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Research article

Uniqueness of 3-adic periodic Gibbs measures for the Ising model on a (2,3)-ary tree

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Abstract

In this paper, translation-invariant 3-adic Gibbs measures for the Ising model on a $(2,3)$-ary tree are studied. The dynamics of the function corresponding to this model is investigated, and the existence of a unique global attracting fixed point is proved.

Keywords: *p-adic numbers; Ising model; Gibbs measure; (i,k)-ary tree.*

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1. Introduction

The Ising model was invented by the physicist Wilhelm Lenz, who gave it as a problem to his student Ernst Ising. The one-dimensional Ising model was solved by Ising in [1] and he proved that the model does not exhibit the phase transition. Using geometric-probabilistic approach, the contour argument was introduced by Peierls for the two-dimensional square-lattice Ising model to show the existence of phase transitions at low temperatures [2]. For this model an analytic description was given by Lars Onsager [3]. The existence of a phase transition for the Ising model on the Cayley trees was established independently by Katsura, Takizawa [4] and Preston [5] (see also [6] for more details).

In the p-adic context, during the last few decades great interest has been devoted to the investigation of various properties of some classical models of statistical mechanics (see, for example, [7, 8, 9, 10, 11]). These investigations lead the development of the p-adic stochastic processes [12, 13]. The p-adic Ising model on a Cayley tree was considered in [15, 14], and for this model the uniqueness of p-adic Gibbs measure was proved. To enlarge the set of p-adic Gibbs measures, the notion of p-adic generalized Gibbs measures for the Ising model were introduced in [14]. In [16] the authors proved the existence of infinitely many periodic p-adic generalized Gibbs measures for the Ising model on the semi-infinite Cayley tree. Furthermore, in [17] $G_k^{(2)}$-periodic p-adic generalized Gibbs measures for the same model on the Cayley trees have been studied. In [18] the authors studied translation-invariant p-adic generalized Gibbs measures for the Ising model on Cayley trees of order k. Moreover, the set of fixed points of the Ising-Potts mapping was described, which allows to detect the existence of the phase transition.

In present paper, we study 3-adic quasi-Gibbs measures for the Ising model on a $(2,3)$-ary tree. It should be noted that for the case $p > 3$, this problem was studied in paper [19].

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2. Preliminaries

2.1. p-adic numbers

For the field of rational numbers $\mathbb{Q}$, and a predetermined prime p, any $x \in \mathbb{Q} \setminus \{0\}$ possesses a representation $x = p^r \frac{n}{m}$, where $r \in \mathbb{Z}$, $m \in \mathbb{Z}^+$, and n, m are integers such that $\gcd(n, p) = \gcd(m, p) = 1$, where $\gcd(a, b)$ is the greatest common divisor of a and b.

The p-adic norm on $\mathbb{Q}$, defined as $| x |_p = p^{-r}$ for non-zero $x = p^r \frac{n}{m}$ and $| 0 |_p = 0$, is a non-Archimedean valuation. Consequently, it adheres to the strong triangle inequality, which states that $| x + y |_p \leq \max\{| x |_p, | y |_p\}$ (see [24]).

The field of rational numbers $\mathbb{Q}$ is incomplete with respect to the p-adic norm. The completion of $\mathbb{Q}$ under this norm results in the construction of the field of p-adic numbers $\mathbb{Q}_p$.

Later we use the following important properties of the p-adic norm:
1) if $| x |_p \neq | y |_p$, then $| x \pm y |_p = \max\{| x |_p, | y |_p\}$;
2) if $| x |_p = | y |_p$, then $| x \pm y |_p \leq | x |_p$.

Every non-zero p-adic number y possesses a unique canonical expansion:

$$y = p^{\gamma(y)}(y_0 + y_1 p + y_2 p^2 + ...),$$

where the p-adic valuation $\gamma(y) \in \mathbb{Z}$ and the coefficients y_j, $j = 0, 1, 2, ...$ satisfy $0 < y_0 \leq p - 1$ and $0 \leq y_j \leq p - 1$ for $j \geq 1$. The p-adic norm is extended to $\mathbb{Q}_p$ as $| y |_p = p^{-\gamma(y)}$, where $\gamma(y)$ is also denoted as $\mathrm{ord}_p(y)$ (see e.g., [24]).

