmetric case $\alpha = \beta_0$ and the Bessel range $\beta_0 < 1$). If $0 < \beta_0 < 1$, the choice $\alpha = \beta_0$ is admissible, since $\beta_0 < (1 + \beta_0)/2$ is equivalent to $\beta_0 < 1$. For this choice the singular term in (4.2) disappears,

$$a_{w,m}(u, v) = \int_\Omega (y^{\beta_0+m} u_x v_x + y^{\beta_0} u_y v_y) dx dy,$$

the form is symmetric, and $\kappa = 1$ in (4.11). This reflects the divergence identity $y^{\beta_0} (u_{yy} + \frac{\beta_0}{y} u_y) = \partial_y (y^{\beta_0} u_y)$, i.e.

$$Lu = y^{-\beta_0} \operatorname{div} (y^{\beta_0+m}, y^{\beta_0}) \nabla u,$$

so that with $\alpha = \beta_0$ Problem II is the Euler–Lagrange equation of $J(u) = \frac{1}{2} \int_\Omega (y^{\beta_0+m} u_x^2 + y^{\beta_0} u_y^2) - \int_\Omega y^{\beta_0} F(u)$, $F' = f$. This is the classical symmetrizing weight for Bessel-type operators [3]; without the x -degeneracy ($m = 0$) it is the operator $\operatorname{div}(y^\alpha \nabla u)$, $\alpha = \beta_0$, of the Caffarelli–Silvestre extension [4], whose weight belongs to A_2 for $-1 < \alpha < 1$ [16]. More generally, for every $\beta_0 < 1$ the set of admissible exponents $\{\alpha \in (0, 1) : \beta_0 < (1 + \alpha)/2\} = (\max\{0, 2\beta_0 - 1\}, 1)$ is nonempty. Thus Problem II covers the whole range $\beta_0 < 1$, which is precisely the range in which a Dirichlet condition can be prescribed on Γ_0 for the Bessel operator: the indicial roots of $\psi'' + \frac{\beta_0}{y} \psi' = 0$ are 0 and $1 - \beta_0$, and for $\beta_0 \geq 1$ boundedness alone selects the solution near $y = 0$, so that the condition on the degeneration line has to be dropped or modified [2, 3]. Consistently, no $\alpha \in (0, 1)$ is admissible when $\beta_0 \geq 1$. Problem I, in contrast, covers only $\beta_0 < 1/2$ (Remark 3.3).

Theorem 4.2 (Existence in $V_{\alpha,m}(\Omega)$). *Suppose that*

$$m > 0, \quad \alpha \in (0, 1), \quad \beta_0 < \frac{1 + \alpha}{2},$$

and that f satisfies (H1)–(H2). Assume, in addition, that the coercivity constant c_* from Lemma 4.5 satisfies $c_* > LC_P$, where L is the Lipschitz constant of f . Then (2.2) admits a weak solution $u \in V_{\alpha,m}(\Omega)$.

Proof. Let $\{e_j\}_{j=1}^\infty$ be a complete orthonormal system in the separable Hilbert space $V_{\alpha,m}(\Omega)$, and set $V_N = \text{span}\{e_1, \dots, e_N\}$. We seek $u_N \in V_N$ such that

$$a_{w,m}(u_N, v) = \int_{\Omega} y^\alpha f(u_N) v \, dx \, dy, \quad v \in V_N.$$

For each N , define the nonlinear map $F_N : V_N \rightarrow V_N$ by

$$(F_N(w), v)_{V_{\alpha,m}} = a_{w,m}(w, v) - \int_{\Omega} y^\alpha f(w) v \, dx \, dy, \quad v \in V_N.$$

Since V_N is finite-dimensional and f is Lipschitz continuous, F_N is continuous. For $w \in V_N$, writing $f(w) = (f(w) - f(0)) + f(0)$ and using coercivity, the Lipschitz condition and (4.7) (which gives $\|w\|_{L^2(\Omega; y^\alpha)}^2 \leq C_P \|w\|_1^2 \leq C_P \|w\|_{V_{\alpha,m}}^2$), we obtain

$$(F_N(w), w)_{V_{\alpha,m}} \geq (c_* - LC_P) \|w\|_{V_{\alpha,m}}^2 - C_P^{1/2} \|f(0)\|_{L^2(\Omega; y^\alpha)} \|w\|_{V_{\alpha,m}}.$$

