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Research article

The singular value function, associated with a Maharam trace

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Abstract

Let M be a finite von Neumann algebra, let $S(M)$ be the $*$-algebra of measurable operators affiliated with M. Maharam traces Φ on a von Neumann algebra M with values in complex Dedekind complete vector lattices are considered. The singular value function of operators from $S(M)$, associated with such a trace Φ are determined. The main properties of these singular value functions, similar to classical singular value functions of measurable operators, are studied.

Keywords: *von Neumann algebra, algebra of measurable operators, vector-valued trace, Dedekind complete vector lattice, singular value function, Maharam trace.*

AMS Subject Classification (2020): *Primary: 28B15; Secondary: 46L52.*

Introduction

The modern theory of noncommutative measure and integration finds its roots in the seminal papers of I.E.Segal [1] and J.Dixmier [2]. The introduced by I.E.Segal noncommutative L^1-space associated with an exact normal semiinfinite trace is the main object of many investigations both in the theory of noncommutative integration and in its multiple applications (for example,[3], [4], [5], [6]). Detailed information on the current state of this theory is presented in [7], [8], [9], [10] and [11].

The existence of the center-valued traces in finite von Neumann algebras makes it natural to construct the theory of integration for traces with values in the complex Dedekind complete vector lattice $F_{\mathbb{C}} = F \oplus iF$. If the von Neumann algebra is commutative, then construction of $F_{\mathbb{C}}$-valued integration for it is the component part for the investigation of the properties of order continuous maps of vector lattices. The theory of such mappings is described rather thoroughly in the monograph [12]. An import role among these mappings is played by operators with the Maharam property. L^p-spaces associated with such operators are profound examples of Banach-Kantorovich lattices.

In [13], [14] and [15] a theory of non-commutative integration for traces Φ with values in the complex Dedekind complete vector lattice $F_{\mathbb{C}}$ was constructed. In particular, for Maharam traces Φ, with the help of the locally measure topology in the algebra $S(M)$ of all measurable operators affiliated with the von Neumann algebra M, the Banach-Kantorovich space $L^p(M,\Phi) \subset S(M), 1 \leq p < \infty$ was constructed and properties of such spaces are considered.

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This article is devoted to a study of singular value function of operators from $S(M)$, associated with a Maharam trace Φ. Also dominated properties of these singular value functions, similar to classical singular value functions of measurable operators, are proved.

In studying the $*$-algebra $S(M,\tau)$ of all τ-measurable operators, the notion of singular value functions plays an important role. There is an intimate relationship between the properties of τ-measurable operators and the properties of the singular value function (see for example ([11], Chapter 3)). For $x \in S(M,\tau)$, the singular value function $\mu(x)$ is defined by

$$\mu(t;x) := \inf\{s \geq 0 : \ \tau(E_{(s,\infty)}(x)) \leq t\}, \ t \geq 0,$$

where $E_{(s,\infty)}(x)$ is the spectral projection of the operator x corresponding to the interval (s,∞). The following expression is classical:

$$\mu(t;x) := \inf\{\|xp\|_M : \ p \in P(M), \ \tau(\mathbf{1}-p) \leq t\}, \ x \in S(M,\tau), \ t \geq 0,$$

where $P(M)$ is the set of all projectors in von Neumann algebra M.

In the present article, we will study the corresponding notion for traces $\Phi : M \to F_{\mathbb{C}}$. More precisely, let M be a finite von Neumann algebra, with center $Z(M)$, on the Hilbert space H. Let $\mathcal{B}$ be a commutative von Neumann algebra, $*$-isomorphic to a von Neumann subalgebra $\mathcal{A}$ in $Z(M)$, and let Φ be a $S(\mathcal{B})$-valued Maharam trace on M. Denote by $\mathcal{P}(\mathcal{B})$ the set of all $f \in S_h(\mathcal{B})$, for which the support $s(f) = \mathbf{1}_{\mathcal{B}}$.

For $x \in S(M)$, the singular value function, associated with a Maharam trace Φ is the map $\Phi(x) : (0,\infty) \to S_h(\mathcal{B})$ defined by the equality

$$\Phi(t;x) := \inf\{g \in \mathcal{P}(\mathcal{B}) : \ \Phi(E_g(x)) \leq t \cdot \mathbf{1}\}, \ t > 0,$$

where $E_g(x) \in P(M)$ is the projector in M, which is a projection onto a closed subspace $\overline{\{\xi \in H : x(\xi) > g(\xi)\}}$.

For all $t > 0$, the singular value function $\Phi(x)$ admits the characterization

$$\Phi(t;x) = \inf\{\|xe\|_{\mathcal{A}} : \ e \in P(M), \ xe \in E(M,\mathcal{A}), \ \Phi(\mathbf{1}-e) \leq t \cdot \mathbf{1}\},$$

where

$$E(M,\mathcal{A}) = \left\{x \in S(M) : |x| \leq a \ \text{ for some } \ a \in S_+(\mathcal{A})\right\}$$

is a Banach-Kantorovich space with $S_h(\mathcal{A})$-valued norm

$$\|x\|_{\mathcal{A}} = \inf\{a \in S_+(\mathcal{A}) : |x| \leq a\}.$$

We use the terminology and results of the theory of von Neumann algebras [8], [9], the theory of measurable operators [1], [10], [11] and of the theory of Dedekind complete vector lattices and Banach-Kantorovich spaces theory [12].