Let a be an element of the p-adic field $\mathbb{Q}_p$ and let $r > 0$. We define the open ball $B(a, r)$, the closed ball $\overline{B(a, r)}$, and the sphere $S(a, r)$ centered at a with radius r as the sets:

$$B(a, r) = \{x \in \mathbb{Q}_p : | x - a |_p < r\},$$
$$\overline{B(a, r)} = \{x \in \mathbb{Q}_p : | x - a |_p \leq r\},$$
$$S(a, r) = \{x \in \mathbb{Q}_p : | x - a |_p = r\}.$$

The sets $\mathbb{Z}_p := \overline{B(0, 1)}$ and $\mathbb{Z}_p^* = S(0, 1)$ are respectively called *p-adic integers* and *p-adic units*.

p-adic *exponential* is defined by

$$\exp_p(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!},$$

which converges for $x \in B(0, p^{-1/(p-1)})$.

Put

$$\mathcal{E}_p = \left\{x \in \mathbb{Q}_p : | x - 1 |_p < p^{-1/(p-1)}\right\}.$$

The set $\mathcal{E}_p$ is a range of exponential. A more detailed see [20], [21].

The symbol $o[x]$ was adopted in the work [22] to facilitate simplified computations. By definition, $z = o[x]$ means that $| z |_p < | x |_p$.

Lemma 2.1. *(Hensel's lemma)[23, 24] Let $F(x)$ be a polynomial whose coefficients are p-adic integers and $F'(x)$ be its formal derivative. If there exists an element $a \in \mathbb{Z}_p$ that satisfies the following strict inequality:*

$$| F(a) |_p < | F'(a) |_p^2,$$

then there exists a unique root α in the ring $\mathbb{Z}_p$ satisfying the following properties:

- $F(\alpha) = 0$;
- $| \alpha - a |_p \leq \frac{|F(a)|_p}{|F'(a)|_p} < | F'(a) |_p$, *i.e.* $\alpha = a + o[F'(a)]$.

2.2. p-adic measure

Let $(X, \mathcal{B})$ be a measurable space, where $\mathcal{B}$ represents an algebra of subsets of X. A function $\mu : \mathcal{B} \to \mathbb{Q}_p$ is termed a p-adic measure if it satisfies the property of finite additivity: for any finite collection of pairwise disjoint sets $A_1, A_2, \dots, A_n \in \mathcal{B}$, the following equality holds:

$$\mu\left(\bigcup_{j=1}^{n} A_j\right) = \sum_{j=1}^{n} \mu(A_j).$$

Furthermore, if the total measure is unity ($\mu(X) = 1$), the p-adic measure is specifically referred to as a p-adic probability measure. A p-adic measure μ is defined as bounded if the supremum of the p-adic norm of the measures of all sets in the algebra is finite:

$$\sup\{|\ \mu(A)\ |_p : A \in \mathcal{B}\} < \infty.$$

2.3. (i, k)-ary tree

Let $T = (V,\ E)$ be a rooted infinite tree, i.e.,

1. T is a connected graph without cycles;
2. $|V|$ is infinite;
3. one vertex (e.g. x_0) has been assigned as the root of the tree.

Definition 2.1. Let k be a positive integer. A *k-ary tree* is the rooted infinite tree such that each vertex has at most $k + 1$ edges.

Definition 2.2. The k-ary tree is called a *regular tree* of order k if every vertex except the root has exactly $k + 1$ edges, the root has k edges.