Hence $(F_N(w), w)_{V_{\alpha,m}} > 0$ whenever

$$\|w\|_{V_{\alpha,m}} > R := \frac{C_P^{1/2}}{c_* - LC_P} \|f(0)\|_{L^2(\Omega; y^\alpha)}.$$

By the acute-angle lemma (a corollary of the Brouwer fixed-point theorem), there exists $u_N \in V_N$ with $F_N(u_N) = 0$ and $\|u_N\|_{V_{\alpha,m}} \leq R$; equivalently, $a_{w,m}(u_N, v) = \int_{\Omega} y^\alpha f(u_N) v$ for $v \in V_N$. Thus $\{u_N\}$ is uniformly bounded in $V_{\alpha,m}(\Omega)$, and after passing to a subsequence, $u_N \rightharpoonup u$ weakly in $V_{\alpha,m}(\Omega)$ for some $u \in V_{\alpha,m}(\Omega)$. By Lemma 4.6 we may assume, after passing to a further subsequence, that $u_N \rightarrow u$ strongly in $L^2(\Omega; y^\alpha)$. Since f is Lipschitz continuous, $\|f(u_N) - f(u)\|_{L^2(\Omega; y^\alpha)} \leq L \|u_N - u\|_{L^2(\Omega; y^\alpha)} \rightarrow 0$, and consequently $\int_{\Omega} y^\alpha f(u_N) v \rightarrow \int_{\Omega} y^\alpha f(u) v$ for every $v \in V_{\alpha,m}(\Omega)$. Moreover, by the weak convergence and the continuity of $a_{w,m}$ (Lemma 4.4), $a_{w,m}(u_N, v) \rightarrow a_{w,m}(u, v)$. Passing to the limit in the Galerkin identities for $v \in V_{N_0}$, $N \geq N_0$, and then using the density of $\bigcup_N V_N$, gives

$$a_{w,m}(u, v) = \int_{\Omega} y^\alpha f(u) v \, dx \, dy, \quad v \in V_{\alpha,m}(\Omega).$$

Thus $u \in V_{\alpha,m}(\Omega)$ is a weak solution of (2.2). $\square$

Theorem 4.3 (Uniqueness in $V_{\alpha,m}(\Omega)$). *Under the hypotheses of Theorem 4.2, the weak solution of (2.2) is unique.*

Proof. Let $u_1, u_2 \in V_{\alpha,m}(\Omega)$ be two weak solutions and set $w = u_1 - u_2$. Subtracting the two weak formulations and taking $v = w$ gives $a_{w,m}(w, w) = \int_{\Omega} y^\alpha (f(u_1) - f(u_2)) w$. By coercivity, the Lipschitz condition and (4.7),

$$c_* \|w\|_{V_{\alpha,m}}^2 \leq a_{w,m}(w, w) \leq L \|w\|_{L^2(\Omega; y^\alpha)}^2 \leq LC_P \|w\|_{V_{\alpha,m}}^2.$$

Since $c_* > LC_P$, it follows that $w \equiv 0$. Hence $u_1 = u_2$. $\square$

Theorem 4.4 (Stability estimates). *Assume the hypotheses of Theorem 4.2. Then the weak solution satisfies*

$$\|u\|_{V_{\alpha,m}} \leq \frac{C_P^{1/2}}{c_* - LC_P} \|f(0)\|_{L^2(\Omega; y^\alpha)}. \quad (4.12)$$

Moreover, let $u_1, u_2 \in V_{\alpha,m}(\Omega)$ be the weak solutions corresponding to two nonlinearities f_1, f_2 satisfying (H1)–(H2) with the same Lipschitz constant L . Then

$$\|u_1 - u_2\|_{V_{\alpha,m}} \leq \frac{C_P^{1/2}}{c_* - LC_P} \|f_1(u_2) - f_2(u_2)\|_{L^2(\Omega; y^\alpha)}. \quad (4.13)$$