1. Preliminaries

Let H be a Hilbert space over the field $\mathbb{C}$ of complex numbers, let $B(H)$ be the $*$-algebra of all bounded linear operators on H, and $\mathbf{1}$ be the identity operator on H. Let M be a von Neumann algebra acting on H, let $Z(M)$ be the center of M and $P(M) = \{p \in M : p^2 = p = p^*\}$ be the lattice of all projectors in M. We denote by $P_{fin}(M)$ the set of all finite projectors in M.

A densely-defined closed linear operator x (possibly unbounded) affiliated with M is said to be *measurable* if there exists a sequence $\{p_n\}_{n=1}^{\infty} \subset P(M)$ such that $p_n \uparrow \mathbf{1}, \ p_n(H) \subset \mathfrak{D}(x)$ and $p_n^{\perp} = \mathbf{1} - p_n \in P_{fin}(M)$ for every $n = 1, 2, \ldots$ (here $\mathfrak{D}(x)$ is the domain of x).

The set $S(M)$ of all measurable with respect to M operators is a complex $*$-algebra with unit element $\mathbf{1}$, with respect to the operations of strong sum, strong product and the $*$-operation of taking adjoints (see [1]). The von Neumann algebra M is a $*$-subalgebra of $S(M)$. The set of all self-adjoint elements in $S(M)$ is denoted by $S_h(M)$, which is a real linear subspace of $S(M)$.

Let $x \in S(M)$ and $x = u|x|$ be the polar decomposition, where $|x| = (x^*x)^{\frac{1}{2}}$, u is a partial isometry in $B(H)$. Then $u \in M$ and $|x| \in S(M)$. If $x \in S_h(M)$ and $\{E_\lambda(x)\}$ are the spectral projections of x, then $\{E_\lambda(x)\} \subset P(M)$.

Let M be a commutative von Neumann algebra. Then M admits a faithful semi-finite normal trace τ, and M is $*$-isomorphic to the $*$-algebra $L^\infty(\Omega, \Sigma, \mu)$ of all bounded complex measurable functions with the identification almost everywhere, where (Ω, Σ, μ) is a measurable space. In addition, $\mu(A) = \tau(\chi_A),\ A \in \Sigma$. Moreover, $S(M) \cong L^0(\Omega, \Sigma, \mu)$, where $L^0(\Omega, \Sigma, \mu)$ is the $*$-algebra of all complex measurable functions with the identification almost everywhere [1].

Let M be an von Neumann algebra, let F be an Dedekind complete vector lattice, and let $F_{\mathbb{C}} = F \oplus iF$ be a complexification of F. If $z = \alpha + i\beta \in F_{\mathbb{C}},\ \alpha, \beta \in F$, then $\overline{z} := \alpha - i\beta$, and $|z| := \sup\{Re(e^{i\theta}z) : 0 \leq \theta < 2\pi\}$ (see[12], 1.3.13).

An $F_{\mathbb{C}}$*-valued trace* on the von Neumann algebra M is a linear mapping $\Phi : M \to F_{\mathbb{C}}$ given $\Phi(x^*x) = \Phi(xx^*) \geq 0$ for all $x \in M$. It is clear that $\Phi(M_h) \subset F,\ \Phi(M_+) \subset F_+ = \{a \in F : a \geq 0\}$. A trace Φ is said to be *faithful* if the equality $\Phi(x^*x) = 0$ implies $x = 0$, *normal* if $\Phi(x_\alpha) \uparrow \Phi(x)$ for every $x_\alpha, x \in M_h,\ x_\alpha \uparrow x$.

If M is a finite von Neumann algebra, then its center-valued trace $\Phi_M : M \to Z(M)$ is an example of a $Z(M)$-valued faithful normal trace.

Let Δ be a separating family of finite normal numerical traces on the von Neumann algebra M, $\mathbb{C}^\Delta = \prod\limits_{\tau \in \Delta} \mathbb{C}_\tau$, where $\mathbb{C}_\tau = \mathbb{C}$ for all $\tau \in \Delta$. Then $\Phi(x) = \{\tau(x)\}_{\tau \in \Delta}$ is also an example of an faithful normal $\mathbb{C}^\Delta$-valued trace on M.

Let us list some properties of the trace $\Phi : M \to F_{\mathbb{C}}$.

Proposition 1.1. *([13])* (i) *Let* $x, y, a, b \in M$. *Then*
$\Phi(x^*) = \overline{\Phi(x)},\ \Phi(xy) = \Phi(yx),\ \Phi(|x^*|) = \Phi(|x|)$,
$|\Phi(axb)| \leq \|a\|_M \|b\|_M \Phi(|x|)$;
(ii) *If* Φ *is a faithful trace, then* M *is finite;*
(iii) *If* M *is a finite von Neumann algebra, then* $\Phi(\Phi_M(x)) = \Phi(x)$ *for all* $x \in M$;
(iv) $\Phi(|x + y|) \leq \Phi(|x|) + \Phi(|y|)$ *for all* $x, y \in M$.

The trace $\Phi : M \to F_{\mathbb{C}}$ possesses *the Maharam property* if for any $x \in M_+,\ 0 \leq f \leq \Phi(x),\ f \in F$, there exists a positive $y \leq x$ such that $\Phi(y) = f$. A faithful normal $F_{\mathbb{C}}$-valued trace Φ with the Maharam property is called *a Maharam trace* (compare with [12], III, 3.4.1). Obviously, any faithful finite numerical trace on M is a $\mathbb{C}$-valued Maharam trace.