Two vertices x and y are called *nearest neighbors* if there exists an edge $l \in L$ connecting them $l = \langle x, y\rangle$.
A collection of nearest neighbor pairs $\langle x, x_1\rangle, \langle x_1, x_2\rangle, \dots, \langle x_m, y\rangle$ is called a *path* from x to y.
The *distance* between the vertices x and y ($d(x, y)$) is the number of edges of the shortest path from x to y.
Let us fix $x_0 \in V$ vertex as a root of the tree. Then for any nonnegative integer n, we introduce the following sets

$$W_n = \{x \in V : d(x, x_0) = n\}, V_n = \bigcup_{m=0}^{n} W_m,$$

$$V_e = \bigcup_{n=0}^{+\infty} W_{2n},\ V_o = \bigcup_{n=0}^{+\infty} W_{2n+1},$$

$$L_n = \{\langle x, y\rangle \in L : \forall x, y \in V_n\}.$$

For a given $x \in W_n$, we denote

$$S(x) = \{y \in W_{n+1} : d(x, y) = 1\}.$$

Definition 2.3. Let $1 \le i < k$. A rooted k-ary tree is called an *(i, k)-ary tree* if each vertex x has exactly $\kappa_{i,k}$ edges, where

$$\kappa_{i,k} = \begin{cases} i, \text{ if } x = x_0, \\ i + 1, \text{ if } x \in V_e \setminus \{x_0\}, \\ k + 1, \text{ if } x \in V_o. \end{cases}$$

2.4. A level function

Let $\mathbb{Q}_p$ be field of p-adic numbers.

Definition 2.4. A function $s : V \to \mathbb{Q}_p$ is called *level function* if $s^{(n)} := s(u) = s(v)$ for all $u, v \in W_n$, $n \in \mathbb{N}$. Any level function has the following form

$$s^{(n)} = \begin{cases} s_e^{(n)}, \text{ if } n \text{ is even}, \\ s_o^{(n)}, \text{ if } n \text{ is odd}, \end{cases}$$

Definition 2.5. The level function $s = \left(s_e^{(2n)}, s_o^{(n+1)}\right)$ is called *d-periodic* if there exists $d \in \mathbb{N}$ such that

$$s_e^{(2n+d)} = s_e^{(2n)}, \; s_o^{(2n+1+d)} = s_o^{(2n+1)},$$

for all $n \in \mathbb{N}$.

Definition 2.6. Any 1-periodic function is called *translation-invariant*.

It should be noted that the definitions in this section are presented in [19].

2.5. p-adic Ising model

Let $\mathbb{Q}_p$ be the field of p-adic numbers, and let $\Phi = \{-1, 1\}$. We consider the p-adic Ising model on a $(2, 3)$-ary tree. A *configuration* σ on V is a mapping $x \in V \mapsto \sigma(x) \in \Phi$. The set of all configurations on V is denoted by $\Omega := \Phi^V$. By Ω_n we denote the set of all configurations on V_n.

The p-adic Ising model's Hamiltonian on Ω_n is defined by

$$\mathcal{H}_n(\sigma_n) = J \sum_{\langle x,y\rangle \in L_n} \sigma_n(x)\sigma_n(y), \quad \forall \sigma_n \in \Omega_n, \tag{2.1}$$

where $J \in B(0, p^{\frac{-1}{p-1}})$ is a constant.

For a given function $\mathbf{h}$ we define a probability measure $\mu_{\mathbf{h}}^{(n)}$ on Ω_n as follows:

Let $\mathbf{h} : V \to \mathbb{Q}_p^2$ be a vector function such that $\mathbf{h} = \{(h_{x,-1}, h_{x,1}) \in \mathbb{Q}_p^2 : \forall x \in V\}$.

$$\mu_{\mathbf{h}}^{(n)}(\sigma_n) = \frac{\exp_p(\mathcal{H}_n(\sigma_n)) \prod_{x \in W_n} h_{x,\sigma_n(x)}}{Z_{n,\mathbf{h}}}, \quad \forall \sigma_n \in \Omega_n, \tag{2.2}$$

here $Z_{n,\mathbf{h}}$ is the corresponding normalizing factor.

If the p-adic probability measures (1.2) are compatible, then by the p-adic analogue of Kolmogorov's theorem there exists a unique limiting measure $\mu_{\mathbf{h}}$ on the set Ω such that $\mu_{\mathbf{h}}\left(\left\{\sigma|_{V_n} \equiv \sigma_n\right\}\right) = \mu_{\mathbf{h}}^{(n)}\,(\sigma_n)$ for all $n \in \mathbb{N}$ and $\sigma_n \in \Omega_{V_n}$.