Proof. Taking $v = u$ in (4.4) and writing $f(u) = (f(u) - f(0)) + f(0)$, Lemma 4.5, (H1), the Cauchy–Schwarz inequality and (4.7) give

$$\begin{aligned} c_* \|u\|_{V_{\alpha,m}}^2 &\leq a_{w,m}(u, u) \leq L \|u\|_{L^2(\Omega; y^\alpha)}^2 + \|f(0)\|_{L^2(\Omega; y^\alpha)} \|u\|_{L^2(\Omega; y^\alpha)} \\ &\leq LC_P \|u\|_{V_{\alpha,m}}^2 + C_P^{1/2} \|f(0)\|_{L^2(\Omega; y^\alpha)} \|u\|_{V_{\alpha,m}}, \end{aligned}$$

and (4.12) follows after division by $\|u\|_{V_{\alpha,m}}$ (the case $u = 0$ being trivial). For (4.13), set $w = u_1 - u_2$; subtracting the weak formulations and taking $v = w$,

$$a_{w,m}(w, w) = \int_{\Omega} y^\alpha (f_1(u_1) - f_1(u_2)) w + \int_{\Omega} y^\alpha (f_1(u_2) - f_2(u_2)) w.$$

The first integral is at most $L \|w\|_{L^2(\Omega; y^\alpha)}^2 \leq LC_P \|w\|_{V_{\alpha,m}}^2$, and the second at most $C_P^{1/2} \|f_1(u_2) - f_2(u_2)\|_{L^2(\Omega; y^\alpha)} \|w\|_{V_{\alpha,m}}$; this norm is finite by (H2), because $u_2 \in L^2(\Omega; y^\alpha)$. Since $a_{w,m}(w, w) \geq c_* \|w\|_{V_{\alpha,m}}^2$, (4.13) follows as before. $\square$

Theorem 4.5 (Monotone dependence of the principal energy on m , for a fixed function). *Suppose $0 < h \leq 1$, let $0 < m_1 < m_2$, and let $u \in V_{\alpha,m_1}(\Omega) \cap V_{\alpha,m_2}(\Omega)$. Then $y^{m_1} \geq y^{m_2}$ for $0 < y \leq h$, and hence*

$$\int_{\Omega} y^{\alpha+m_1} u_x^2 dx dy \geq \int_{\Omega} y^{\alpha+m_2} u_x^2 dx dy.$$

Since the remaining terms of $a_{w,m}(u, u)$ (the u_y -term and the singular term) do not depend on m , it follows that $a_{w,m_1}(u, u) \geq a_{w,m_2}(u, u)$.

Remark 4.4. This monotonicity statement is deliberately modest. The norm $\|u\|_{V_{\alpha,m}}$ itself changes with m through the term $y^{\alpha+m} u_x^2$, so dividing the pointwise inequality of Theorem 4.5 by a fixed norm does not yield monotonicity of the optimal coercivity constant $c_*(m) := \inf_{u \neq 0} a_{w,m}(u, u) / \|u\|_{V_{\alpha,m}}^2$ in m , and no such claim is made here. For a related reason, a bound $\|u_m\|_{V_{\alpha,m}} \leq C$ that is uniform in m does not by itself control the corresponding $m = 0$ energy $\int_{\Omega} y^\alpha u_{m,x}^2$ as $m \rightarrow 0^+$: the quantity $\int y^{\alpha+m} u_{m,x}^2 \leq C^2$ discounts u_x near $y = 0$ by a factor y^m that can be arbitrarily small even for small m , leaving room for mass concentrating near $y = 0$ to push $\int y^\alpha u_{m,x}^2$ up without bound along some sequence $m_k \rightarrow 0$. Convergence of u_m as $m \rightarrow 0$ is accordingly not asserted in this setting.