Let us give another examples of Maharam traces. Let M be a finite von Neumann algebra, let $\mathcal{A}$ be a von Neumann subalgebra in $Z(M)$, and let $T : Z(M) \to \mathcal{A}$ be an injective linear positive normal operator. If $f \in S(\mathcal{A})$ is a reversible positive element, then $\Phi(T, f)(x) = fT(\Phi_M(x))$ is an $S(\mathcal{A})$-valued faithful normal trace on M. In addition, if $T(ab) = aT(b)$ for all $a \in \mathcal{A}, b \in Z(M)$, then $\Phi(T, f)$ is a Maharam trace on M.

Note that if τ is a faithful normal finite numerical trace on M and $\dim(Z(M)) > 1$, then $\Phi(x) = \tau(x)\mathbf{1}$ is a $Z(M)$-valued faithful normal trace. In addition, Φ does not possess the Maharam property (see [13]).

Let F have an order unit $\mathbf{1}_F$. Denote by $B(F)$ the complete Boolean algebra of unitary elements with respect to $\mathbf{1}_F$, and let Q be the Stone representation space of the Boolean algebra $B(F)$. Let $C_\infty(Q)$ be the order complete vector lattice of all continuous functions $a : Q \to [-\infty, +\infty]$ such that $a^{-1}(\{\pm\infty\})$ is a nowhere dense subset of Q. We identify F with the order-dense ideal in $C_\infty(Q)$ containing algebra $C(Q)$ of all continuous real functions on Q. In addition, $\mathbf{1}_F$ is identified with the function equal to 1 identically on Q ([12], 1.4.4).

The next theorem gives the description of Maharam traces on von Neumann algebras.

Theorem 1.1. *([13]) Let Φ be an $F_{\mathbb{C}}$-valued Maharam trace on a von Neumann algebra M. Then there exists a von Neumann subalgebra $\mathcal{A}$ in $Z(M)$, a $*$-isomorphism ψ from $\mathcal{A}$ onto the $*$-algebra $C(Q)_{\mathbb{C}}$, an injective positive linear normal operator $\mathcal{E}$ from $Z(M)$ onto $\mathcal{A}$ with $\mathcal{E}(\mathbf{1}) = \mathbf{1}$, $\mathcal{E}^2 = \mathcal{E}$, such that*

1) $\Phi(x) = \Phi(\mathbf{1})\psi(\mathcal{E}(\Phi_M(x)))$ *for all* $x \in M$;
2) $\Phi(zy) = \Phi(z\mathcal{E}(y))$ *for all* $z, y \in Z(M)$;
3) $\Phi(zy) = \psi(z)\Phi(y)$ *for all* $z \in \mathcal{A}$, $y \in M$.

Due to Theorem 1.1, the $*$-algebra $\mathcal{B} = C(Q)_{\mathbb{C}}$ is $*$-isomorphic to a von Neumann subalgebra in $Z(M)$. Therefore $\mathcal{B}$ is a commutative von Neumann algebra, and $*$-algebra $C_\infty(Q)_{\mathbb{C}}$ is identified with $*$-algebra $S(\mathcal{B})$. It is clear that the $*$-isomorphism ψ from $\mathcal{A}$ onto $\mathcal{B}$ can be extended to a $*$-isomorphism from $S(\mathcal{A})$ onto $S(\mathcal{B})$. We denote this mapping also by ψ.

Let Φ be an $S(\mathcal{B})$-valued Maharam trace on a von Neumann algebra M. Next we will need the concept of a central extension of a von Neumann algebra from [16].

A set $\{z_j\}_{j\in J}$ of pairwise orthogonal nonzero central projections from M will be called a partition of unity $\mathbf{1}$, if $\sup_{j\in J} z_j = \mathbf{1}$. Following [16], denote by $E(M, \mathcal{A})$ the set of all those operators $x \in S(M)$ for which there exists a partition of unity $\{z_j\}_{j\in J} \subset P(\mathcal{A})$ and a set $\{x_j\}_{j\in J} \subset M$ such that $xz_j = x_j z_j$ for all $j \in J$. It is clear that $M \subset E(M, \mathcal{A}), S(\mathcal{A}) \subset E(M, \mathcal{A})$ and $E(M, \mathcal{A})$ is an $*$-subalgebra of $S(M)$ with respect to the natural operations in $S(M)$. $E(M, \mathcal{A})$ is called the central extension of the algebra M with respect to the subalgebra $\mathcal{A} \subset Z(M)$.

Proposition 1.2. *([17], Proposition 3.4) For the operator $x \in S(M)$ the following conditions are equivalent:*
(i) $x \in E(M, \mathcal{A})$;
(ii) *there exists* $a \in S_+(\mathcal{A})$ *such that* $|x| \leq a$.

According to proposition 1.2, for each $x \in E(M, \mathcal{A})$, an element $\|x\|_{\mathcal{A}} = \inf\{a \in S_+(\mathcal{A}) : |x| \leq a\}$ from $S_+(\mathcal{A})$ is defined. The following theorem follows from the results of [17].

Theorem 1.2. $(E(M, \mathcal{A}), \|\cdot\|_{\mathcal{A}})$ *is a Banach-Kantorovich space over* $S_h(\mathcal{A})$.

It follows directly from Theorem 1.2 that the mapping $\|x\|_{\mathcal{B}} = \Psi(\|x\|_{\mathcal{A}})$ defines an $S_h(\mathcal{B})$-valued norm on $E(M, \mathcal{A})$, with respect to which $E(M, \mathcal{A})$ becomes a Banach-Kantorovich space over $S_h(\mathcal{B})$.