Such a measure $\mu_{\mathbf{h}}$ is called *p-adic quasi Gibbs measure* Moreover, if $h_{x,\sigma(x)} \in \mathcal{E}_p$ for every $x \in V$ and every configuration $\sigma \in \Omega$, then corresponding measure $\mu_{\mathbf{h}}$ is called *p-adic Gibbs measure*.

Proposition 2.1. *[15, 14] The sequence of p-adic probability measures $\{\mu_h^{(n)}\}_{n\geq 1}$, determined by formula (1.2) is consistent if and only if for any $x \in V \setminus \{x_0\}$, the following equation holds*

$$h_x = \prod_{y \in S(x)} \frac{\theta h_y + 1}{h_y + \theta}, \tag{2.3}$$

where $\theta = \exp_p(2J), \theta \neq 1$.

3. Main results

Using the eqution (1.3), the problem of finding p-adic quasi-Gibbs measures for the Ising model on a $(2,3)$-ary tree leads to the analysis of the dynamics of the following function:

$$f(u) = \frac{au^6 + 2u^3 + 1}{u^6 + 2u^3 + a}, \tag{3.1}$$

where $a = \frac{\theta^2 - \theta + 1}{\theta} \in \mathcal{E}_3 \setminus \{1\}$.

Remark 3.1. Note that $\sqrt{a-1} = \frac{\theta - 1}{\sqrt{\theta}} \in \mathbb{Q}_3$.

We are going to find fixed points of $f(u)$. We consider the equation $f(u) = u$. The last equation follows that

$$(u-1)\left(u^6 - (a-1)u^5 - (a-1)u^4 - (a-3)u^3 - (a-1)u^2 - (a-1)u + 1\right) = 0. \tag{3.2}$$

Clear that, one of the solution of (3.2) is $u_0 = 1$.

Now we consider the follow equation

$$u^6 - (a-1)u^5 - (a-1)u^4 - (a-3)u^3 - (a-1)u^2 - (a-1)u + 1 = 0. \tag{3.3}$$

We divide u^3 ($u = 0$ is not solution of (3.3)) both sides of (3.3), and using the notation

$$t = u + \frac{1}{u}, \tag{3.4}$$

we get

$$t^3 - (a-1)t^2 - (a+2)t + a + 1 = 0. \tag{3.5}$$

Lemma 3.1. *If t^* is a solution of* (3.5) *then $t^* \in \mathcal{E}_3$.*

Proof. Clear that $t = 0$ is not solution of (3.5). Let t^* be solution of (3.5).

Case 1. Assume that $0 < |t^*|_3 < 1$. Then

$$\left|t^{*3} - (a-1)t^{*2} - (a+2)t^* + a + 1\right|_3 = 1 \neq 0.$$

This case will not occur.

Case 2. Assume that $|t^*|_3 > 1$. It follows that

$$|t^{*3}|_3 = |t^*|_3^3 > |t^{*2}|_3 > |(a-1)t^{*2}|_3,$$

$$|t^{*3}|_3 > |t^*|_3 > |(a+2)t^*|_3,$$

$$|t^{*3}|_3 > 1 = |a+1|_3.$$

The strong triangle inequality yields that

$$\left|t^{*3} - (a-1)t^{*2} - (a+2)t^* + a + 1\right|_3 = |t^{*3}|_3 > 1 \neq 0.$$

This case also will not occur.

Case 3. Assume that $|t^*|_3 = 1$ and $t^* \notin \mathcal{E}_3$. It gives t^* has $2 + o[1]$ form, i.e., $t^* = 2 + o[1]$. Substituting this expression into (3.5), and after some algebras, we get

$$1 + o[1] = 0,$$

which is impossible. This contradiction proves the assumption of the lemma. □

Now we are going to solve (3.5).