Theorem 4.6 (Interior regularity away from Γ_0). *Let $u \in V_{\alpha,m}(\Omega)$ be a weak solution of (2.2), i.e. of $L_m u = y^\alpha f(u)$ with L_m defined in (4.3). If $\Omega' \Subset \Omega$, then $u \in H^2(\Omega')$ and $\|u\|_{H^2(\Omega')} \leq C_{\Omega'} (\|f(u)\|_{L^2(\Omega)} + \|u\|_{V_{\alpha,m}})$. If, in addition, $f \in C^\infty(\mathbb{R})$, then $u \in C^\infty(\Omega)$.*

Proof. By (4.5) and $y^{\alpha-2} \geq h^{\alpha-2}$, $u \in L^2(\Omega)$, hence $f(u) \in L^2(\Omega)$ by (H2). Let $\Omega' \Subset \Omega'' \Subset \Omega$ with $y \geq \delta > 0$ on Ω'' . There the principal coefficients of L_m satisfy $y^{\alpha+m} \geq \delta^{\alpha+m}$, $y^\alpha \geq \delta^\alpha$ and are smooth and bounded, while

the lower-order coefficient obeys $|(\alpha - \beta_0)y^{\alpha-1}| \leq |\alpha - \beta_0|\delta^{\alpha-1}$; moreover $u \in H^1(\Omega'')$ with norm bounded by $C\|u\|_{V_{\alpha,m}}$. Thus L_m is uniformly elliptic with smooth bounded coefficients on Ω'' , and standard interior regularity [22] gives the H^2 estimate; the C^∞ statement follows by the bootstrap argument of Theorem 3.3. Global $H^2(\Omega)$ -regularity should not be expected, since the coefficients degenerate and $(\alpha - \beta_0)y^{\alpha-1}$ becomes singular as $y \rightarrow 0$ (unless $\alpha = \beta_0$). $\square$

5. Placing the two settings side by side

Both settings act on the identical operator, both work for every $m > 0$, and each delivers existence, uniqueness and interior regularity; they differ in the admissible range of β_0 , in the strength of the energy norm, and in what can be said up to Γ_0 .

Range of m . By Remark 3.1, neither setting imposes any restriction on $m > 0$ for well-posedness: the Muckenhoupt condition $y^m \in A_2$ ($0 < m < 1$) is not needed. In $H_0^1(\Omega, y^m)$, the exponent m enters only the conditional weighted regularity result (Theorem 3.4, $m > 1$); in $V_{\alpha,m}(\Omega)$ it does not enter the coercivity threshold at all.

Character of the condition on β_0 . In $H_0^1(\Omega, y^m)$ the coercivity condition (A0) reads $\beta_0 < 2/C_H(\Omega)$, with $C_H(\Omega) \leq 4$; when Γ_0 contains a segment, $C_H(\Omega) = 4$ and the condition $\beta_0 < 1/2$ is sharp (Remark 3.3). In $V_{\alpha,m}(\Omega)$ the condition is

$$\beta_0 < \beta_0^*(\alpha) = \frac{1 + \alpha}{2},$$

which depends only on α and is sharp for every $m > 0$ (Theorem 4.1). In both settings there is no lower bound on β_0 : for $\beta_0 \leq 0$, respectively $\beta_0 \leq \alpha$, the singular term is nonnegative by Lemmas 3.1 and 4.3 and only helps.

What the auxiliary weight costs. The wider range of β_0 in $V_{\alpha,m}(\Omega)$ has a price. First, the energy norm is weaker near Γ_0 (all derivatives carry the extra factor y^α), so the solution is found in a larger space. Second, the Poincaré constant $C_P = h^2/(2(1 - \alpha))$ blows up as $\alpha \rightarrow 1^-$, so the smallness condition $c_* > LC_P$ on the Lipschitz constant deteriorates exactly where the admissible range of β_0 is widest. Third, the norm of $V_{\alpha,m}(\Omega)$ depends on m ; this is why the energy-monotonicity statement (Theorem 4.5) is confined to a fixed function, and why neither the m -monotonicity of $c_*(m)$ nor convergence of solutions as $m \rightarrow 0^+$ is established.