2. Spectral distribution functions and singular value functions, associated with a Maharam trace

Let $\mathcal{B}$ be a commutative von Neumann algebra $*$-isomorphic to the von Neumann subalgebra $\mathcal{A}$ in the center $Z(M)$ of M and $\Phi : M \to S(\mathcal{B})$ be the Maharam trace on M (see Theorem 1.1). We suppose that $\Phi(\mathbf{1}) = \mathbf{1}_{\mathcal{B}}$.

For each $f \in S_h(\mathcal{B})$, we denote by $s(f)$ the support of f, i.e. $s(f) = \mathbf{1}_{\mathcal{B}} - \sup\{e \in P(\mathcal{B}) : fe = 0\}$. It is clear that $s(\Phi(\mathbf{1})) = \mathbf{1}_{\mathcal{B}}$. Let $\mathcal{P}(\mathcal{B})$ denote the set of all $f \in S_+(\mathcal{B})$ for which the support $s(f) = \mathbf{1}_{\mathcal{B}}$. It is clear that each element $f \in \mathcal{P}(\mathcal{B})$ is invertible in the algebra $S_h(\mathcal{B})$, i.e. there exists an element $g \in S_h(\mathcal{B})$ such that $f \cdot g = \mathbf{1}_{\mathcal{B}}$, and $g \in \mathcal{P}(\mathcal{B})$.

Definition 2.1. Let $0 \leq x \in S(M)$ and $g \in \mathcal{P}(\mathcal{B})$. The $S(\mathcal{B})$-valued spectral distribution function $d(.;x) : \mathcal{P}(\mathcal{B}) \to S_h(\mathcal{B})$ is defined by

$$d(g;x) := \Phi(E_g(x)),$$

where $E_g(x) \in P(M)$ is the projector in M, which is a projection onto a closed subspace $\overline{\{\xi \in H : x(\xi) > g(\xi)\}}$.

Evidently, the mapping $d(.;x)$ is decreasing. If $g, g_n \in \mathcal{P}(\mathcal{B})$, $n = 1, 2, ...,$ and $g_n \downarrow g$, then $E_g(x) = \sup_{n\geq 1} E_{g_n}(x)$ in M_+, and so, $\Phi(E_g(x)) = \sup_{n\geq 1} \Phi(E_{g_n}(x))$. Hence, $d(.;x)$ is right-continuous on $\mathcal{P}(\mathcal{B})$.

Definition 2.2. For $x \in S(M)$, the singular value function, associated with a Maharam trace Φ is the map $\Phi(.;x) : (0, \infty) \to S_h(\mathcal{B})$ defined by the equality

$$\Phi(t;x) := \inf\{g \in \mathcal{P}(\mathcal{B}) : \ d(g;|x|) \leq t \cdot \mathbf{1}\}, \ t > 0. \tag{2.1}$$

It is clear that $\Phi(t;x) \leq \Phi(s;x)$ at $s < t$. In addition, the map $\Phi(t;x)$ has the following useful continuity property.

Proposition 2.1. *If* $t_n, t > 0, n = 1, 2, ...,$ *and* $t_n \downarrow t$, *then* $\Phi(t;x) = \sup_{n\geq 1} \Phi(t_n;x)$.

Proof. Let $t_n, t > 0, n = 1, 2, ...,$ and $t_n \downarrow t$. Then

$$\Phi(t;x) = \inf\{g \in \mathcal{P}(\mathcal{B}) : d(g;|x|) \leq \inf_{n\geq 1}(t_n \cdot \mathbf{1})\}$$

$$= \inf\left\{g \in \mathcal{P}(\mathcal{B}) : d(g;|x|) \leq t_n \cdot \mathbf{1} \ \text{for all} \ n \geq 1\right\}$$

$$= \sup_{n\geq 1}\left\{\inf\{g \in \mathcal{P}(\mathcal{B}) : d(g;|x|) \leq t_n \cdot \mathbf{1}\}\right\} = \sup_{n\geq 1} \Phi(t_n;x),$$

i.e. $\Phi(t;x) = \sup_{n\geq 1} \Phi(t_n;x)$. $\triangleright$

Example 2.1. Let $x = p \in P(M)$ and $g \in \mathcal{P}(\mathcal{B})$. Then
$d(g;p) = \Phi(E_g(p)) = \Phi(p)$, if $g < \mathbf{1}$, and $d(g;p) = \mathbf{0}$, if $g \geq \mathbf{1}$. Hence by (1) $\Phi(t;p) = \mathbf{1}$, if $\mathbf{0} < t \cdot \mathbf{1} \leq \Phi(p)$, and $\Phi(t;p) = \mathbf{0}$, if $t \cdot \mathbf{1} > \Phi(p)$.

Example 2.2. Let M be a finite von Neumann algebra and suppose that $x = \sum_{j=1}^{m} \alpha_j p_j$, where $p_1, \ldots, p_m \in P(M)$ with $p_j p_k = 0$ whenever $j \neq k$, and $\alpha_1, \ldots, \alpha_m \in \mathbb{R}^+$ are such that $\alpha_j \neq \alpha_k$ whenever $j \neq k$. For the computation of $\Phi(x)$, it may be assumed that $\alpha_1 > \alpha_2 > \ldots > \alpha_m > 0$. If $g \in \mathcal{P}(\mathcal{B})$ and $g \geq \alpha_1 \cdot \mathbf{1}$, then clearly $d(g;x) = 0$. However, if $\alpha_2 \cdot \mathbf{1} \leq g < \alpha_1 \cdot \mathbf{1}$, then $E_g(x) = p_1$, and so $d(g;x) = \Phi((p_1))$. Similarly, if $\alpha_3 \cdot \mathbf{1} \leq g < \alpha_2 \cdot \mathbf{1}$, then $E_g(x) = p_1 + p_2$, and so $d(g;x) = \Phi(p_1 + p_2) = \Phi(p_1) + \Phi(p_2)$. In general, we have

$$d(g;x) = \sum_{i=1}^{j} \Phi(p_i), \quad \text{if} \ \alpha_{j+1} \cdot \mathbf{1} \leq g < \alpha_j \cdot \mathbf{1} \ (g \in \mathcal{P}(\mathcal{B})),$$

where $j = 1, 2, \ldots, m$, and $\alpha_{m+1} = 0$.