Lemma 3.2. *The equation* (3.5) *is solvable iff* $|a-1|_3 < \frac{1}{27}$. *Moreover, the equation* (3.5) *has a unique solution and the solution has the following form*

$$t = -2 + 3^{ord_3(a-1)-2} + o[3^{ord_3(a-1)-2}].$$

Proof. Thanks to Lemma 3.1, the solution of (3.5) has the form $1 + o[1]$. Denote

$$g(t) = t^3 - (a-1)t^2 - (a+2)t + a + 1. \tag{3.6}$$

Then $g(1) = 1 - a$, $g'(1) = 3(1-a)$. Since $0 < |1-a|_3 < 1$, clear that

$$|g'(1)|_3^2 = \frac{1}{9}|1-a|_3^2 = \frac{1}{9}|g(1)|_3^2 < |g(1)|_3,$$

i.e. the condition of Hensel's lemma does not hold.

We rewrite (3.5) as follows

$$(t-1)(t-a)(t+2) = a - 1. \tag{3.7}$$

According to Lemma 3.1, we have

$$t = 1 + 3^r T, \ a = 1 + 3^l A, \tag{3.8}$$

where $r, l \in \mathbb{N}$, $T, A \in \mathbb{Z}_p^*$. According to Remark 3.1, we have that l is even and $A \equiv 1 (\mathrm{mod}\, 3)$.

Substituting (3.8) into (3.7), we have

$$3^r T(3^r T - 3^l A)(3 + 3^r T) = 3^l A. \tag{3.9}$$

Case 1. Suppose that $l \le r$. Then 3^{r+l+1} divides LHS of (3.9), however 3^{r+l+1} does not divide RHS of (3.9). It means that (3.5) does not have any solution in closed ball $\overline{B}(1, |a-1|_3)$.

Case 2. Suppose that $1 < r < l$. Then (3.9) gives that

$$3^{2r+1} T(T - 3^{l-r} A)(1 + 3^{r-1} T) = 3^l A.$$

Last equation is solvable iff $l = 2r + 1$ and

$$3^{r-1} T^3 + (1 - 9^r A) T^2 - 3^{r+1} AT - A = 0. \tag{3.10}$$

This contrary to l is even.

Case 3. Suppose that $r = 1 < l$. Then (3.9) gives that

$$3^3 T(T - 3^{l-1} A)(1 + T) = 3^l A$$

i.e.

$$T(T - 3^{l-1} A)(1 + T) = 3^{l-3} A. \tag{3.11}$$

The equation (3.11) follows that $l \ge 3$ and

$$H(T) := T^3 + (1 - 3^{l-1} A) T^2 - 3^{l-1} AT - 3^{l-3} A = 0. \tag{3.12}$$

Since l is even. It yields that $l > 3$. The (3.11) is solvable iff

$$|T + 1|_3 = 3^{3-l}.$$

It gives that

$$T = -1 + 3^{l-3} T_1, \ T_1 \in \mathbb{Z}_3^*.$$

Substituting the last expression into (3.12) and after some algebras we have

$$H(T_1) := 3^{2l-6}T_1^3 + 3^{l-3}(-2 - 3^{l-1}A)T_1^2 + (3^{l-1}A + 1)T_1 - A = 0. \tag{3.13}$$

Since $A \equiv 1 (\mathrm{mod}\, 3)$, the equation (3.13) gives that $|H(2)|_3 = 1$, $|H(1)|_3 < 1$, $|H'(1)|_3 = 1$. By Hensel's lemma, there exists a unique solution of (3.13), such that $T_1 \equiv 1 (\mathrm{mod}\, 3)$. It gives that the equation (3.12) has a unique solution such that

$$T = -1 + 3^{l-3} + o[3^{l-3}],$$

i.e.

$$t = -2 + 3^{l-2} + o[3^{l-2}].$$

□

Theorem 3.1. *The equation* (3.2) *has a unique solution* $u_0 = 1$.

Proof. Clear that the equation (3.2) always has a solution $u_0 = 1$.