5.1. A quantitative comparison of the two thresholds

Both thresholds trace back to the same one-dimensional Hardy inequality, which makes a direct comparison possible. Condition (A0) in $H_0^1(\Omega, y^m)$ rests on the unweighted constant $C_H(\Omega)$ of (3.1), whose sharp value is 4, approached by $\psi_k(y) \sim y^{1/2+1/k}$; this is exactly the value of the weighted constant $C_H(\alpha) = 4/(1 - \alpha)^2$ of (4.5) at $\alpha = 0$. Moreover, for $\alpha = 0$ the identity (4.9) reduces formally to $a_m(u, u) = \|u\|_{H_0^1(\Omega, y^m)}^2 - \frac{\beta_0}{2} \int_\Omega u^2/y^2$, which is (3.5). Hence the threshold $\beta_0^*(\alpha) = (1 + \alpha)/2$ of Theorem 4.1 tends, as $\alpha \rightarrow 0^+$, to the threshold $1/2$ of Problem I: the two independently derived conditions, including the sign of the singular term, agree exactly in this limit, even though $V_{\alpha,m}(\Omega)$ is only defined for $\alpha > 0$. Table 1 lists $C_H(\alpha)$, $\beta_0^*(\alpha)$ and the normalized Poincaré constant C_P/h^2 for a range of α , and Figure 1 displays $\beta_0^*(\alpha)$ graphically. The trade-off encoded in this monotone increase is worth spelling out: taking α small keeps $V_{\alpha,m}(\Omega)$ close to $H_0^1(\Omega, y^m)$, with a moderate Poincaré constant, but leaves the admissible range of β_0 close to that of Problem I; taking α larger widens the admissible range up to the Bessel limit $\beta_0 < 1$, but weakens the norm near Γ_0 and increases C_P , and with it the restriction on the Lipschitz constant of f .

A practical summary. Neither formulation subsumes the other. $H_0^1(\Omega, y^m)$ is preferable when $\beta_0 < 1/2$, since it gives the stronger norm, a moderate Poincaré constant and the weighted second-order regularity of

Table 1. The sharp threshold $\beta_0^*(\alpha) = (1 + \alpha)/2 = \alpha + C_H(\alpha)^{-1/2}$, the Hardy constant $C_H(\alpha) = 4/(1 - \alpha)^2$ and the normalized Poincaré constant $C_P/h^2 = 1/(2(1 - \alpha))$ for a sample of values of the auxiliary exponent α . Coercivity of $a_{w,m}$ holds if and only if $\beta_0 < \beta_0^*(\alpha)$; the value $\alpha = 0$ corresponds to Problem I.

α	$C_H(\alpha)$	$\beta_0^*(\alpha)$	C_P/h^2
0 (Problem I)	4.000	0.500	0.500
0.1	4.938	0.550	0.556
0.2	6.250	0.600	0.625
0.3	8.163	0.650	0.714
0.4	11.111	0.700	0.833
0.5	16.000	0.750	1.000
0.6	25.000	0.800	1.250
0.7	44.444	0.850	1.667
0.8	100.000	0.900	2.500
0.9	400.000	0.950	5.000

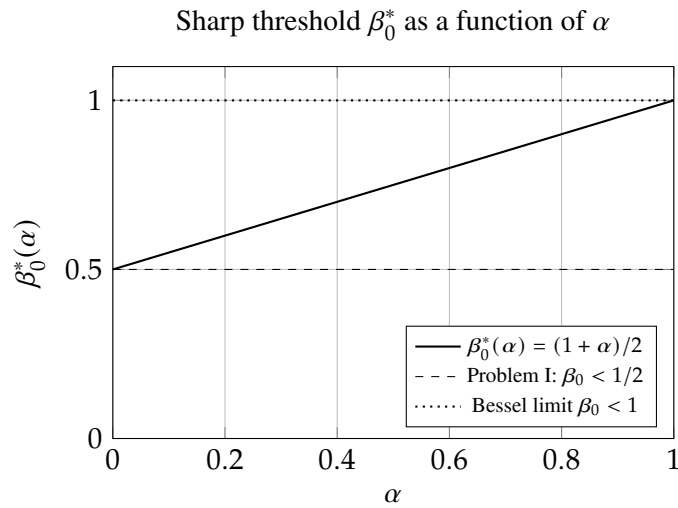


Figure 1. $\beta_0^*(\alpha)$ increases linearly from $1/2$ as $\alpha \rightarrow 0^+$ (the sharp threshold of Problem I, dashed) to 1 as $\alpha \rightarrow 1^-$ (the Bessel limit, dotted). The admissible set is the half-line $\beta_0 < \beta_0^*(\alpha)$, lying below the solid line.