Define $\rho_k = \sum_{i=1}^{j} \Phi(p_i)$ for $j = 1, 2, \ldots, m$. Referring to (2.1), we see that $\Phi(t;x) = \mathbf{0}$ if $t \cdot \mathbf{1} \geq \rho_m$. Also, if $\rho_m > t \cdot \mathbf{1} \geq \rho_{m-1}$, then $\Phi(t;x) = \alpha_m \cdot \mathbf{1}$, and if $\rho_{m-1} > t \cdot \mathbf{1} \geq \rho_{m-2}$, then $\Phi(t;x) = \alpha_{m-1} \cdot \mathbf{1}$, and so on. Hence,

$$\Phi(t;x) = \begin{cases} \alpha_1 \cdot \mathbf{1}, & \mathbf{0} < t \cdot \mathbf{1} \leq \rho_1; \\ \alpha_j \cdot \mathbf{1}, & \rho_j > t \cdot \mathbf{1} \geq \lambda_{j-1}, \ 2 \leq j \leq m-1; \\ \mathbf{0}, & t \cdot \mathbf{1} > \rho_m. \end{cases}$$

Theorem 2.1. *Let* $x \in S(M)$. *For all* $t > 0$, *the singular value function* $\Phi(., x)$ *admits the characterization*

$$\Phi(t;x) = \inf\{\|xe\|_{\mathcal{B}} : \ e \in P(M), \ xe \in E(M, \mathcal{A}), \ \Phi(\mathbf{1} - e) \leq t \cdot \mathbf{1}\}. \tag{2.2}$$

Proof. We fix $t > 0$ and put

$$G(x) = \{g \in \mathcal{P}(\mathcal{B}) : \ d(g; |x|) \le t \cdot \mathbf{1}\}.$$

If $g_1, g_2 \in G(x)$, $e = s((g_2 - g_1)_+)$, then $g = g_1 \wedge g_2 = g_1 \cdot e + g_2 \cdot (\mathbf{1} - e) \in \mathcal{P}(\mathcal{B})$, at the same time

$$d(g; |x|) = \Phi(E_g(x)) = \Phi(E_{g_1}(x)) \cdot e + \Phi(E_{g_2}(x)) \cdot (\mathbf{1} - e)$$
$$\le t \cdot e + t \cdot (\mathbf{1} - e) = t \cdot \mathbf{1},$$

i.e. $g_1 \wedge g_2 \in G(x)$. Using mathematical induction, we obtain that for any finite set $\{g_i\}_{i=1}^n \subset G(x)$ the inclusion holds true $\bigwedge\limits_{i=1}^{n} g_i \in G(x)$. Since the Boolean algebra $P(\mathcal{B})$ has a countable type, there exists a sequence $\{g_k\}_{k=1}^\infty \subset G(x)$ for which $g_k \downarrow f$, where

$$f = \Phi(t; x) := \inf\{g \in \mathcal{P}(\mathcal{B}) : \ d(g; |x|) \le t \cdot \mathbf{1}\} \in S_h(\mathcal{B}).$$

Since $g_k \downarrow f$ and for all $k \in \mathbb{N}$ the inequality $d(g_k; |x|) \le t \cdot \mathbf{1}$ is true, then

$$t \cdot \mathbf{1} \ge d(g_k; |x|) \uparrow d(f; |x|).$$

In particular, the inequality is true

$$\Phi(E_f(x)) = d(f; |x|) \le t \cdot \mathbf{1} \ \text{ i.e. } \ \Phi(\mathbf{1} - e) \le t \cdot \mathbf{1},$$

where $e = \mathbf{1} - E_f(x) \in P(M)$. Since $|x|e \le f \in S_h(\mathcal{B})$, then $xe \in E(M, \mathcal{A})$. Moreover, using the polar decomposition $x = u|x|$ of the operator x, we obtain $\|xe\|_{\mathcal{B}} = \|u|x|e\|_{\mathcal{B}} \le \||x|e\|_{\mathcal{B}} \le f$. Therefore,

$$\Psi(t; x) = \inf\{\|xe\|_{\mathcal{B}} : \ e \in P(M), \ xe \in E(M, \mathcal{A}), \ \Phi(\mathbf{1} - e) \le t \cdot \mathbf{1}\} \le f = \Phi(t; x).$$

To prove the reverse inequality, we set

$$\Lambda(x) = \{e \in P(M) : xe \in E(M, \mathcal{A}), \ \Phi(\mathbf{1} - e) \le t \cdot \mathbf{1}\}.$$