Using notation (3.4), we have

$$u_{1,2} = \frac{t \pm \sqrt{t^2 - 4}}{2}. \tag{3.14}$$

The solution $u_{1,2}$ exist in $\mathbb{Q}_3$ if and only if $\sqrt{t^2 - 4}$ exists in $\mathbb{Q}_3$.

Assume that $|a - 1|_3 < \frac{1}{27}$. According to Lemma 3.2, $t = -2 + 3^{l-2} + o[3^{l-2}]$. It follows that

$$\sqrt{t^2 - 4} = \sqrt{3^{l-2}(2 + o[1])} \notin \mathbb{Q}_3.$$

Thus, the equation (3.2) has a unique solution $u_0 = 1$. □

Theorem 3.2. *Let f_a be a function defined by* (3.1)*. Then $u_0 = 1$ is a global attractor for f_a.*

Proof. It is easy to see that

$$f'_a(u) = \frac{6u^2(a-1)(u^6 + (a+1)u^3 + 1)}{(u^6 + 2u^3 + a)^2}$$

and

$$|f'_a(1)|_3 = \left|\frac{6(a-1)}{a+3}\right|_3 < 1.$$

Thus $u_0 = 1$ is attractive fixed point.

Let us assume that $u \in \mathbb{Q}_3 \setminus \mathbb{Z}_3^*$. Then

$$f_a(u) = \begin{cases} \frac{a+o[1]}{1+o[1]}, & \text{if } u \in \mathbb{Q}_3 \setminus \mathbb{Z}_3, \\ \frac{1+o[1]}{a+o[1]}, & \text{if } u \in 3\mathbb{Z}_3, \end{cases}$$

which implies $f_a(u) \in \mathcal{E}_3$.

Moreover, for all $u \in \mathcal{E}_3$ the following holds:

$$|f_a(u) - f(1)|_p = |f_a(u) - 1|_p = \frac{1}{3} \cdot |a - 1|_3 |u - 1|_3.$$

This relation shows that f acts as a contraction mapping on $\mathcal{E}_3$. Since $\mathcal{E}_3$ is compact, the contraction mapping principle guarantees the existence of a unique fixed point in $\mathcal{E}_3$, and consequently, we have

$$\lim_{n\to\infty} f_a^n(u) = 1,$$

for every $u \in \mathbb{Q}_3 \setminus B(2,1)$.

Assume that $u \in B(2,1)$, i.e. $u = 2 + o[1]$. Then

$$|f_a(u) - 1|_3 = \left| \frac{(a-1)(u^3-1)(u^3+1)}{(u^3+1)^2 + a - 1} \right|_3 =$$

$$= \begin{cases} |u^3+1|_3, & \text{if } |(u^3+1)^2|_3 \leq |a-1|_3 \\ \frac{|a-1|_3}{|u^3+1|_3}, & \text{if } |(u^3+1)^2|_3 > |a-1|_3 \end{cases}$$

$$= \begin{cases} |u^3+1|_3, & \text{if } |(u^3+1)^2|_3 \leq |a-1|_3 \\ \leq |u^3+1|_3, & \text{if } |(u^3+1)^2|_3 > |a-1|_3 \end{cases}$$

$$< 1.$$

From the last one, we get $f_a(u) \in \mathcal{E}_3$. This completes the proof. □

Corollary 3.1. *Any periodic measure* 3*-adic quasi Gibbs measure for the Ising model on* $(2,3)$*-ary tree is translation-invariant. Moreover, there is a unique translation-invariant measure.*

Proof. The proof follows directly from Theorem 3.2. □

Conclusion

As we previously mentioned, this problem was resolved for the case $p > 3$ in [19]. In that work, it was proven that if the condition $p \equiv 1 \pmod 4$ holds, there exist at least 2^d distinct d-periodic measures; whereas if the condition $p \not\equiv 1 \pmod 4$ holds, any d-periodic measure is translation-invariant, and furthermore, there exists a unique translation-invariant p-adic quasi Gibbs measure. In the present paper, a result similar to that obtained under the condition $p \not\equiv 1 \pmod 4$ in [19] is established for the case $p = 3$. The case $p = 2$ has been omitted from this paper as it requires extensive computations.

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Competing interests

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