Theorem 3.4; $V_{\alpha,m}(\Omega)$ is needed when $1/2 \leq \beta_0 < 1$, or when a threshold independent of the domain geometry is required, and for $0 < \beta_0 < 1$ the symmetric choice $\alpha = \beta_0$ of Remark 4.3 is the canonical one.

6. Open questions

The theory developed above leaves the following questions unanswered: (1) whether, for $m > 1$, the hypothesis $y^{m/2}u_{yy} \in L^2(\Omega)$ of Theorem 3.4 can be derived from (2.1), and whether the estimate extends to non-rectangular domains uniformly up to the points where Γ_0 meets Γ_+ ; (2) which formulation is well posed for $\beta_0 \geq 1$, where no $\alpha \in (0, 1)$ is admissible and the Dirichlet condition on Γ_0 has to be modified; (3) whether the optimal coercivity constant $c_*(m)$ in $V_{\alpha,m}(\Omega)$ depends monotonically on m ; (4) whether solutions of (2.2) converge as $m \rightarrow 0^+$; (5) whether, when Problem I and Problem II (for some α) are both well posed, their weak solutions coincide.

7. Concluding remarks

We have built and compared two weighted variational treatments of the degenerate–singular operator $Lu = \partial_x(y^m u_x) + u_{yy} + \frac{\beta_0}{y} u_y$. Working in $H_0^1(\Omega, y^m)$ yields existence, uniqueness and interior regularity for every $m > 0$ under the sharp one-sided condition $\beta_0 < 1/2$ (more precisely $\beta_0 < 2/C_H(\Omega)$), and, for $m > 1$, a conditional regularity result in the explicitly defined weighted space $\mathcal{H}_m^2(\Omega)$. Working instead in $V_{\alpha,m}(\Omega)$, built from an exponent $\alpha \in (0, 1)$ chosen independently of m , the derivative of the auxiliary weight shifts the singular coefficient to $\alpha - \beta_0$, and the exact evaluation of the singular term produces the coercivity criterion $\beta_0 < (1 + \alpha)/2$, which is necessary and sufficient, independent of m and of the domain, and covers the whole Bessel range $\beta_0 < 1$ as α varies, with the symmetric choice $\alpha = \beta_0$ as its natural representative. The price is a weaker, m -dependent energy norm and a Poincaré constant that grows as $\alpha \rightarrow 1^-$. The two thresholds coincide as $\alpha \rightarrow 0^+$. Placing the two treatments side by side pins down what each buys and what it costs, and isolates the specific open questions attached to each.

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Author's contributions

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References

- [1] Chaplain, M.A.J., Preziosi, L.: *Mathematical Oncology*. Springer, Cham (2025).
- [2] Keldysh, M.V.: *On certain cases of degeneration of equations of elliptic type on the boundary of a domain*. Dokl. Akad. Nauk SSSR **77**(2), 181–183 (1951) (in Russian).
- [3] Kipriyanov, I.A.: *Singular Elliptic Boundary Value Problems*. Nauka, Moscow (1997) (in Russian).
- [4] Caffarelli, L., Silvestre, L.: *An extension problem related to the fractional Laplacian*. Comm. Partial Differential Equations **32**(8), 1245–1260 (2007). <https://doi.org/10.1080/03605300600987306>
- [5] Salakhitdinov, M.S., Mirsaburov, M.: *Nelokal'nye zadachi dlia uravnenii smeshannogo tipa s singuliarnymi koeffitsientami*. Tashkent University Press, Tashkent (2005).
- [6] Mirsaburov, M.: *A boundary value problem for a class of mixed-type equations with the Bitsadze–Samarskii condition on degeneration lines*. Uzbek Math. J. **2**, 48–57 (2012).
- [7] Mirsaburov, M., Ruziev, M.Kh.: *A problem with conditions on both characteristics and on a degeneration line for a class of hyperbolic–parabolic equations*. Differ. Equ. **50**, 1347–1359 (2014).
- [8] Aharrouch, B., Bennouna, J.: *Nonlinear degenerate elliptic equations in weighted Sobolev spaces*. Electron. J. Differential Equations **2020**, No. 105 (2020).
- [9] El Ouaraabi, M., Allalou, C., Melliani, S., et al.: *On a class of nonlinear degenerate elliptic equations in weighted Sobolev spaces*. Georgian Math. J. **30**(1), 81–94 (2023). <https://doi.org/10.1515/gmj-2022-2191>