Then, we have $\Psi(t; x) = \inf\{\|xe\|_{\mathcal{B}} : e \in \Lambda(x)\}$. Suppose that the inequality $\Phi(t; x) = f \le \Psi(t; x)$ is not satisfied. Therefore, there exists an $e \in \Lambda(x)$ for which the inequality $f \le \|xe\|_{\mathcal{B}}$ does not hold. In particular, this means that there exist $0 \ne q \in P(M)$, $\varepsilon > 0$, for which the inequalities

$$|xeq| \le \|xeq\|_{\mathcal{B}} = \|xe\|_{\mathcal{B}} \cdot q \le qf + \varepsilon q.$$

are true. Therefore, there exists an $e \in \Lambda(x)$ for which the inequality $f \le \|xe\|_{\mathcal{B}}$ does not hold. In particular, this means that there exist $0 \ne q \in P(M)$, $\varepsilon > 0$, for which the inequalities

$$|xeq| \le \|xeq\|_{\mathcal{B}} = \|xe\|_{\mathcal{B}} \cdot q \le qf + \varepsilon q$$

are true. Let $q_1 = qg + 2\varepsilon \cdot q$. Consider the element $r = s((|xe| - q_1)_+)$ from $P(M)$. From the relations

$$|xe|rq \ge (qg + 2\varepsilon \cdot q)q = qg + 2\varepsilon \cdot q > qg + \varepsilon \cdot q \ge |xeq| = |xe|q$$

it follows that $|xe|rq > |xe|q$, which is impossible.

Thus, the inequality $\Phi(t; x) \le \Psi(t; x)$ is satisfied. Consequently, the equality $\Phi(t; x) = \Psi(t; x)$ is true. ▷

Recall that for each $x \in S(M)$, the support projection of x is denoted by $s(x)$, that is, $s(x) = \mathbf{1} - n(x)$, where $n(x)$ is the projection onto $Ker(x)$. For $t > 0$, define

$$R_t = \{x \in S(M) : \Phi(s(x)) \le t \cdot \mathbf{1}\}.$$

The following proposition presents a geometric interpretation of the singular value function in terms of what might be called generalized approximation numbers.

Proposition 2.2. *If* $x \in S(M)$, *then*

$$\Phi(t;x) = \inf\{\|x-y\|_{\mathcal{B}} : y \in R_t,\ (x-y) \in E(M,\mathcal{A})\}$$

for all $t > 0$.

Proof. Let $t > 0$ and $x \in S(M)$ be fixed. Let $x = u|x|$ is the polar decomposition of x, $\Phi(t;x) = f$ and $p = E_f(|x|) \in P(M)$. Defining $y = xp$, $e = \mathbf{1} - p$, it follows that $x - y = x(\mathbf{1} - p) = xe$, and so, $x - y \in E(M,\mathcal{A})$ with

$$\|x-y\|_{\mathcal{B}} = \|ue|x|e\,\|_{\mathcal{B}} = \|e|x|\|_{\mathcal{B}} \leq \Phi(t;x).$$

Moreover, $s(y) \leq p$, which implies that

$$\Phi(s(y)) \leq \Phi(p) = \Phi(E_f(|x|)) = d(f;|x|) \leq t\cdot\mathbf{1}.$$

Consequently,

$$\inf\{\|x-y\|_{\mathcal{B}} : y \in R_t,\ (x-y) \in E(M,\mathcal{A})\} \leq \Phi(t;x).$$

To obtain the reverse inequality, suppose that $y \in R_t$ is such that $x - y \in E(M,\mathcal{A})$. Since $xn(y) = (x-y)n(y)$, it is clear that $\|xn(y)\|_{\mathcal{B}} \leq \|x-y\|_{\mathcal{B}}$. Furthermore, $\Phi(\mathbf{1} - n(y)) = \Phi(s(y) \leq t\cdot\mathbf{1}$, and so, by Theorem 2.1 implies that

$$\Phi(t;x) = \inf\{\|xe\|_{\mathcal{B}} :\ e \in P(M),\ xe \in E(M,\mathcal{A}),\ \Phi(\mathbf{1}-e) \leq t\cdot\mathbf{1}\} \leq \|x-y\|_{\mathcal{B}}.$$

This shows that

$$\Phi(t;x) \leq \inf\{\|x-y\|_{\mathcal{B}} : y \in R_t,\ (x-y) \in E(M,\mathcal{A})\},$$

which concludes that proof of the proposition. ▷

In the next theorem, some basic properties of singular value functions are collected.

Theorem 2.2. *Let* $t > 0$. *For all* $x, y \in S(M)$, *the following hold:*

(i). $\Phi(t;x) = \Phi(t;|x|) = \Phi(t;x^*)$ *and* $\Phi(t;\alpha x) = |\alpha|\Phi(t;x)$ *for all* $\alpha \in \mathbb{C}$.
(ii). $\Phi(t;xe) = \mathbf{0}$ *whenever* $\Phi(e) \leq t\cdot\mathbf{1}$ *for all* $e \in P(M)$. *In particular,* $\Phi(t;x) = \mathbf{0}$ *for all* $t\cdot\mathbf{1} \geq \Phi(s(x))$.
(iii). *If* $|x| \leq |y|$, *then* $\Phi(t;x) \leq \Phi(t;y)$.
(iv). $\Phi(t_1+t_2;x+y) \leq \Phi(t_1;x) + \Phi(t_2;y)$ *for all* $t_1, t_2 > 0$.
(v). *If* $x \in E(M,\mathcal{A})$, $t_n > 0$, $n = 1,2,...,$ *and* $t_n \downarrow 0$, *then*

$$\|x\|_{\mathcal{B}} = \sup_{n\geq 1}\Phi(t_n;x).$$

Proof. (i). The equality $\Phi(t;x) = \Phi(t;|x|)$ follows directly from the definition of the mapping $\Phi(t;x)$.
Let $x = u|x|$ be the polar decomposition of the operator x. Then $|x^*| = u|x|u^*$. Therefore, for $g \in \mathcal{P}(\mathcal{B})$ we have

$$d(g;x^*) = \Phi(E_g(x^*)) = \Phi(uE_g(x)u^*) = \Phi(E_g(x) = d(g;x).$$

Consequently, $\Phi(t;x) = \Phi(t;x^*)$.
Finally, for any $\alpha \in \mathbb{C}$, $t > 0$ we have that

$$\Phi(t;\alpha x) = \inf\{\|\alpha xe\|_{\mathcal{B}} :\ e \in P(M),\ \alpha xe \in E(M,\mathcal{A}),\ \Phi(\mathbf{1}-e) \leq t\cdot\mathbf{1}\}$$

$$= |\alpha|\inf\{\|xe\|_{\mathcal{B}} :\ e \in P(M),\ xe \in E(M,\mathcal{A}),\ \Phi(\mathbf{1}-e) \leq t\cdot\mathbf{1}\} = |\alpha|\Phi(t;x).$$

(ii). If $\Phi(e) \leq t\cdot\mathbf{1}$, then, trivially, $xe(\mathbf{1}-e) = 0$ and $\Phi(\mathbf{1}-(\mathbf{1}-e)) \leq t\cdot\mathbf{1}$; hence, $\Phi(t;xe) = \mathbf{0}$ follows immediately from (2.2).

(iii). If $|x| \le |y|$, then $d(g;|x|) \le d(g;|y|)$ for all $g \in \mathcal{P}(\mathcal{B})$. Hence $\Phi(t;x) \le \Phi(t;y)$.

(iv). For all $g_1, g_2 \in \mathcal{P}(\mathcal{B})$ the following inequality holds

$$E_{g_1+g_2}(|x+y|) \le E_{g_1}(|x|) \vee E_{g_2}(|y|).$$

It follows that

$$\Phi(E_{g_1+g_2}(|x+y|)) \le \Phi(E_{g_1}(|x|)) + \Phi(E_{g_2}(|y|)).$$

We fix $\varepsilon > 0$ and set $g_1 = \Phi(t_1;x)$, $g_2 = \Phi(t_2;y) + \varepsilon \cdot \mathbf{1}$. Using the inequality $\Phi(E_{g_2}(|y|) \le \Phi(E_{\Phi(t_2;y)}(|y|)$ we have

$$\Phi(E_{\Phi(t_1;x)+\Phi(t_2;y)+\varepsilon\cdot\mathbf{1}} \le \Phi(E_{\Phi(t_1;x)}(|x|)) + \Phi(E_{\Phi(t_2;y)+\varepsilon\cdot\mathbf{1}}(|y|))$$

$$\le t_1 \cdot \mathbf{1} + \Phi(E_{\Phi(t_2;y)}(|y|)) \le (t_1+t_2)\cdot\mathbf{1},$$

i.e. $\Phi(E_{\Phi(t_1;x)+\Phi(t_2;y)+\varepsilon\cdot\mathbf{1}} \le (t_1+t_2)\cdot\mathbf{1}$.

Since $\Phi(t_1;x) + \Phi(t_2;y) + \varepsilon\cdot\mathbf{1} \in \mathcal{P}(\mathcal{B})$, then from the definition of the mapping $\Phi(t;x)$ the following inequality follows

$$\Phi((t_1+t_2);x+y) \le \Phi(t_1;x) + \Phi(t_2;y) + \varepsilon\cdot\mathbf{1}.$$

From here, at $\varepsilon \downarrow 0$, we obtain the required inequality

$$\Phi((t_1+t_2);x+y) \le \Phi(t_1;x) + \Phi(t_2;y).$$

(vi). First we show that for all $q \in P(\mathcal{B})$, $x \in E(M,\mathcal{A})$, $t > 0$ the equality $\Phi(t;qx) = q\Phi(t;x)$ is true. Because

$$\Phi(t;qx) = \inf\{\|qxe\|_{\mathcal{B}} : e \in P(M),\ qxe \in E(M,\mathcal{A}),\ \Phi(\mathbf{1}-e) \le t\cdot\mathbf{1}\}$$

$$= \inf\{q\|qxe\|_{\mathcal{B}} : e \in P(M), qxe \in E(M,\mathcal{A}),\ \Phi(\mathbf{1}-e) \le t\cdot\mathbf{1}\} = q\Phi(t;qx),$$

then from the inequality $|qx| \le |x|$ follows the inequality $q\Phi(t;qx) \le q\Phi(t;x)$ (see the property (iii) proved above).

On the other hand, if $e \in P(M)$, $qxe \in E(M,\mathcal{A})$ and $\Phi(\mathbf{1}-e) \le t\cdot\mathbf{1}$, then

$$\Phi(t;x) = q\Phi(t;x) + (\mathbf{1}-q)\Phi(t;x) \le q\|qxe\|_{\mathcal{B}} + (\mathbf{1}-q)\Phi(t;x).$$

Therefore,

$$q\Phi(t;x) \le q\|qxe\|_{\mathcal{B}} \le \|qxe\|_{\mathcal{B}},$$

and hence $q\Phi(t;x) \le \Phi(t;qx)$. Thus, the equality is true $\Phi(t;qx) = q\Phi(t;x)$.

If $x \in E(M,\mathcal{A})$, then $x\cdot\mathbf{1} \in E(M,\mathcal{A})$, and it follows directly from Proposition 2.1 that $\Phi(t;x) \le \|x\|_{\mathcal{B}}$ for all $t > 0$. In addition, the inequality $\Phi(t_1;x) \le \Phi(t_2;x)$ is true for $0 < t_2 < t_1$.