- [10] Alimohammady, M., Cattani, C., Koozehgar Kalleji, M.: *Existence and regularity results for a degenerate elliptic nonlinear equation on weighted Sobolev spaces*. Quaestiones Mathematicae **47**(1), 157–171 (2024). <https://doi.org/10.2989/16073606.2023.2205033>
- [11] Cavalheiro, A.C.: *Weighted Sobolev Spaces and Degenerate Elliptic Equations*. Cambridge Scholars Publishing, Newcastle upon Tyne (2023).
- [12] Beisebay, P., Akzhigitov, Y., Akhazhanov, T., Kenzhebekova, G., Matin, D.: *Smoothness of the solution of a boundary value problem for degenerate elliptic equations*. Symmetry **17**(9), 1562 (2025). <https://doi.org/10.3390/sym17091562>
- [13] Di Fazio, G., Fanciullo, M.S., Zamboni, P.: *Regularity up to the boundary for some degenerate elliptic operators*. Applicable Analysis **101**(10), 3563–3575 (2022). <https://doi.org/10.1080/00036811.2020.1769079>
- [14] Do, T.D.: *Harnack inequality for degenerate Hormander sub-elliptic operators with potentials*. Expositiones Mathematicae **40**(3), 605–627 (2022). <https://doi.org/10.1016/j.exmath.2021.12.004>
- [15] Muckenhoupt, B.: *Weighted norm inequalities for the Hardy maximal function*. Trans. Amer. Math. Soc. **165**, 207–226 (1972).
- [16] Fabes, E.B., Kenig, C.E., Serapioni, R.P.: *The local regularity of solutions of degenerate elliptic equations*. Comm. Partial Differential Equations **7**(1), 77–116 (1982). <https://doi.org/10.1080/03605308208820218>
- [17] Balci, A.Kh., Diening, L., Giova, R., Passarelli di Napoli, A.: *Elliptic equations with degenerate weights*. SIAM J. Math. Anal. **54**(2), 2373–2412 (2022). <https://doi.org/10.1137/21M1412529>
- [18] Opic, B., Kufner, A.: *Hardy-Type Inequalities*. Pitman Research Notes in Mathematics Series, vol. 219. Longman, Harlow (1990).
- [19] Wang, B.-S., Hou, G.-L., Ge, B.: *Existence and uniqueness of solutions for the $p(x)$ -Laplacian equation with convection term*. Mathematics **8**(10), 1768 (2020). <https://doi.org/10.3390/math8101768>
- [20] Tiarimti Alaoui, A., Jourhmane, M.: *Existence and uniqueness of weak solutions for novel anisotropic nonlinear diffusion equations related to image analysis*. J. Math. **2021**, Art. 5553126 (2021). <https://doi.org/10.1155/2021/5553126>
- [21] Ghaffari Laleh, N., Loeffler, C.M.L., Grajek, J., et al.: *Classical mathematical models for prediction of response to chemotherapy and immunotherapy*. PLoS Comput. Biol. **18**(2), e1009822 (2022). <https://doi.org/10.1371/journal.pcbi.1009822>
- [22] Gilbarg, D., Trudinger, N.S.: *Elliptic Partial Differential Equations of Second Order*. Springer, Berlin–Heidelberg (2001).
- [23] Grisvard, P.: *Elliptic Problems in Nonsmooth Domains*. Pitman, Boston (1985).

Affiliations

OCHILOVA N.K.

ADDRESS: Tashkent State Transport University, Dept. of Higher Mathematics, 100167, Tashkent, Uzbekistan.

E-MAIL: ochilovanozima51@gmail.com

ORCID ID: <https://orcid.org/0009-0003-5958-9577>

ESHMAMATOVA D.B.

ADDRESS: Tashkent State Transport University, Dept. of Higher Mathematics, 100167, Tashkent, Uzbekistan;
V.I. Romanovskiy Institute of Mathematics, Uzbekistan Academy of Sciences, 100174, Tashkent, Uzbekistan.

E-MAIL: 24dil@mail.ru

ORCID ID: <https://orcid.org/0000-0002-1096-2751>