Thus, $\Phi(t_n;x) \uparrow z \le \|x\|_{\mathcal{B}}$ as $t_n \downarrow 0$ for some $0 \le z \in S_h(\mathcal{B})$.

If $z \ne \|x\|_{\mathcal{B}}$, then for any $\varepsilon > 0$ there is $q_\varepsilon \in P(\mathcal{B})$, such that

$$\Phi(t_n;xq_\varepsilon) = \Phi(t_n;x)q_\varepsilon \le zq_\varepsilon < q_\varepsilon \cdot \|x\|_{\mathcal{B}}$$

for all $t_n \in (0,\varepsilon)$. From here, by virtue of proposition 2.1, we obtain that

$$\Phi(t_n;xq_\varepsilon) \le zq_\varepsilon \text{ for all } t_n \in (0,\varepsilon).$$

Again using proposition 2.1, we have,

$$\Phi(\{|xq_\varepsilon| > zq_\varepsilon\}) \le \Phi(\{|xq_\varepsilon| > \Phi(t_n, xq_\varepsilon)\}) \le t_n \cdot \mathbf{1}$$

for all $t_n \in (0, \varepsilon)$. Hence, $\Phi(\{|xq_\varepsilon| > zq_\varepsilon\}) = 0$. This means that $|xq_\varepsilon| \le zq_\varepsilon$, in particular, $\|xq_\varepsilon\|_{\mathcal{B}} \le zq_\varepsilon$, which is not the case. Thus,

$$\|x\|_{\mathcal{B}} = \sup_{n\ge 1} \Phi(t_n; x).$$

▷

Corollary 2.1. *For any* $x, y \in E(M, \mathcal{A})$, $t > 0$ *the inequality holds*

$$|\Phi(t; x) - \Phi(t; y)| \le \|x - y\|_{\mathcal{B}}.$$

Proof. By Theorem 2.2 (iv) we have that

$$\Phi(t_1 + t_2; x) = \Phi(t_1 + t_2; y + (x - y)) \le \Phi(t_1; y) + \Phi(t_2; x - y) \le \Phi(t_1; y) + \|x - y\|_{\mathcal{B}}.$$

Similarly,

$$\Phi(t_1 + t_2; y) = \Phi(t_1 + t_2; x + (y - x)) \le \Phi(t_2; x) + \Phi(t_1; y - x) \le \Phi(t_2; x) + \|x - y\|_{\mathcal{B}}.$$

Assuming in these inequalities $t_1 = t$, $t_2 = 0$, we obtain

$$\Phi(t; x) \le \Phi(t; y) + \|x - y\|_{\mathcal{B}} \text{ and } \Phi(t; y) \le \Phi(t; x) + \|x - y\|_{\mathcal{B}},$$

from which it follows that $|\Phi(t; x) - \Phi(t; y)| \le \|x - y\|_{\mathcal{B}}$. ▷

The following proposition establishes the relation between ordinal convergence in $S(M)$ and ordinal convergence of singular values functions.

Proposition 2.3. *If* $x_n, x \in S(M)$, $n \in \mathbb{N}$ *and* $0 \le x_n \uparrow x$, *then* $\Phi(t, x_n) \uparrow \Phi(t, x)$ *for all* $t > 0$.

Proof. Let $x_n, x \in S(M)$ and $0 \le x_n \uparrow x$. First, let us show that

$$d(g, x_n) \uparrow d(g, x), g \in \mathcal{P}(\mathcal{B}) \tag{2.3}$$

We fix $g \in \mathcal{P}(\mathcal{B})$ and put $G_g(x) = \{\xi \in H : x(\xi) > g(\xi)\}$, $G_g(x_n) = \{\xi \in H : x_n(\xi) > h(\xi)\}$, $(n = 1, 2, ...)$. Since $x_n \le x_{n+1}$, then $G_g(x_n) \subset G_g(x_{n+1})$. Furthermore, the condition $x_n \uparrow x$ imply that $G_g(x) = \bigcup_{n=1}^{\infty} G_g(x_n)$ and $E_g(x_n) \uparrow E_g(x)$. Hence, by normality of trace Φ,

$$d(h; x_n) = \Phi(E_g(x_n)) \uparrow \Phi(E_g(x)) = d(h; x).$$

Since

$$\Phi(t, x_n) \le \Phi(t, x_{n+1}) \le \Phi(t, x)$$

for all $n = 1, 2, \ldots$ and $t > 0$, it is clear that $\Phi(t, x_n) \uparrow_n$ and that

$$\sup_{n\ge 1} \Phi(t, x_n) \le \Phi(t, x)$$

for all $t > 0$. For the proof of the reverse inequality, it may be assumed that $t > 0$ is such that $\Phi(t, x_n) < g$, for all n and some $g \in \mathcal{P}(\mathcal{B})$. Hence $d(g, x_n) \le t \cdot \mathbf{1}$ for all n. Then by (2.3), it follows that $d(g, x) \le t \cdot \mathbf{1}$. Consequently, $\Phi(t, x) < g$. This suffices to show that $\Phi(t, x) \le \sup_{n\ge 1} \Phi(t, x_n)$. The proof is complete. ▷

Conclusion

In this paper, the Maharam trace Φ on a von Neumann algebra M with values in complex Dedekind complete vector lattice is considered. For an operator x from the $*$-algebra $S(M)$ of measurable operators affiliated with M, the singular value function of x, associated with such a trace Φ are determined. The main properties of these singular value functions, similar to classical singular value functions of measurable operators with respect numerical trace, are studied.

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